International Journal of Mathematical Archive-5(7), 2014, 118-128
IJMA Available online through www.ijma.info ISSN 2229 – 5046

BIPOLAR ANTI-FUZZY NORMAL HX SUBGROUPS

R. Muthuraj*[1] and M. Sridharan[2]

[1]PG & Research Department of Mathematics,
H. H. The Rajah's College, Pudukkottai – 622 001, Tamilnadu, India.

[2]Department of Mathematics,
PSNA College of Engineering and Technology, Dindigul-624 622, Tamilnadu, India.

(Received on: 22-06-14; Revised & Accepted on: 17-07-14)

ABSTRACT

In this paper, we define a new algebraic structure of a bipolar anti fuzzy normal HX subgroup and some properties of intersection and union bipolar anti fuzzy normal HX subgroups are discussed. We establish the necessary condition for intersection of any two bipolar anti fuzzy normal HX subgroups is a bipolar anti fuzzy normal HX subgroup.The concept of an anti image, anti pre-image of a bipolar fuzzy subsets were discuss in detail a series of homomorphic and anti homomorphic properties of bipolar anti fuzzy normal HX subgroups. We define lower level subset of a bipolar anti fuzzy normal HX subgroup and discuss some of its properties. Characterizations of lower level subsets of a bipolar anti fuzzy normal HX subgroup of a HX group are given.

AMS Subject Classification (2000): *20N25, 03E72, 03F055, 06F35, 03G25.*

Keywords: *fuzzy set, bipolar-valued fuzzy set, bipolar fuzzy HX subgroup, bipolar anti fuzzy HX subgroup, bipolar fuzzy normal HX subgroup ,bipolar anti fuzzy normal HX subgroup, homomorphism and anti homomorphism ,anti image and anti pre-image of bipolar fuzzy subsets, lower level subset of bipolar anti fuzzy normal HX subgroup.*

1. INTRODUCTION

The concept of fuzzy sets was initiated by Zadeh [16]. Then it has become a vigorous area of research in engineering, medical science, social science, graph theory etc. Rosenfeld [15] gave the idea of fuzzy subgroups. Li Hongxing [3] introduced the concept of HX group and the authors Luo Chengzhong, Mi Honghai, Li Hongxing [5] introduced the concept of fuzzy HX group. In fuzzy sets the membership degree of elements ranges over the interval [0, 1]. The membership degree expresses the degree of belongingness of elements to a fuzzy set. The membership degree 1 indicates that an element completely belongs to its corresponding fuzzy set and membership degree 0 indicates that an element does not belong to fuzzy set. The membership degrees on the interval (0, 1) indicate the partial membership to the fuzzy set. Sometimes, the membership degree means the satisfaction degree of elements to some property or constraint corresponding to a fuzzy set. Bipolar valued fuzzy set, which was introduced by K.M.Lee [2] are an extension of fuzzy sets whose membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bipolar-valued fuzzy set, the membership degree 0 means that the elements are irrelevant to the corresponding property, the membership degree (0, 1] indicates that elements somewhat satisfy the property and the membership degree [−1, 0) indicates that elements somewhat satisfy the implicit counter-property. The author W.R.Zhang [17] commenced the concept of bipolar fuzzy sets as a generalization of fuzzy sets.R.Muthuraj *et.al* [9] introduced the concept of bipolar anti fuzzy HX subgroup of a HX group.In this paper we define a new algebraic structure of a bipolar anti fuzzy normal HX group and its lower level normal sub HX subgroups and study some of their properties. We define the concept of union and intersection of bipolar fuzzy subsets of a bipolar anti fuzzy normal HX subgroups of a HX group and discuss some of their related properties. We [10] introduced the concept of an anti image and anti pre-image of a bipolar fuzzy subsets and discuss some of its properties with bipolar anti fuzzy normal HX subgroup under homomorphism and anti homomorphism.

Corresponding author: R. Muthuraj*[1]
[1]PG & Research Department of Mathematics, H.H.The Rajah's College,
Pudukkottai – 622 001, Tamilnadu, India.

2. PRELIMINARIES

In this section, we site the fundamental definitions that will be used in the sequel. Throughout this paper, G = (G,*) is a finite group, e is the identity element of G, and xy we mean x * y.

Definition 2.1 [3]: In $2^G - \{\phi\}$, a non-empty set $\vartheta \subset 2^G - \{\phi\}$ is called a HX group on G, if ϑ is a group with respect to the algebraic operation defined by
AB = { ab / a $\in$ A and b $\in$ B}, which its unit element is denoted by E.

Definition 2.2 [1]: Let X be any non-empty set. A fuzzy subset μ of X is a function $\mu : X \to [0,1]$.

Definition 2.3 [11]: Let G be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set μ in G is an object having the form $\mu = \{\langle x, \mu^+(x), \mu^-(x) \rangle : x \in G\}$,where $\mu^+ : G \to [0,1]$ and $\mu^- : G \to [-1,0]$ are mappings. The positive membership degree $\mu^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to a bipolar-valued fuzzy set $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$ and the negative membership degree $\mu^-(x)$ denotes the satisfaction degree of an element x to some implicit counter property corresponding to a bipolar-valued fuzzy set $\mu = \{\langle x, \mu^+(x), \mu^-(x) \rangle : x \in G\}$. If $\mu^+(x) \neq 0$ and $\mu^-(x) = 0$, it is the situation that x is regarded as having only positive satisfaction for $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$. If $\mu^+(x) = 0$ and $\mu^-(x) \neq 0$, it is the situation that x does not satisfy the property of $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$, but somewhat satisfies the counter property of $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$. It is possible for an element x to be such that $\mu^+(x) \neq 0$ and $\mu^-(x) \neq 0$ when the membership function of property overlaps that its counter property over some portion of G. For the sake of simplicity, we shall use the symbol $\mu = (\mu^+, \mu^-)$ for the bipolar-valued fuzzy set $\mu = \{\langle x, \mu^+(x), \mu^-(x)\rangle : x \in G\}$.

Definition 2.4 [11]: A bipolar-valued fuzzy set or bipolar fuzzy set μ of G is a bipolar fuzzy subgroup of G if for all x, y $\in$G,

i. $\mu^+(xy) \geq \min\{\mu^+(x), \mu^+(y)\}$,
ii. $\mu^-(xy) \leq \max\{\mu^-(x), \mu^-(y)\}$,
iii. $\mu^+(x^{-1}) = \mu^+(x)$, $\mu^-(x^{-1}) = \mu^-(x)$.

Definition 2.5 [11]: A bipolar-valued fuzzy set or bipolar fuzzy set μ of G is a bipolar anti fuzzy subgroup of G if for all x, y $\in$G,

i. $\mu^+(xy) \leq \max\{\mu^+(x), \mu^+(y)\}$,
ii. $\mu^-(xy) \geq \min\{\mu^-(x), \mu^-(y)\}$,
iii. $\mu^+(x^{-1}) = \mu^+(x)$, $\mu^-(x^{-1}) = \mu^-(x)$.

Definition 2.6 [11]: Let $\mu = (\mu^+, \mu^-)$ be a bipolar fuzzy subset of G. Let ϑ be a non-empty set. A bipolar-valued fuzzy set or bipolar fuzzy set λ_μ in ϑ is an object having the form $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$ where $\lambda_\mu^+ : \vartheta \to [0,1]$ and $\lambda_\mu^- : \vartheta \to [-1,0]$ are mappings. The positive membership degree $\lambda_\mu^+(A)$ denotes the satisfaction degree of an element A to the property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$ and the negative membership degree $\lambda_\mu^-(A)$ denotes the satisfaction degree of an element A to some implicit counter property corresponding to a bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$. If $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) = 0$, it is the situation that A is regarded as having only positive satisfaction for $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$. If $\lambda_\mu^+(A) = 0$ and $\lambda_\mu^-(A) \neq 0$, it is the situation that A does not satisfy the property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$, but somewhat satisfies the counter property of $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$. It is possible for an element A to be such that $\lambda_\mu^+(A) \neq 0$ and $\lambda_\mu^-(A) \neq 0$ when the membership function of property overlaps that its counter property over some portion of ϑ. For the sake of simplicity, we shall use the symbol $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ for the bipolar-valued fuzzy set $\lambda_\mu = \{\langle A, \lambda_\mu^+(A), \lambda_\mu^-(A)\rangle : A \in \vartheta\}$.

Definition 2.7 [11]: Let G be a group. Let μ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. A bipolar fuzzy set λ_μ defined on ϑ is said to be a bipolar fuzzy subgroup induced by μ on ϑ or a bipolar fuzzy HX subgroup on ϑ if for A , B $\in \vartheta$,

i. $\lambda_\mu^+(AB) \geq \min\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$
ii. $\lambda_\mu^-(AB) \leq \max\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$
iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$.

where, $\lambda_\mu^+(A) = \max\{\mu^+(x) \,/\, \text{for all } x \in A \subseteq G\}$ and
$\lambda_\mu^-(A) = \min\{\mu^-(x) \,/\, \text{for all } x \in A \subseteq G\}$.

Definition 2.8 [11]: Let G be a group. Let μ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. A bipolar fuzzy set λ_μ defined on ϑ is said to be a bipolar anti fuzzy subgroup induced by μ on ϑ or a bipolar anti fuzzy HX subgroup on ϑ if for A, B $\in \vartheta$,

i. $\lambda_\mu^+(AB) \leq \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}$
ii. $\lambda_\mu^-(AB) \geq \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}$
iii. $\lambda_\mu^+(A^{-1}) = \lambda_\mu^+(A)$, $\lambda_\mu^-(A^{-1}) = \lambda_\mu^-(A)$.

where, $\lambda_\mu^+(A) = \max\{\mu^+(x) \,/\, \text{for all } x \in A \subseteq G\}$ and
$\lambda_\mu^-(A) = \min\{\mu^-(x) \,/\, \text{for all } x \in A \subseteq G\}$.

Definition 2.9: Let G be a group. Let μ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy subset on ϑ, then λ_μ is said to be bipolar fuzzy normal HX subgroup on ϑ, if

$\lambda_\mu^+(ABA^{-1}) \geq \lambda_\mu^+(B)$, $\lambda_\mu^-(ABA^{-1}) \leq \lambda_\mu^-(B)$, for all A , B $\in \vartheta$.

where, $\lambda_\mu^+(A) = \max\{\mu^+(x) \,/\, \text{for all } x \in A \subseteq G\}$ and
$\lambda_\mu^-(A) = \min\{\mu^-(x) \,/\, \text{for all } x \in A \subseteq G\}$.

Definition 2.10: Let G be a group. Let μ be a bipolar fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar fuzzy subset on ϑ, then λ_μ is said to be bipolar anti fuzzy normal HX subgroup on ϑ, if

$\lambda_\mu^+(ABA^{-1}) \leq \lambda_\mu^+(B)$, $\lambda_\mu^-(ABA^{-1}) \geq \lambda_\mu^-(B)$, for all A, B $\in \vartheta$

where, $\lambda_\mu^+(A) = \min\{\mu^+(x) \,/\, \text{for all } x \in A \subseteq G\}$ and
$\lambda_\mu^-(A) = \max\{\mu^-(x) \,/\, \text{for all } x \in A \subseteq G\}$.

Theorem 2.11: Let G be a group and $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of a HX group ϑ, then the following conditions are equivalent

i. λ_μ is a bipolar anti fuzzy normal HX subgroup on ϑ.
ii. $\lambda_\mu(ABA^{-1}) = \lambda_\mu(B)$, for all A , B $\in \vartheta$.
iii. $\lambda_\mu(AB) = \lambda_\mu(BA)$, for all A , B $\in \vartheta$.

Proof: i $\Rightarrow$ ii: Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy normal HX subgroup on ϑ, then $\lambda_\mu^+(ABA^{-1}) \leq \lambda_\mu^+(B)$, $\lambda_\mu^-(ABA^{-1}) \geq \lambda_\mu^-(B)$, for all A, B $\in \vartheta$

Thus, taking advantage of the arbitrary property of A, we get

$\lambda_\mu^+(A^{-1}BA) = \lambda_\mu^+(A^{-1}B(A^{-1})^{-1}) \leq \lambda_\mu^+(B)$

Therefore, $\lambda_\mu^+(B) = \lambda_\mu^+(A^{-1}ABA^{-1}A) = \lambda_\mu^+(A^{-1}(ABA^{-1})A) \leq \lambda_\mu^+(ABA^{-1}) \leq \lambda_\mu^+(B)$

Hence, $\lambda_\mu^+(ABA^{-1}) = \lambda_\mu^+(B)$.

Similarly,
$\lambda_\mu^-(A^{-1}BA) = \lambda_\mu^-(A^{-1}B(A^{-1})^{-1}) \geq \lambda_\mu^-(B)$

Therefore, $\lambda_\mu^-(B) = \lambda_\mu^-(A^{-1}ABA^{-1}A) = \lambda_\mu^-(A^{-1}(ABA^{-1})A) \geq \lambda_\mu^-(ABA^{-1}) \geq \lambda_\mu^-(B)$

Hence, $\lambda_\mu^-(ABA^{-1}) = \lambda_\mu^-(B)$.

Hence , $\lambda_\mu(ABA^{-1}) = \lambda_\mu(B)$, for all A, B $\in \vartheta$.

ii $\Rightarrow$ iii: Substituting BA for B in (ii) ,we can easily get (iii).

iii $\Rightarrow$ i: According to $\lambda_\mu(AB) = \lambda_\mu(BA)$, for all A, B $\in \vartheta$ we obtain $\lambda_\mu(ABA^{-1}) = \lambda_\mu(BAA^{-1}) = \lambda_\mu^+(B) \leq \lambda_\mu(B)$.

That is, $\lambda_\mu(ABA^{-1}) \leq \lambda_\mu(B)$ implies that $\lambda_\mu^+(ABA^{-1}) \leq \lambda_\mu^+(B)$ and $\lambda_\mu^-(ABA^{-1}) \geq \lambda_\mu^-(B)$ for arbitrary A, B $\in \vartheta$.

Hence, λ_μ is a bipolar anti fuzzy normal HX subgroup on ϑ.

Theorem 2.12: Let G be an classical abelian group and $\vartheta \subset 2^G - \{\phi\}$ be a HX group on G. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of an abelian HX group ϑ, then λ_μ is a bipolar anti fuzzy normal HX subgroup on ϑ.

Proof: Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of an abelian HX group ϑ.Then AB = BA for every A,B$\in\vartheta$.

Hence, $\lambda_\mu(AB) = \lambda_\mu(BA)$, for all A, B $\in\vartheta$.

Therefore, λ_μ is a bipolar anti fuzzy normal HX subgroup on ϑ.

Theorem 2.13: Let μ is a bipolar anti fuzzy normal subgroup on G then the bipolar fuzzy set λ_μ is a bipolar anti fuzzy normal HX subgroup on ϑ.

Proof: Let μ be a bipolar anti fuzzy normal subgroup of G then $\mu^+(xy) = \mu^+(yx)$, $\mu^-(xy) = \mu^-(yx)$,

Now λ_μ be a bipolar fuzzy subset of ϑ for all A, B $\in \vartheta$

$$\begin{aligned}\lambda_\mu^+(AB) &= \min\{\mu^+(xy) / \text{for all } x \in A \subseteq G, y \in B \subseteq G\}\\ &= \min\{\mu^+(yx) / \text{for all } x \in A \subseteq G, y \in B \subseteq G\}\\ &= \lambda_\mu^+(BA)\end{aligned}$$

$$\begin{aligned}\lambda_\mu^-(AB) &= \max\{\mu^-(xy) / \text{for all } x \in A \subseteq G, y \in B \subseteq G\}\\ &= \max\{\mu^-(yx) / \text{for all } x \in A \subseteq G, y \in B \subseteq G\}\\ &= \lambda_\mu^-(BA)\end{aligned}$$

Hence, $\lambda_\mu(AB) = \lambda_\mu(BA)$, for all A, B $\in\vartheta$.

Therefore, λ_μ is a bipolar anti fuzzy normal HX subgroup on ϑ.

Theorem 2.14: Let λ_μ is a bipolar fuzzy normal HX subgroup of ϑ if and only if $(\lambda_\mu)^c$ is a bipolar anti fuzzy normal HX subgroup of ϑ.

Proof: Let λ_μ is a bipolar fuzzy HX subgroup of ϑ if and only if $(\lambda_\mu)^c$ is a bipolar anti fuzzy HX subgroup of ϑ.by Theorem 3.4 [12]

Let us show the normality condition, we have λ_μ is a bipolar fuzzy normal HX subgroup of ϑ, For every A, B$\in \vartheta$

$$\lambda_\mu^+(AB) = \lambda_\mu^+(BA)$$

$$\Leftrightarrow 1 - \lambda_\mu^+(AB) = 1 - \lambda_\mu^+(BA)$$

$$\Leftrightarrow (\lambda_\mu^+)^c(AB) = (\lambda_\mu^+)^c(BA)$$

and

$$\lambda_\mu^-(AB) = \lambda_\mu^-(BA)$$

$$\Leftrightarrow -1 - \lambda_\mu^-(AB) = -1 - \lambda_\mu^-(BA)$$

$$\Leftrightarrow (\lambda_\mu^-)^c(AB) = (\lambda_\mu^-)^c(BA)$$

Hence, λ_μ be a bipolar fuzzy normal HX subgroup of a HX group ϑ if and only if $(\lambda_\mu)^c$ is a bipolar anti fuzzy normal HX subgroup of HX group ϑ.

Definition 2.15[12]: Let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are bipolar fuzzy subsets of G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group of G. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar fuzzy subsets of ϑ. The union of λ_μ and σ_φ is $(\lambda_\mu \cup \sigma_\varphi) = ((\lambda_\mu \cup \sigma_\varphi)^+, (\lambda_\mu \cup \sigma_\varphi)^-)$ defined as

i. $(\lambda_\mu \cup \sigma_\varphi)^+(A) = \max\{\lambda_\mu^+(A), \sigma_\varphi^+(A)\}$
ii. $(\lambda_\mu \cup \sigma_\varphi)^-(A) = \min\{\lambda_\mu^-(A), \sigma_\varphi^-(A)\}$.

where $\lambda_\mu^+(A) = \max\{\mu^+(x) \,/\, \text{for all } x \in A \subseteq G\}$
$\lambda_\mu^-(A) = \min\{\mu^-(x) \,/\, \text{for all } x \in A \subseteq G\}$
$\sigma_\varphi^+(A) = \max\{\varphi^+(x) \,/\, \text{for all } x \in A \subseteq G\}$ and
$\sigma_\varphi^-(A) = \min\{\varphi^-(x) \,/\, \text{for all } x \in A \subseteq G\}$.

Definition 2.16[12]: Let $\mu = (\mu^+,\mu^-)$ and $\varphi = (\varphi^+,\varphi^-)$ are bipolar fuzzy subsets of G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group of G. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar fuzzy subsets of ϑ.The intersection of λ_μ and σ_φ is $(\lambda_\mu \cap \sigma_\varphi) = ((\lambda_\mu \cap \sigma_\varphi)^+, (\lambda_\mu \cap \sigma_\varphi)^-)$ defined as

i. $(\lambda_\mu \cap \sigma_\varphi)^+(A) = \min\{\lambda_\mu^+(A), \sigma_\varphi^+(A)\}$
ii. $(\lambda_\mu \cap \sigma_\varphi)^-(A) = \max\{\lambda_\mu^-(A), \sigma_\varphi^-(A)\}$.

where $\lambda_\mu^+(A) = \max\{\mu^+(x) \,/\, \text{for all } x \in A \subseteq G\}$
$\lambda_\mu^-(A) = \min\{\mu^-(x) \,/\, \text{for all } x \in A \subseteq G\}$
$\sigma_\varphi^+(A) = \max\{\varphi^+(x) \,/\, \text{for all } x \in A \subseteq G\}$ and
$\sigma_\varphi^-(A) = \min\{\varphi^-(x) \,/\, \text{for all } x \in A \subseteq G\}$.

Theorem 2.17: If λ_μ and σ_φ are bipolar anti fuzzy normal HX subgroups of ϑ then $\lambda_\mu \cup \sigma_\varphi$ is also a bipolar anti fuzzy normal HX subgroup of ϑ.

Proof: Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar anti fuzzy HX subgroups of ϑ then $\lambda_\mu \cup \sigma_\varphi$ is also a bipolar anti fuzzy HX subgroup of ϑ. by Theorem 3.8 [13].

Now λ_μ and σ_φ are bipolar anti fuzzy normal HX subgroups of ϑ. Let us show the normality condition, For every A, $B \in \vartheta$

$$\begin{aligned}(\lambda_\mu \cup \sigma_\varphi)^+(ABA^{-1}) &= \max\{\lambda_\mu^+(ABA^{-1}), \sigma_\varphi^+(ABA^{-1})\}\\ &\le \max\{\lambda_\mu^+(B), \sigma_\varphi^+(B)\}\\ &= (\lambda_\mu \cup \sigma_\varphi)^+(B)\end{aligned}$$

There fore, $(\lambda_\mu \cup \sigma_\varphi)^+(ABA^{-1}) \le (\lambda_\mu \cup \sigma_\varphi)^+(B)$

$$\begin{aligned}(\lambda_\mu \cup \sigma_\varphi)^-(ABA^{-1}) &= \min\{\lambda_\mu^-(ABA^{-1}), \sigma_\varphi^-(ABA^{-1})\}\\ &\ge \min\{\lambda_\mu^-(B), \sigma_\varphi^-(B)\}\\ &= (\lambda_\mu \cup \sigma_\varphi)^-(B)\end{aligned}$$

There fore, $(\lambda_\mu \cup \sigma_\varphi)^-(ABA^{-1}) \ge (\lambda_\mu \cup \sigma_\varphi)^-(B)$

Hence, $\lambda_\mu \cup \sigma_\varphi$ is a bipolar anti fuzzy normal HX subgroup of ϑ.

Remark 2.18: Let G be a group. Let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are bipolar anti fuzzy normal subgroups of G and $\mu \cup \varphi$ is also a bipolar anti fuzzy normal subgroup of G then $\eta_{\mu\cup\varphi} = ((\eta_{\mu\cup\varphi})^+,(\eta_{\mu\cup\varphi})^-)$ is a bipolar anti fuzzy normal HX subgroup of ϑ induced by $\mu \cup \varphi$ of G (By Theorem 2.13).

Theorem 2.19: If $\lambda_\mu, \sigma_\varphi, \eta_{\mu\cup\varphi}$ are bipolar anti fuzzy normal HX subgroups of ϑ induced by bipolar anti fuzzy normal subgroups μ, φ and $\mu \cup \varphi$ of G then $\eta_{\mu\cup\varphi} = \lambda_\mu \cup \sigma_\varphi$

Proof: Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$,$\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar anti fuzzy normal HX subgroups of ϑ then $\lambda_\mu \cup \sigma_\varphi$ are bipolar anti fuzzy normal HX subgroup of ϑ by theorem 2.17 and $\eta_{\mu\cup\varphi}$ is a bipolar anti fuzzy normal HX subgroup of ϑ induced by $\mu \cup \varphi$ of G.

i. $$\begin{aligned}(\eta_{\mu\cup\varphi})^+(AB) &= \min\{(\mu \cup \varphi)^+(xy) \,/\, x \in A \subseteq G, y \in B \subseteq G\}\\ &= \min\{\max\{\mu^+(xy), \varphi^+(xy)\} \,/\, x \in A \subseteq G, y \in B \subseteq G\}\\ &= \max\{\min\{\mu^+(xy)/x\in A\subseteq G, y\in B\subseteq G\}, \min\{\varphi^+(xy)/x\in A\subseteq G, y\in B\subseteq G\}\}\\ &= \max\{\lambda_\mu^+(AB), \sigma_\varphi^+(AB)\}\\ &= (\lambda_\mu \cup \sigma_\varphi)^+(AB)\end{aligned}$$

ii. $$\begin{aligned}(\eta_{\mu\cup\varphi})^-(AB) &= \max\{(\mu \cup \varphi)^-(xy) \,/\, x \in A \subseteq G, y \in B \subseteq G\}\\ &= \max\{\min\{\mu^-(xy), \varphi^-(xy)\} \,/\, x \in A \subseteq G, y \in B \subseteq G\}\\ &= \min\{\max\{\mu^-(xy)/x\in A\subseteq G, y\in B\subseteq G\}, \max\{\varphi^-(xy)/x\in A\subseteq G, y\in B\subseteq G\}\}\\ &= \min\{\lambda_\mu^-(AB), \sigma_\varphi^-(AB)\}\\ &= (\lambda_\mu \cup \sigma_\varphi)^-(AB)\end{aligned}$$

Hence, $\eta_{\mu\cup\varphi} = \lambda_\mu \cup \sigma_\varphi$.

Theorem 2.20: Let $\mu = (\mu^+, \mu^-)$ and $\varphi = (\varphi^+,\varphi^-)$ are bipolar anti fuzzy normal subgroups and $\mu \cap \varphi$ is also bipolar anti fuzzy normal subgroup of G. Let $\vartheta \subset 2^G - \{\phi\}$ be a HX group of the group G. Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar anti fuzzy normal HX subgroups of ϑ then $\lambda_\mu \cap \sigma_\varphi$ is a bipolar anti fuzzy normal HX subgroup of ϑ.

Proof: Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar anti fuzzy HX subgroups of ϑ then $\lambda_\mu \cup \sigma_\varphi$ is also a bipolar anti fuzzy HX subgroup of ϑ. by Theorem 3.12 [13].

Let us show the normality condition,

Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar anti fuzzy normal HX subgroups of ϑ. For every A, B$\in\vartheta$,

i. $$\begin{aligned}(\lambda_\mu \cap \sigma_\varphi)^+ (ABA^{-1}) &= \min\{\lambda_\mu^+ (ABA^{-1}), \sigma_\varphi^+ (ABA^{-1})\}\\ &\le \min\{\lambda_\mu^+ (B), \sigma_\varphi^+ (B)\}\}\\ &= (\lambda_\mu \cap \sigma_\varphi)^+ (B)\end{aligned}$$

Therefore, $(\lambda_\mu \cap \sigma_\varphi)^+ (ABA^{-1}) \le (\lambda_\mu \cap \sigma_\varphi)^+ (B)$

ii. $$\begin{aligned}(\lambda_\mu \cap \sigma_\varphi)^- (ABA^{-1}) &= \max\{\lambda_\mu^- (ABA^{-1}), \sigma_\varphi^- (ABA^{-1})\}\\ &\ge \max\{\lambda_\mu^- (B), \sigma_\varphi^- (B)\}\\ &= (\lambda_\mu \cap \sigma_\varphi)^- (B)\end{aligned}$$

Therefore, $(\lambda_\mu \cap \sigma_\varphi)^- (ABA^{-1}) \ge (\lambda_\mu \cap \sigma_\varphi)^- (B)$

Hence, $\lambda_\mu \cap \sigma_\varphi$ is a bipolar anti fuzzy normal HX subgroup of ϑ.

Remark 2.21: Let $\mu = (\mu^+,\mu^-)$ and $\varphi = (\varphi^+,\varphi^-)$ are bipolar anti fuzzy normal subgroups and $\mu \cap \varphi$ is a bipolar fuzzy subset of G, but need not be a bipolar anti fuzzy normal subgroup of G.

If $\eta_{\mu\cap\varphi} = ((\eta_{\mu\cap\varphi})^+,(\eta_{\mu\cap\varphi})^-)$ is a bipolar fuzzy subset induced by $\mu \cap \varphi$ then $\eta_{\mu\cap\varphi}$ is a bipolar anti fuzzy normal HX subgroup of ϑ with $|X| \ge 2$ for all $X\in\vartheta$.

Theorem 2.22: Let λ_μ, σ_φ, $\eta_{\mu\cap\varphi}$ are bipolar anti fuzzy normal HX subgroups of ϑ induced by bipolar fuzzy subsets μ, φ and $\mu \cap \varphi$ of G respectively then $\eta_{\mu\cap\varphi} = \lambda_\mu \cap \sigma_\varphi$.

Proof: Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$, $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar anti fuzzy normal HX subgroups of ϑ then $\lambda_\mu \cap \sigma_\varphi$ are bipolar anti fuzzy normal HX subgroup of ϑ by theorem 2.20 and $\eta_{\mu\cap\varphi}$ is a bipolar fuzzy normal HX subgroup of ϑ induced by a bipolar fuzzy subset $\mu \cap \varphi$ of G.

i. $$\begin{aligned}(\eta_{\mu\cap\varphi})^+ (AB) &= \min\{(\mu \cap \varphi)^+ (xy) / x \in A \subseteq G, y\in B\subseteq G\}\\ &= \min\{\min\{\mu^+ (xy), \varphi^+ (xy)\} / x \in A \subseteq G, y\in B\subseteq G\}\\ &= \min\{\min\{\mu^+ (xy)/x\in A\subseteq G,y\in B\subseteq G\},\min\{\varphi^+(xy)/x\in A\subseteq G,y\in B\subseteq G\}\\ &= \min\{\lambda_\mu^+ (AB), \sigma_\varphi^+ (AB)\}\\ &= (\lambda_\mu \cap \sigma_\varphi)^+ (AB)\end{aligned}$$

ii. $$\begin{aligned}(\eta_{\mu\cap\varphi})^- (AB) &= \max\{(\mu \cap \varphi)^- (xy) / x\in A \subseteq G, y\in B\subseteq G\}\\ &= \max\{\max\{\mu^- (xy), \varphi^- (xy)\} / x\in A\subseteq G, y\in B\subseteq G\}\\ &= \max\{\max\{\mu^-(xy)/x\in A\subseteq G,y\in B\subseteq G\},\max\{\varphi^-(xy)/x\in A\subseteq G,y\in B\subseteq G\}\\ &= \max\{\lambda_\mu^- (AB), \sigma_\varphi^- (AB)\}\\ &= (\lambda_\mu \cap \sigma_\varphi)^- (AB)\end{aligned}$$

Hence, $\eta_{\mu\cap\varphi} = \lambda_\mu \cap \sigma_\varphi$.

Definition 2.23[9]: A mapping f from a HX group ϑ_1 to a HX group ϑ_2 is said to be a homomorphism if f (AB) = f(A) f(B) for all A,B $\in \vartheta_1$.

Definition 2.24[9]: A mapping f from a HX group ϑ_1 to a HX group ϑ_2 (ϑ_1 and ϑ_2 are not necessarily commutative) is said to be an anti homomorphism if f (AB) = f(B) f(A) for all A,B $\in \vartheta_1$.

3. PROPERTIES OF A BIPOLAR ANTI FUZZY NORMAL HX SUBGROUP OF A HX GROUP UNDER HOMOMORPHISM AND ANTI HOMOMORPHISM

In this section, we introduce the notion of an anti image and anti pre-image of the bipolar fuzzy subsets of a HX group, and discuss the properties of a bipolar anti fuzzy normal HX subgroup of a HX group under homomorphism and anti homomorphism .Throughout this section, We mean that ϑ_1 and ϑ_2 are HX groups (ϑ_1 and ϑ_2 are not necessarily commutative), E_1, E_2 are the identity elements of ϑ_1 and ϑ_2 respectively, and XY we mean X *Y.

Definition 3.1[13]: Let G_1 and G_2 be any two groups. Let $\vartheta_1 \subset 2^{G1}-\{\phi\}$ and $\vartheta_2 \subset 2^{G2}-\{\phi\}$are HX groups defined on G_1 and G_2 respectively. Let $\mu = (\mu^+,\mu^-)$ and $\varphi = (\varphi^+, \varphi^-)$ are bipolar fuzzy subsets in G_1 and G_2 respectively, let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$, and $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ are bipolar fuzzy subsets defined on ϑ_1 and ϑ_2 respectively induced by μ and φ. Let $f : \vartheta_1 \to \vartheta_2$ be a mapping then the anti image $f_a(\lambda_\mu)$ of λ_μ is a bipolar fuzzy subset $f_a(\lambda_\mu) = ((f_a(\lambda_\mu))^+ ,(f_a(\lambda_\mu))^-)$ of ϑ_2 defined by for each $U \in \vartheta_2$,

$$(f_a(\lambda_\mu))^+(U) = \begin{cases} \min\{(\lambda_\mu)^+(X): X \in f^{-1}(U)\}, & \text{if } f^{-1}(U) \neq \phi \\ 1, & \text{otherwise} \end{cases}$$

and

$$(f_a(\lambda_\mu))^-(U)) = \begin{cases} \min\{(\lambda_\mu)^-(X): X \in f^{-1}(U)\}, & \text{if } f^{-1}(U) \neq \phi \\ -1, & \text{otherwise} \end{cases}$$

also the anti pre-image $f^{-1}(\sigma_\varphi)$ of σ_φ under f is a bipolar fuzzy subset of ϑ_1 defined by for $X \in \vartheta_1$, $(f^{-1}(\sigma_\varphi)^+)(X) = \sigma_\varphi^+(f(X))$, $(f^{-1}(\sigma_\varphi)^-)(X) = \sigma_\varphi^-(f(X))$.

Theorem 3.2: Let f be a homomorphism from a HX group ϑ_1 into a HX group ϑ_2. If $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ is a bipolar anti fuzzy normal (BAFN) HX subgroup of ϑ_1 then the anti image $f_a(\lambda_\mu)$ of λ_μ under f is a bipolar anti fuzzy normal HX subgroup of ϑ_2.

Proof: Let $\lambda_\mu = (\lambda_\mu^+, \lambda_\mu^-)$ be a bipolar anti fuzzy HX subgroup of ϑ_1. Then $f_a(\lambda_\mu)$ is a bipolar anti fuzzy HX subgroup of ϑ_2. by Theorem 3.3 [13].

Let us show the normality condition,

Now, Let $U,V \in \vartheta_2$, since f is onto, f(X) = U and f(Y) = V for some $X,Y \in \vartheta_1$.

$$\begin{aligned}(f_a(\lambda_\mu))^+(UVU^{-1}) &= \min\{\lambda_\mu^+(Z): Z \in f^{-1}(UVU^{-1})\} \\ &\leq \min\{\lambda_\mu^+(X^{-1}YX): X \in f^{-1}(U), Y \in f^{-1}(V)\} \\ &\leq \min\{\lambda_\mu^+(Y): Y \in f^{-1}(V)\} \\ &= (f_a(\lambda_\mu))^+(V)\end{aligned}$$

Therefore, $(f_a(\lambda_\mu))^+(UVU^{-1}) \leq (f_a(\lambda_\mu))^+(V)$

$$\begin{aligned}(f_a(\lambda_\mu))^-(UVU^{-1}) &= \min\{\lambda_\mu^-(Z): Z \in f^{-1}(UVU^{-1})\} \\ &\geq \min\{\lambda_\mu^-(XYX^{-1}): X \in f^{-1}(U), Y \in f^{-1}(V)\} \\ &\geq \min\{\lambda_\mu^-(Y): Y \in f^{-1}(V)\} \\ &= (f_a(\lambda_\mu))^-(V)\end{aligned}$$

Therefore, $(f_a(\lambda_\mu))^-(UVU^{-1}) \geq (f_a(\lambda_\mu))^-(V)$

Therefore $f_a(\lambda_\mu)$ is a bipolar anti fuzzy normal HX subgroup of ϑ_2.

Hence, if λ_μ is a bipolar anti fuzzy normal HX subgroup of ϑ_1 then the anti image $f_a(\lambda_\mu)$ of λ_μ under f is a bipolar anti fuzzy normal HX subgroup of ϑ_2.

Theorem 3.3: The homomorphic anti pre-image of a bipolar anti fuzzy normal (BAFN) HX subgroup $\sigma_\varphi = (\sigma_\varphi^+, \sigma_\varphi^-)$ of a HX group ϑ_2 is a bipolar anti fuzzy normal HX subgroup of a HX group ϑ_1.

Proof: Let σ_φ be a bipolar anti fuzzy HX subgroup on ϑ_2 then anti pre-image $f^{-1}(\sigma_\varphi)$ of σ_φ under f is a bipolar anti fuzzy HX subgroup of ϑ_1.by Theorem 3.4[13]

Let us show the normality condition,

$$\begin{aligned}(f^{-1}(\sigma_\varphi))^{+}(XYX^{-1}) &= \sigma_\varphi^{+}(f(XYX^{-1}))\\ &= \sigma_\varphi^{+}(f(X)\,f(Y)\,f(X^{-1}))\\ &= \sigma_\varphi^{+}(f(X)\,f(Y)\,f(X)^{-1})\\ &\le \sigma_\varphi^{+}(f(Y)),\ \text{Since } \sigma_\varphi \text{ is a BAFN HX subgroup}\\ &= (f^{-1}(\sigma_\varphi))^{+}(Y)\end{aligned}$$

Therefore, $(f^{-1}(\sigma_\varphi))^{+}(XYX^{-1}) \le (f^{-1}(\sigma_\varphi))^{+}(Y)$

$$\begin{aligned}(f^{-1}(\sigma_\varphi))^{-}(XYX^{-1}) &= \sigma_\varphi^{-}(f(XYX^{-1}))\\ &= \sigma_\varphi^{-}(f(X)\,f(Y)\,f(X^{-1}))\\ &= \sigma_\varphi^{-}(f(X)\,f(Y)\,f(X)^{-1})\\ &\ge \sigma_\varphi^{-}(f(Y)),\ \text{Since } \sigma_\varphi \text{ is a BAFN HX subgroup}\\ &= (f^{-1}(\sigma_\varphi))^{-}(Y)\end{aligned}$$

Therefore, $(f^{-1}(\sigma_\varphi))^{-}(XYX^{-1}) \ge (f^{-1}(\sigma_\varphi))^{-}(Y)$

Therefore $f^{-1}(\sigma_\varphi)$ is a bipolar anti fuzzy normal HX subgroup of ϑ_1.

Hence, if σ_φ is a bipolar anti fuzzy normal HX subgroup of ϑ_2 then the anti pre-image $f^{-1}(\sigma_\varphi)$ of σ_φ under f is a bipolar anti fuzzy normal HX subgroup of ϑ_1.

Theorem 3.4: Let f be an anti homomorphism from a HX group ϑ_1 into a HX group ϑ_2. If $\lambda_\mu = (\lambda_\mu^{+}, \lambda_\mu^{-})$ be a bipolar anti fuzzy normal HX subgroup of ϑ_1 then the anti image $f_a(\lambda_\mu)$ of λ_μ under f is a bipolar anti fuzzy normal HX subgroup of ϑ_2.

Proof: Let $\lambda_\mu = (\lambda_\mu^{+}, \lambda_\mu^{-})$ be a bipolar anti fuzzy HX subgroup of ϑ_1. Then $f_a(\lambda_\mu)$ is a bipolar anti fuzzy HX subgroup of ϑ_2.by Theorem 3.5 [13]

Let us show the normality condition,

Let U, V$\in \vartheta_2$, since f is an anti homomorphism and so there exist X,Y $\in \vartheta_1$ such that f(X) = U and f(Y) = V it follows that XY = $f^{-1}(VU)$.

$$\begin{aligned}(f_a(\lambda_\mu))^{+}(UVU^{-1}) &= \min\{\lambda_\mu^{+}(Z) : Z\in f^{-1}(UVU^{-1})\}\\ &\le \min\{\lambda_\mu^{+}(XYX^{-1}) : Y\in f^{-1}(V),\ X\in f^{-1}(U)\}\\ &\le \min\{\lambda_\mu^{+}(Y) : Y\in f^{-1}(V)\}\\ &= (f_a(\lambda_\mu))^{+}(V)\end{aligned}$$

Therefore, $(f_a(\lambda_\mu))^{+}(UVU^{-1}) \le (f_a(\lambda_\mu))^{+}(V)$

$$\begin{aligned}(f_a(\lambda_\mu))^{-}(UVU^{-1}) &= \min\{\lambda_\mu^{-}(Z) : Z\in f^{-1}(UVU^{-1})\}\\ &\ge \min\{\lambda_\mu^{-}(XYX^{-1}) : Y\in f^{-1}(V),\ X\in f^{-1}(U)\}\\ &\ge \min\{\lambda_\mu^{-}(Y) : Y\in f^{-1}(V)\}\\ &= (f_a(\lambda_\mu))^{-}(V)\end{aligned}$$

Therefore, $(f_a(\lambda_\mu))^{-}(UVU^{-1}) \ge (f_a(\lambda_\mu))^{-}(V)$

Therefore $f_a(\lambda_\mu)$ is a bipolar anti fuzzy normal HX subgroup of ϑ_2.

Hence, if λ_μ is a bipolar anti fuzzy normal HX subgroup of ϑ_1 then the anti image $f_a(\lambda_\mu)$ of λ_μ under f is a bipolar anti fuzzy normal HX subgroup of ϑ_2.

Theorem 3.5: The anti homomorphic anti pre-image of a bipolar anti fuzzy normal (BAFN) HX subgroup $\sigma_\varphi = (\sigma_\varphi^{+}, \sigma_\varphi^{-})$ of a HX group ϑ_2 is a bipolar anti fuzzy normal HX subgroup of a HX group ϑ_1.

Proof: Let σ_φ be a bipolar anti fuzzy HX subgroup on ϑ_2 then anti pre-image $f^{-1}(\sigma_\varphi)$ of σ_φ under f is a bipolar anti fuzzy HX subgroup of ϑ_1.by Theorem 3.6[13]

Let us show the normality condition,

$$\begin{aligned}(f^{-1}(\sigma_\varphi))^+(XYX^{-1}) &= \sigma_\varphi^+(f(XYX^{-1}))\\ &= \sigma_\varphi^+(f(X^{-1})f(Y)\,f(X))\\ &= \sigma_\varphi^+(f(X)^{-1}f(Y)\,f(X))\\ &\le \sigma_\varphi^+(f(Y)),\ \text{Since } \sigma_\varphi \text{ is a BAFN HX subgroup}\\ &= (f^{-1}(\sigma_\varphi))^+(Y)\end{aligned}$$

Therefore, $(f^{-1}(\sigma_\varphi))^+(XYX^{-1}) \le (f^{-1}(\sigma_\varphi))^+(Y)$

$$\begin{aligned}(f^{-1}(\sigma_\varphi))^-(XYX^{-1}) &= \sigma_\varphi^-(f(XYX^{-1}))\\ &= \sigma_\varphi^-(f(X^{-1})f(Y)\,f(X))\\ &= \sigma_\varphi^-(f(X)^{-1}f(Y)\,f(X))\\ &\ge \sigma_\varphi^-(f(Y)),\ \text{Since } \sigma_\varphi \text{ is a BAFN HX subgroup}\\ &= (f^{-1}(\sigma_\varphi))^-(Y)\end{aligned}$$

Therefore, $(f^{-1}(\sigma_\varphi))^-(XYX^{-1}) \ge (f^{-1}(\sigma_\varphi))^-(Y)$

Therefore $f^{-1}(\sigma_\varphi)$ is a bipolar anti fuzzy normal HX subgroup of ϑ_1.

Hence, if σ_φ is a bipolar anti fuzzy normal HX subgroup of ϑ_2 then the anti pre-image $f^{-1}(\sigma_\varphi)$ of σ_φ under f is a bipolar anti fuzzy normal HX subgroup of ϑ_1.

4. PROPERTIES OF LOWER LEVEL SUBSETS OF A BIPOLAR ANTI FUZZY NORMAL HX SUBGROUP OF A HX GROUP

In this section, we introduce the concept of lower level subsets of a bipolar anti fuzzy normal HX subgroup and discuss some of its properties.

Definition 4.1: Let λ_μ be a bipolar anti fuzzy normal HX subgroup of a HX group ϑ. For any $<\alpha, \beta> \in [0,1]\times[-1,0]$, we define the set $\lambda_{\mu <\alpha,\beta>} = \{A \in \vartheta \,/\, \lambda_\mu^+(A) \le \alpha$ and $\lambda_\mu^-(A) \ge \beta\}$ is called the $<\alpha, \beta>$ lower level subset of λ_μ or simply the lower level subset of λ_μ.

Theorem 4.2: Let G be a classical group. Then the bipolar anti fuzzy HX subgroup λ_μ of a HX group ϑ is a bipolar anti fuzzy normal HX subgroup on ϑ if and only if for any $<\alpha,\beta> \in [0,1]\times[-1,0]$, $\lambda_{\mu <\alpha,\beta>}$ is a normal sub HX group of ϑ.

Proof: Let λ_μ be a bipolar anti fuzzy normal HX subgroup on ϑ.Then,
$\lambda_\mu^+(ABA^{-1}) \le \lambda_\mu^+(B)$, $\lambda_\mu^-(ABA^{-1}) \ge \lambda_\mu^-(B)$, for all $A, B \in \vartheta$.

For all $A, B \in \lambda_{\mu <\alpha,\beta>}$ we have $\lambda_\mu^+(A) \le \alpha$, $\lambda_\mu^-(A) \ge \beta$ and $\lambda_\mu^+(B) \le \alpha$, $\lambda_\mu^-(B) \ge \beta$

$$\begin{aligned}\text{Now } \lambda_\mu^+(AB^{-1}) &\le \max\{\lambda_\mu^+(A), \lambda_\mu^+(B)\}\\ &\le \max\{\alpha, \alpha\}\\ &= \alpha\end{aligned}$$

$$\Rightarrow \lambda_\mu^+(AB^{-1}) \le \alpha$$

$$\begin{aligned}\lambda_\mu^-(AB^{-1}) &\ge \min\{\lambda_\mu^-(A), \lambda_\mu^-(B)\}\\ &\ge \min\{\beta, \beta\}\\ &= \beta\end{aligned}$$

$$\Rightarrow \lambda_\mu^-(AB^{-1}) \ge \beta$$

Hence $AB^{-1} \in \lambda_{\mu <\alpha,\beta>}$. Hence $\lambda_{\mu <\alpha,\beta>}$ is a sub HX group.

For all $B \in \lambda_{\mu <\alpha,\beta>}$, $A \in \vartheta$, we have $\lambda_\mu^+(B) \le \alpha$ and $\lambda_\mu^-(B) \ge \beta$.

Since, λ_μ be a bipolar anti fuzzy normal HX subgroup of a HX group ϑ.Then,

$\lambda_\mu^+(ABA^{-1}) \le \lambda_\mu^+(B) \le \alpha$ and $\lambda_\mu^-(ABA^{-1}) \ge \lambda_\mu^-(B) \ge \beta$, for all $A, B \in \vartheta$.

$ABA^{-1} \in \lambda_{\mu <\alpha, \beta>}$. Hence $\lambda_{\mu <\alpha, \beta>}$, is a classical normal sub HX group.

Conversely, for any $<\alpha, \beta> \in [0, 1] \times [-1, 0]$, $\lambda_{\mu <\alpha, \beta>} \neq \phi$ and $\lambda_{\mu <\alpha, \beta>}$, is a classical normal sub HX group. Then we have,

$\lambda_\mu^+(ABA^{-1}) \leq \lambda_\mu^+(B)$ and $\lambda_\mu^-(ABA^{-1}) \geq \lambda_\mu^-(B)$, for all A, $B \in \vartheta$.

Otherwise, if there exist A_0 or $B_0 \in \vartheta$ such that,

$\lambda_\mu^+(A_0B_0A_0^{-1}) < \lambda_\mu^+(B_0)$ and $\lambda_\mu^-(A_0B_0A_0^{-1}) > \lambda_\mu^-(B_0)$.

Take $\alpha_0 = 0.5\,[\lambda_\mu^+(B_0) + \lambda_\mu^+(A_0B_0A_0^{-1})]$ and $\beta_0 = 0.5\,[\lambda_\mu^-(B_0) + \lambda_\mu^-(A_0B_0A_0^{-1})]$.

Evidently $< \alpha_0, \beta_0 > \in [0, 1] \times [-1, 0]$, we can infer that,

$\lambda_\mu^+(B_0) < \alpha_0$, $\lambda_\mu^+(A_0B_0A_0^{-1}) > \alpha_0$ and $\lambda_\mu^-(B_0) > \beta_0$ $\lambda_\mu^-(A_0B_0A_0^{-1}) < \beta_0$.

Consequently, we have $B_0 \in \lambda_{\mu < \alpha_0, \beta_0 >}$ and $A_0B_0A_0^{-1} \notin \lambda_{\mu < \alpha_0, \beta_0 >}$

This contradicts that $\lambda_{\mu < \alpha_0, \beta_0 >}$ is a classical normal sub HX group.

Hence, we get $\lambda_\mu^+(ABA^{-1}) \leq \lambda_\mu^+(B)$ and $\lambda_\mu^-(ABA^{-1}) \geq \lambda_\mu^-(B)$, for all A, $B \in \vartheta$.

Hence, λ_μ be a bipolar anti fuzzy normal HX subgroup of a HX group ϑ.

Definition 4.3: Let λ_μ is a bipolar anti fuzzy normal HX subgroup of a HX group ϑ. The normal sub HX groups $\lambda_{\mu <\alpha, \beta>}$ for $< \alpha, \beta > \in [0,1] \times [-1,0]$ and $\lambda_\mu^+(E) \leq \alpha$, $\lambda_\mu^-(E) \geq \beta$ are called lower level normal sub HX groups of λ_μ.

Theorem 4.4: Let ϑ be a HX group and λ_μ be a bipolar anti fuzzy normal HX subgroup of ϑ. If two bipolar lower level normal sub HX groups $\lambda_{\mu < \alpha, \gamma >}$, $\lambda_{\mu < \beta, \delta >}$ with $\alpha < \beta$ and $\delta < \gamma$ of λ_μ are equal if and only if there is no $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$.

Proof: Let $\lambda_{\mu < \alpha, \gamma >} = \lambda_{\mu < \beta, \delta >}$. Suppose that there exists $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$. Then $\lambda_{\mu < \alpha, \gamma >} \subset \lambda_{\mu < \beta, \delta >}$ Since $A \in \lambda_{\mu < \beta, \delta >}$ but not in $\lambda_{\mu < \beta, \delta >}$ which contradicts the hypothesis. Hence there exists no $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$.

Conversely, Let there be no $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$ Since $\alpha < \beta$ and $\delta < \gamma$, we have, $\lambda_{\mu < \alpha, \gamma >} \subset \lambda_{\mu < \beta, \delta >}$ Let $A \in \lambda_{\mu < \beta, \delta >}$, then $\lambda_\mu^+(A) \leq \beta$ and $\lambda_\mu^-(A) \geq \delta$. Since there exists no $A \in \vartheta$ such that $\alpha < \lambda_\mu^+(A) \leq \beta$ and $\delta \leq \lambda_\mu^-(A) < \gamma$, we have $\lambda_\mu^+(A) \leq \alpha$ and $\lambda_\mu^-(A) \geq \gamma$ which implies $A \in \lambda_{\mu < \alpha, \gamma >}$ i.e. $\lambda_{\mu < \beta, \delta >} \subseteq \lambda_{\mu < \alpha, \gamma >}$. Hence, $\lambda_{\mu < \alpha, \gamma >} = \lambda_{\mu < \beta, \delta >}$.

Theorem 4.5: A bipolar fuzzy subset λ_μ of ϑ is a bipolar anti fuzzy normal HX subgroup of HX group ϑ if and only if the lower level subsets $\lambda_{\mu <\alpha, \beta>}, <\alpha, \beta> \in$ Image λ_μ are normal HX subgroups of ϑ.

Proof: It is clear.

Remark 4.6: As a consequence of the Theorem 4.4, Theorem 4.5, the lower level normal sub HX groups of a bipolar anti fuzzy normal HX subgroup λ_μ of a HX group ϑ form a chain. Since $\lambda_\mu^+(E) \leq \lambda_\mu^+(A)$ and $\lambda_\mu^-(E) \geq \lambda_\mu^-(A)$ for all A in ϑ. Therefore, $\lambda_{\mu < \alpha_0, \beta_0 >}$, $\alpha \in [0,1]$ and $\beta \in [-1,0]$ where $\lambda_\mu^+(E) = \alpha_0$, $\lambda_\mu^-(E) = \beta_0$ is the smallest sub HX group and we have the chain: $\{E\} \subseteq \lambda_{\mu < \alpha_0, \beta_0 >} \subseteq \lambda_{\mu < \alpha_1, \beta_1 >} \subseteq \lambda_{\mu < \alpha_2, \beta_2 >} \subseteq$ ………….. $\subseteq \lambda_{\mu < \alpha_n, \beta_n >} = \vartheta$, where $\alpha_0 < \alpha_1 < \alpha_2 < \ldots. < \alpha_n$ and $\beta_0 > \beta_1 > \beta_2 >$. ………..…$> \beta_n$.

REFERENCES

[1]. R.Biswas, *Fuzzy subgroups and ant fuzzy subgroups*, Fuzzy Sets and Systems, 35, 1990, 121 – 124.

[2]. K. M. Lee, *Bipolar–valued fuzzy sets and their operations*, Proc.Int.conf.on Intelligent Technologies, Bangkok, Thailand, 2000, 307-312.

[3]. Li Hongxing, *HX group*, BUSEFAL, 33, 1987, 31 – 37.

[4]. Li Xiaoping,*The intuitionistic fuzzy normal subgroup and its some equivalent proposition*, BUSEFAL, 82, 2000, 40 – 44.

[5]. Luo Chengzhong, Mi Honghai, Li Hongxing, *Fuzzy HX group*, BUSEFAL, 41(14), 1989, 97 – 106.

[6]. M.Marudai,V.Rajendran, *New Constructions on Bipolar Anti Q-fuzzy Groups and Bipolar Anti Q-fuzzy d-ideals under (t,s) norms*, Advances in Fuzzy Mathematics , Volume 6, Number 1, 2011,145 – 153.
[7]. Mehmet sait Eroglu, *The homomorphic image of a fuzzy subgroup is always a fuzzy subgroup*, Fuzzy sets and System, 33,1989, 255-256.
[8]. R.Muthuraj, M.S.Muthuraman, M.Sridharan, *Bipolar fuzzy Sub-Bi HX group and its Bi Level Sub-Bi HX groups*,Antarctica J.Math., 8(5),2011, 427 - 434.
[9]. R.Muthuraj, M.Sridharan, *Bipolar Anti fuzzy HX group and its Lower Level Sub HX groups*, Journal of Physical Sciences, Volume 16, December 2012, 157- 169.
[10].R.Muthuraj, M.Sridharan, *Homomorphism and anti Homomorphism of a bipolar anti fuzzy HX subgroups*, SVRM Science Journal, Volume 2, Issue 1, Jan-March 2014, 34-42.
[11].R.Muthuraj, M.Sridharan, *Bipolar fuzzy HX group and its Level Sub HX Groups,* International Journal of Mathematical Archive, 5-(1), 2014, 230-239.
[12].R.Muthuraj, M.Sridharan, *Operations on bipolar fuzzy HX subgroups*, International Journal of Engineering Associates, Volume 3 , Issue 5 , 2014, 12-18.
[13].R.Muthuraj, M.Sridharan, *Operations on bipolar anti fuzzy HX subgroups* Journal of Uncertain System (Communicated).
[14].N.Palaniappan, R.Muthuraj, *Anti fuzzy group and Lower level subgroups*, Antarctica J.Math., 1(1), 2004, 71-76.
[15].A. Rosenfeld., *Fuzzy Groups*, J.Math. Anal. Appl. 35, 1971, 512-517.
[16].L.A. Zadeh, *Fuzzy Sets*, Information and Control, 8, 1965, 338-365.
[17].W.R. Zhang., *Bipolar fuzzy sets,* Proc. of FUZZ-IEEE,1998,835-840.

Source of support: Nil, Conflict of interest: None Declared