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COMPACT STANDARD FUZZY METRIC SPACE

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ABSTRACT

In this paper we recall the definition of standard fuzzy metric space then we define a compact standard fuzzy metric space and F-totally bounded after that we prove that F-totally bounded complete standard fuzzy metric space is compact. Moreover we recall the definition of continuous and uniform continuous function to prove that continuous function and uniform continuous functions are equivalent on a compact standard fuzzy metric spaces.

Key Words: *Standard fuzzy metric space, compact standard fuzzy metric space, F-bounded, F-totally bounded, continuous function.*

INTRODUCTION

Theory of fuzzy sets was introduced by Zadeh in 1965 [1]. Many authors have introduced the concept of fuzzy metric in different ways [2,3, 4 and 5]. Kramosil and Michalek in 1975 [6] introduced the definition of fuzzy metric space which is called later KM-fuzzy metric space .George and Veeramani in 1994[3] introduced the definition of continuous $*$ t-norm to modify the concept of KM-fuzzy metric space which was introduced by Kramosil and Michalek which is called later GV-fuzzy metric space.

In section one of this paper we recall the definition of standard fuzzy metric space [9] which is a modification of the definition GV-fuzzy metric space after that we introduce basic definitions, basic concepts and properties of standard fuzzy metric space.

In section two the notion of compact standard fuzzy metric space is introduced, we try to prove results similar to that in the ordinary case.

The aim of studying a continuous function on compact spaces in section three is to prove that continuous function and uniform continuous functions are equivalent on compact standard fuzzy metric space.

1. STANDARD FUZZY METRIC SPACE

Definition 1.1: [3] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

1. $*$ is associative and commutative.
2. $*$ is continuous.
3. $a*1 = a$ for all $a \in [0,1]$.
4. $a*b \leq c*d$ whenever $a \leq c$ and $b \leq d$ where $a,b, c,d \in[0,1]$.

Remark 1.2: [3] For any $r_1 > r_2$ we can find r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find an r_5 such that $r_5*r_5 \geq r_4$ where $r_1,r_2,r_3,r_4,r_5 \in(0,1)$.

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We introduce the following definition.

Definition 1.3:[9] A triple (X, M,$*$) is said to be standard fuzzy metric space if X is an arbitrary set, $*$ is a continuous t- norm and M is a fuzzy set on X^2 satisfying the following conditions:
(FM_1) M(x, y) > 0 for all x, y $\in$ X
(FM_2) M(x, y) = 1 if and only if x = y
(FM_3) M(x, y) = M(y, x) for all x, y $\in$ X
(FM_4) M(x,z) $\geq$ M(x,y) $*$M(y,z) for all x, y and z $\in$ X
(FM_5) M(x, y) is a continuous fuzzy set

Example 1.4: [9] Let X= ℕ, and let a$*$b = a.b for all a, b $\in$[0, 1].

Define $M(x,y)=\begin{cases}\dfrac{x}{y} & if \quad x\leq y\\ \dfrac{y}{x} & if \quad y\leq x\end{cases}$ for all x, y$\in$ℕ.

Then (ℕ, M,.) is a standard fuzzy metric space.

Example 1.5: [9] Let X=ℝ and let a$*$b = a.b for all a, b $\in$ [0, 1].

Define M(x, y) = $\frac{1}{e^{|x-y|}}$ for all x, y$\in$ ℝ.

Then (ℝ,M,.) is a standard fuzzy metric space.

Definition 1.6: [9] Let (X, M,$*$) be a standard fuzzy metric space then M is continuous if whenever $x_n \to x$ and $y_n \to y$ in X then M(x_n, y_n) $\to$M(x, y) that is $\lim_{n\to\infty}$ M(x_n, y_n) =M(x, y).

Definition 1.7: [9] Let (X, M,$*$) be a standard fuzzy metric space .Then B(x, r) ={y$\in$X: M(x, y) > 1-r} is an open ball with center x$\in$X and radius r, 0 < r < 1**.**

Proposition 1.8: [9] Let B(x,r_1) and B(x,r_2) be two open balls with same center x in a standard fuzzy metric space (X,M,$*$).Then either B(x,r_1) $\subseteq$ B(x,r_2) or B(x,r_2) $\subseteq$ B(x,r_1) where r_1,$r_2$$\in$(0,1).

Definition 1.9:[9] A subset A of a standard fuzzy metric space (X, M,$*$) is said to be open if given any point a in A there exists r , 0 < r < 1 such that B(a, r) $\subseteq$ A. A subset B is said to be closed if B^c is open.

Theorem 1.10: [9] Every open ball in a standard fuzzy metric space (X, M,$*$) is an open set.

Theorem 1.11: [9] Let (X, M,$*$) is a standard fuzzy metric space. Define τ_M= {A$\subset$X: x$\in$A if and only if there exists 0 < r < 1 such that B(x, r) $\subset$ A} Then τ_M is a topology on X.

Theorem 1.12: [9] Every standard fuzzy metric space is a Hausdorff space.

Definition 1.13:[9] A sequence (x_n) in a standard fuzzy metric space (X,M,$*$) is said to be converge to a point x in X if for each r,0 < r < 1 there exists a positive number N such that M(x_n, x) > (1-r), for each n $\geq$ N.

Theorem 1.14: [9] Let (X, M,$*$) be a standard fuzzy metric space then for a sequence (x_n) in X converge to x if and only if $\lim_{n\to\infty}$ M(x_n, x) = 1.

Definition 1.15: [9] A sequence (x_n) in a standard fuzzy metric space (X,M,$*$) is Cauchy if for each r, 0 < r < 1, there exists a positive number N such that M(x_n,x_m) > (1-r), for each m, n $\geq$ N.

Proposition 1.16: [9] Let (X, d) be an ordinary metric space and let a$*$b = a.b for all a, b$\in$[0,1].
Define M_d(x, y) = $\frac{1}{1+d(x,y)}$, then (X,M_d,$*$) is a standard fuzzy metric space and it is called the standard fuzzy metric induced by the metric d.

Proposition 1.17: [9] Let (X, d) be a metric space and let (X,M_d,$*$) be the standard fuzzy metric space induced by d. Let (x_n) be a sequence in X. Then (x_n) converges to x$\in$X in (X, d) if and only if (x_n) converges to x in (X,M_d,$*$).

Proposition 1.18: [9] Let (X, d) be a metric space and let $M_d(x,y) = \frac{1}{1+d(x,y)}$. Then (x_n) is a Cauchy sequence in (X, d) if and only if (x_n) is a Cauchy sequence in (X, M_d,∗).

Definition 1.19: [9] Let (X, M,∗) be a standard fuzzy metric space. A subset A of X is said to be F-bounded if there exists $0 < r < 1$ such that, $M(x, y) > 1- r$, for all x, y∈A.

Proposition 1.20: [9] Let (X, d) be a metric space and let $M_d(x,y) = \frac{1}{1+d(x,y)}$ then a subset A of X is F-bounded if and only if it is bounded.

Definition1.21: [9] A standard fuzzy metric space (X, M,∗) is complete if every Cauchy sequence in X converges to a point in X.

Definition 1.21: [9] Let (X,M_X,∗) and (Y,M_Y,∗) be standard fuzzy metric spaces and A ⊆ X. A function f:A→Y is said to be continuous at a∈A, if for every $0 < \varepsilon < 1$, there exist some $0 < \delta < 1$, such that $M_Y(f(x),f(a)) > (1- \varepsilon)$ whenever x∈A and $M_X(x,a) > (1- \delta)$. If f is continuous at every point of A, then it is said to be continuous on A.

Theorem 1.22: [9] Let (X,M_X,∗) and (Y,M_Y,∗) be standard fuzzy metric spaces and A⊆X. A function f: A→Y is continuous at a∈A if and only if whenever a sequence (x_n) in A converge to a, the sequence $(f(x_n))$ converges to f(a).

Theorem 1.23:[9] A function f:X→Y is continuous on X if and only if $f^{-1}(G)$ is open in X for all open subset G of Y.

Theorem 1.24: [9] A mapping f: X→Y is continuous on X if and only if $f^{-1}(F)$ is closed in X for all closed subset F of Y.

2. COMPACTNESS

Definition 2.1: Let (X, M,∗) be a standard fuzzy metric space and Y⊆ X. Let $\widehat{G}$ be a collection of open sets in X with the property that $Y \subseteq \bigcup_{G\in\widehat{G}} G$. Equivalently, for each x∈Y there is G∈$\widehat{G}$ such that x∈G. Then $\widehat{G}$ is called an open cover or an open covering of Y. A finite subcollection of $\widehat{G}$. which itself a cover is called a finite subcover or a finite subcovering of Y.

Definition 2.2: A standard fuzzy metric space (X,M,∗) is said to be compact if every open covering $\widehat{G}$ of X has a finite subcovering that is there is a finite subcollection $\{G_1, G_2, G_3,\ldots, G_n\} \subseteq \widehat{G}$ such that $X = \bigcup_{i=1}^{n} G_i$.

Definition 2.3: A nonempty subset Y of X is said to be compact if it is compact with the standard fuzzy metric induced on it by M.

Example 2.4: The interval (0, 1) in the standard fuzzy metric space (ℝ, M,∗) where $M(x, y) = \frac{1}{1+|x-y|}$, a∗b = a.b for all a, b∈[0,1], is not compact. In order to prove the assertion suffices to exhibit an open covering from which no finite subcovering can be selected. The open covering $\{(\frac{1}{n}, 1): n = 2, 3...\}$ is one such covering of (0, 1), from which no finite subcovering can be selected.

Remark 2.5: Let Y be a finite subset of a standard fuzzy metric space (X, M,∗). Then Y is compact.

Definition 2.6: A collection F of sets in X is said to have the finite intersection property if every finite subcollection of F has a nonempty intersection.

Definition 2.7: Let S be a subset of X where (X,M,∗) is a standard fuzzy metric space then S is called F-totally bounded if for each $0 < \varepsilon < 1$, there is a finite set of points $\{y_1,y_2,y_3,\ldots,y_n\} \subset S$ such that whenever x in X, $M(x, y_i) > (1- \varepsilon)$ for some $y_i \in \{y_1,y_2,y_3,\ldots,y_n\}$. This set of points $\{y_1,y_2,y_3\ldots y_n\}$ is called F- ε - net.

Proposition 2.8: An F-totally bounded standard fuzzy metric space is F-bounded.

Proof: Let (X, M,∗) be F-totally bounded and suppose $0 < \varepsilon < 1$ is given. Then there exists a finite F-ε- net for X, say A. Since A is a finite set of points $0 < M(A) < 1$, where M(A) = sup{M(y,z): y, z∈A}. Now let $x_1$and x_2 be any two points of X. There exists points y and z in A such that $M(x_1,y) > (1- \varepsilon)$ and $M(x_2,z) > (1- \varepsilon)$. Now for M(A) and ε there is 1- r, where $0 < r < 1$ such that $M(A) * (1- \varepsilon) * (1- \varepsilon) \geq (1- r)$. It follows that

$M(x_1,x_2) \geq M(x_2,y) * M(y,z) * M(z,x_2)$

$\geq (1- \varepsilon) * M(A) * (1- \varepsilon) \geq (1- r)$

So, $M(X) = \sup\{M(x_1,x_2):x_1, x_2 \in X\} \geq (1- r)$. Hence, X is F-bounded.

Theorem 2.9: Let Y be a subset of a standard fuzzy metric space (X, M,∗). Then Y is F-totally bounded if and only if every sequence in Y contains a Cauchy subsequence.

Proof: Suppose Y is F- totally bounded. Let (y_n) be a sequence in Y whose range may be assumed to be infinite. Choose a finite F- $\frac{1}{2}$ -net in Y. Then one of the balls of radius $\frac{1}{2}$ with center in the F- $\frac{1}{2}$ -net contains infinitely many elements of the range of the sequence. We shall denote the subsequence formed by these elements by $(y_n^{(1)})$. Choose a finite F- $\frac{1}{4}$ -net in Y. Then one of the balls of radius $\frac{1}{4}$ with center in the finite F- $\frac{1}{4}$ -net contains infinitely many elements of the range of $(y_n^{(1)})$.

We shall denote the subsequence formed as $(y_n^{(2)})$. Proceeding in this way, we obtain a sequence of sequences, each a subsequence of the preceding one, so that at the kth stage, the terms $(y_n^{(k)})$ lie in the ball of radius $\frac{1}{2^k}$ with center in the F- $\frac{1}{2^k}$ -net.

Now $(y_n^{(n)})$ is a subsequence of (y_n). Let $0 < \varepsilon < 1$ be given. Choose N so large that
$(1-\frac{1}{2^n}) * (1-\frac{1}{2^{n+1}}) * \ldots * (1-\frac{1}{2^{m-1}}) > (1-\varepsilon)$.

Then for $m > n > N$, we have

$$
\begin{aligned}
M(y_n^{(n)}, y_m^{(m)}) &\geq M(y_n^{(n)}, y_{n+1}^{(n+1)}) * M(y_{n+1}^{(n+1)}, y_{n+2}^{(n+2)}) * \ldots * M(y_{m-1}^{(m-1)}, y_m^{(m)}) \\
&\geq (1-\frac{1}{2^n}) * (1-\frac{1}{2^{n+1}}) * \ldots (1-\frac{1}{2^{m-1}}) \\
&> (1-\varepsilon)
\end{aligned}
$$

So that the sequence $(y_n^{(n)})$ is a Cauchy sequence.

Conversely, suppose that every sequence in Y has a Cauchy subsequence

We shall show that Y is F-totally bounded. Let $0 < \varepsilon < 1$ and let $y_1 \in Y$. If $Y - B(y_1,\varepsilon) = \emptyset$, we have found an F-ε-net, namely, the set $\{y_1\}$. Otherwise choose $y_2 \in Y - B(y_1,\varepsilon)$. If $Y - [B(y_1,\varepsilon) \cup B(y_2,\varepsilon)] = \emptyset$. We have found an F- ε-net, namely the set $\{y_1, y_2\}$. It is enough to show that this process terminates, after a finite number of steps. If it dose not terminate, we shall obtain an infinite sequence (y_n) with property that $M(y_n, y_m) \leq (1-\varepsilon)$, $n \neq m$. Consequently, the sequence (y_n) would have no Cauchy subsequence, contrary to hypothesis.

We give below a characterization of compact standard fuzzy metric space.

Proposition 2.10: Let (X, M,∗) be a compact standard fuzzy metric space. Then X is F-totally bounded.

Proof: For any given $0 < \varepsilon < 1$, the collection of all balls $B(x,\varepsilon)$ for $x \in X$ is an open cover of X. The compactness of X implies that this open cover contains a finite sub cover. Hence for $0 < \varepsilon < 1$, X is covered by a finite number of open balls of radius ε i.e the centers of the balls in the finite subcover form a finite F-ε-net for X. So, X is F-totally bounded.

Proposition 2.11: Let (X, M,∗) be a compact standard fuzzy metric space. Then (X, M,∗) is complete.

Proof: Suppose, if possible, that (X, M,∗) is a compact standard fuzzy metric space is not complete. Then there exists a Cauchy sequence (x_n) in (X,M,∗) not having a limit in X. Let $y \in X$, since (x_n) dose not converge to y there exists $0 < r < 1$. Such that $M(x_n,y) \leq (1-r)$ for infinitely many values of n, since (x_n) is Cauchy, there exists an integer N such that n, m ≥ N .That implies $M(x_n, x_m) > (1-\varepsilon)$. Choose m≥N for which $M(x_m,y) > (1-\varepsilon)$. So, the open ball $B(y,\varepsilon)$ contains x_n for only finitely many values of n. In this manner, we can associate with each $y \in X$ a ball $B(y,\varepsilon(y))$, where $0 < \varepsilon(y) < 1$ depends on y, and the ball $B(y, \varepsilon(y))$ contains x_n for only finitely many values of n. Observe that
$X = \cup_{y \in X} B(y,\varepsilon(y))$ which means that $\{B(y,\varepsilon(y)) : y \in X\}$ is a covering of X. Since X is compact there exists a finite subcovering $B(y_i,\varepsilon(y_i))$, i = 1, 2, …, n.

Since each ball contains x_n for only a finite number of values of n, therefore the balls are in the finite subcovering and, hence, also X, must contains x_n for only a finite number of values of n. This however, is impossible. Hence (X, M,∗) must be complete

Theorem 2.12: Let (X, M,∗) be F-totally bounded and complete standard fuzzy metric space. Then (X, M,∗) is compact.

Proof: Suppose, if possible, that (X, M,∗) is F-totally bounded and complete but not compact. Then there exists an open covering $\{G_\lambda: \lambda\in\Lambda\}$ of X that dose not admit a finite subcovering. Since (X, M,∗) is F-totally bounded, it is F-bounded, hence for some $0 < r < 1$ and some $x\in X$, we have $X \subseteq B(x, r)$. Observe that $X \subseteq B(x, r)$ implies $X = B(x, r)$.

Let $\varepsilon_n = \frac{r}{2^n}$. We know that X being F-totally bounded can be covered by finite many balls of radius ε_1. By our hypothesis at least one of these balls, say $B(x_1,\varepsilon_1)$, cannot be covered by a finite number of sets G_λ.

[for if each had a finite subcovering, the same would be true for X]

Because $B(x_1,\varepsilon_1)$ is itself F-totally bounded [any nonempty subset of F-totally bounded set is F-totally bounded], we can find an $x_2\in B(x_1,\varepsilon_1)$ such that $B(x_2,\varepsilon_2)$ cannot be covered by a finite number of sets G_λ.

In this way, a sequence (x_n) may be defined with the property that for each n, $B(x_n,\varepsilon_n)$ cannot be covered by a finite number of sets G_λ and $x_{n+1} \in B(x_n,\varepsilon_n)$.We next show that the sequence (x_n) is convergent.

Since $x_{n+1} \in B(x_n,\varepsilon_n)$ it follows that $M(x_n,x_{n+1}) > (1- \varepsilon_n)$.

Let $0 < \varepsilon < 1$ such that $(1- \varepsilon_n) * (1- \varepsilon_{n+1}) *\ldots.. *(1- \varepsilon_m) > (1- \varepsilon)$

Hence

$$\begin{aligned} M(x_n,x_m) &\geq M(x_n,x_{n+1}) *\ldots\ldots* M(x_{m-1},x_m) \\ &\geq (1- \varepsilon_n) * (1- \varepsilon_{n+1}) *\ldots.. *(1- \varepsilon_m) \\ &> (1- \varepsilon) \end{aligned}$$

So (x_n) is a Cauchy sequence in X and since X is complete, it converges to $y\in X$, say. Since $y\in X$ there exists $\lambda_0\in \Lambda$ such that $y\in G_{\lambda_0}$. Because G_{λ_0} is open it contains $B(y,\delta)$ for some $0 < \delta < 1$. Choose N so large that, $M(x_n,y) > (1- \delta)$ and $(1- \varepsilon_n) > (1- \delta)$. Then, for any $x\in X$ such that

$M(x, x_n) > (1- \varepsilon_n)$. It follows that

$$\begin{aligned} M(x, y) &\geq M(x, x_n) * M(x_n,y) \\ &\geq (1- \delta) * (1- \delta) > (1- r), \end{aligned}$$

for some $0 < r < 1$. So that $B(x_n,\varepsilon_n) \subseteq B(y, r)$. Therefore $B(x_n,\varepsilon_n)$ admits a finite subcovering, namely by the set G_{λ_0}.

Since this contradicts $B(x_n,\varepsilon_n)$ cannot be covered by a finite number of sets G_λ, the proof is complete.

Theorem 2.13: A standard fuzzy metric space is compact if and only if it is complete and F-totally bounded.

Theorem 2.14: (X, d) is a compact metric space if and only if $(X,M_d,*)$ is a compact standard fuzzy metric space where $M_d(x,y) = \frac{1}{1+d(x,y)}$.

Proof: Suppose that (X, d) is compact. Let (x_n) be a sequence in $(X,M_d,*)$ then (x_n) is a sequence in (X,d). But (X,d) is compact hence (x_n) has a convergent subsequence. Then (x_n) has a convergent sequence in $(X,M_d,*)$ by Proposition 1.17. Hence $(X,M_d,*)$ is compact. In similar way we can prove that if $(X,M_d,*)$ is compact then (X, d) is compact by using Proposition 1.17.

Proposition 2.15: Let (X, M,∗) be a standard fuzzy metric space. Then the following tatements are equivalent:
(i) Every infinite set in (X, M,∗) has at least one limit point in X.
(ii) Every infinite sequence in (X, M,∗) contains a convergent subsequence.

Proof: (i)⇒(ii): Let (x_n) be a sequence in X. If the set $\{x_1, x_2,\ldots,\}$ is finite, then one of the points, say x_{i_0}, satisfies $x_{i_0}= x_j$, for infinitely many $j\in N$. Hence the constant sequence (x_{i_0}) is a subsequence of (x_n), which converges to the point x_{i_0}. Suppose that the set $\{x_1, x_2,\ldots,\}$ is infinite.

In view of (i), the infinite set $\{x_1, x_2,\ldots,\}$ has at least one limit point $x\in X$. A subsequence of (x_n) that converges to x may be obtained as follows: Let n_1 be any integer such that $M(x_{n_1}, x) > 0$.

Having defined n_k, let n_{k+1} be the smallest integer such that $n_{k+1} > n_k$ and $M(x_{n_{k+1}},x) > 1- \frac{1}{k+1}$. Then the sequence (x_{n_k}) converges to x.

Proof (ii)⇒(i): Let Y be an infinite subset of X. Then there exists a sequence (y_n) in X of distinct terms. In view of (ii) (y_n) contains a subsequence (y_{n_i}) of distinct terms that converges to y∈X. Hence every open ball with center y contains an infinite number of terms of the convergent subsequence (y_{n_i}). But the terms are distinct; hence every open ball centered at y contains an infinite number of points of Y. y∈X is a limit point of Y.

Theorem 2.16: The standard fuzzy metric space (X, M,∗) is compact if and only if every sequence of points in X has a subsequence converging to a point in X.

Proof: Suppose first that X is compact (equivalently, F-totally bounded and complete see Theorem 2.13) and that (x_n) is any sequence of points in X. Since X is F-totally bounded, it follows, using Theorem 2.13, that (x_n) contains a Cauchy subsequence (x_{n_i}). But (x_{n_i}) converges to a point x∈X because X is complete. Thus, if X is complete, then every sequence in X contains a convergent subsequence.

Conversely, suppose every sequence in X has a convergent subsequence, it follows in view of the fact that every convergent sequence is Cauchy and Theorem 2.13 that X is F-totally bounded. It remains to show that X is complete. To this end let (x_n) be a Cauchy sequence in X. By assumption (x_n) has a subsequence (x_{n_i}) that converges to a point x∈X. We shall show that $x_n \to x$. Let $0 < \varepsilon < 1$ be given by Remark 1.2, there is $0 < r < 1$ such that
$(1- r) *(1- r) > (1- \varepsilon)$.

Now $x_{n_i} \to x$, there exist N_1 such that $M(x_{n_i},x) > (1- r)$ for all $n_i \geq N_1$.

Since the sequence (x_n) is Cauchy, there exists N_2 such that

$M(x_n,x_m) > (1- r)$ for all m, n ≥ N_2.

Let N = minN_1, N_2} then $M(x_n,x) \geq M(x_n,x_{n_i}) * M(x_{n_i},x)$
$> (1- r) *(1- r) > (1- \varepsilon)$

For all n ≥ N. This complete the proof.

The results of this section can be summed up as follows:

Theorem 2.17: Let (X, M,∗) be a standard fuzzy metric space. The following statements are equivalent:

a. (X,M,∗) is compact
b. (X,M,∗) is complete and F-totally bounded
c. Every infinite set in X has at least one limit point
d. Every sequence in X contains a convergent subsequence.

Corollary 2.18: Let Y be a closed subset of the compact standard fuzzy metric space (X, M,∗). Then Y is compact**.**

Proof: Let (y_n) be a sequence of points in Y. Then (y_n) considered as sequence of points in X, has a subsequence converging to a point x∈X. But then x∈Y since Y is closed. Thus any sequence in Y has a subsequence converging to a point in Y. By Theorem 2.13 Y is compact.

Theorem 2.19: Let Y be a subset of a standard fuzzy metric space (X, M,∗). If Y is compact then Y is a closed subset of X.

Proof: Let x∈X be a limit point of Y. Then there is a sequence (y_n) in Y converging to x. But then (y_n) is a Cauchy sequence in Y. Since Y is complete,(y_n) converges to a point y in Y. This point y must be x and so x∈Y. Thus Y contains all its limit points and is therefore closed.

3. CONTINUOUS FUNCTION ON COMPACT SPACE

Theorem 3.1: Let f be a continuous function from a compact standard fuzzy metric space $(X,M_X,*)$ into a standard metric space $(Y,M_Y,*)$. Then the range f(X) of f is also compact.

Proof: Let $\{G_\lambda: \lambda\in\Lambda\}$ be an open covering of f(X). Since f is continuous $f^{-1}(G_\lambda)$ is open in X. Moreover $\{f^{-1}(G_\lambda):\lambda\in\Lambda\}$ is an open covering of X. Since X is compact, there exists $\lambda_1, \lambda_2, \lambda_3,\ldots, \lambda_n$ in Λ such that $X = \cup_{i=1}^{n} f^{-1}(G_{\lambda_i})$.

Now
$f(X) = f(\cup_{i=1}^{n} f^{-1}(G_{\lambda_i})) = \cup_{i=1}^{n} f(f^{-1}(G_{\lambda_i})) \subseteq \cup_{i=1}^{n} G_{\lambda_i}$

So $\{G_{\lambda_i}$, i = 1, 2, …, n$\}$ is a finite sub covering of f(X). Consequently, f(X) is compact

Corollary 3.2: Let f be a homeomorphism of a standard fuzzy metric space $(X,M_X,*)$ onto a standard fuzzy metric space $(Y,M_Y,*)$. Then X is compact if and only if Y is compact.

Corollary 3.3: Let f be a continuous function from a compact standard fuzzy metric space $(X,M_X,*)$ onto a standard fuzzy metric space $(Y,M_Y,*)$. Then f(X) is F-bounded and closed subset of Y.

Theorem 3.4: If f is a one-to-one continuous mapping of a compact standard fuzzy metric space $(X,M_X,*)$ onto a standard fuzzy metric space $(Y,M_Y,*)$ then f^{-1} is continuous on Y and, hence, f is homeomorphism of $(X,M_X,*)$ onto $(Y,M_Y,*)$.

Proof: Suppose f: X→Y is one-to-one and onto. Its inverse f^{-1}:Y→X is well defined . Let F be a closed subset of X. By Theorem 2.19, F is compact. By Theorem 3.1 f(F) is compact and hence, a closed subset of Y by Theorem 2.19. But f(F) = $(f^{-1})^{-1}$(F) and so $(f^{-1})^{-1}$(F) is closed in Y. Hence by Theorem 1.24 f^{-1} is continuous.

Definition 3.5: Let $\{G_\lambda : \lambda\in\Lambda\}$ be an open covering of the standard fuzzy metric space (X,M,∗). Any number $0 < \delta < 1$ such that for each x∈X there exists λ∈Λ (dependent on x) for which $B(x,\delta) \subseteq G_\lambda$ is called a Lebesgue number of the covering $\{G_\lambda : \lambda\in\Lambda\}$.

Theorem 3.6: Let $(X,M_X,*)$ be a compact standard fuzzy metric space, $(Y,M_Y,*)$ be an arbitrary standard fuzzy metric space and f: X→Y be continuous.

Then for each $0 < \varepsilon < 1$, there exists a δ, $0 < \delta < 1$ (δ depending on ε only) such that $f(B(x,\delta)) \subseteq B(f(x),\varepsilon)$ for every x∈X. That is f is uniformly continuous on X.

Proof: Let $0 < r < 1$ such that $(1- r) * (1- r) > (1- \varepsilon)$ .The collection of ball {B(y, r): y∈Y} constitutes an open cover of Y. The set $\{f^{-1}(B(y, r)): y\in Y\}$ therefore form an open cover of the compact standard fuzzy metric space X. Let δ be a Lebesgue number of this open cover of X.

Since each open ball B(x,δ) lies in one of these sets,

$f(B(x,\delta)) \subseteq B(y, r)$ for some y∈Y.

Because f(x)∈ B(y, r), we find for any z∈ B(x,δ) that

$$M_Y(f(z),f(x)) \geq M_Y(f(z),y) * M_Y(y,f(x))$$
$$> (1- r) * (1- r) > (1- \varepsilon)$$

i.e, $f(B(x,\delta)) \subseteq B(f(x),\varepsilon)$ this complete the proof.

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