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USING SQUARED-LOG ERROR LOSS FUNCTION TO ESTIMATE THE SHAPE PARAMETER AND THE RELIABILITY FUNCTION OF PARETO TYPE I DISTRIBUTION

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ABSTRACT

In this paper, we derived Bayes estimators for the shape parameter and the reliability function of the Pareto type I distribution under Squared-Log error loss function. In order to get better understanding of our Bayesian analysis, we consider non-informative prior for the shape parameter Using Jeffery prior Information as well as informative prior density represented by Exponential prior. According to Monte-Carlo simulation study, the performance of these estimators is compared depending on the mean square Errors (MSE's).

Key words: *Pareto distribution, Reliability function, Maximum Likelihood Estimator, Bayes estimator, Squared-Log error loss function, Jeffery prior and Exponential prior.*

1. INTRODUCTION

The Pareto distribution is named after the economist Vilfredo Pareto (1848-1923), this distribution is first used as a model for the distribution of incomes a model for city population within a given area, failure model in reliability theory [1] and a queuing model in operation research [5].

A random variable X, is said to follow the two parameter Pareto distribution if its pdf is given by:

$$f(x;\alpha,\theta)=\begin{cases}\dfrac{\theta\alpha^{\theta}}{x^{\theta+1}}; & x\geq\alpha,\alpha>0,\theta>0\\ 0, & \text{otherwise}\end{cases} \tag{1}$$

where α and θ are the scale and shape parameters respectively.

The cumulative distribution function (CDF) in its simplest form is given by:

$$F(x;\alpha,\theta)=\begin{cases}1-\left(\dfrac{\alpha}{x}\right)^{\theta}, & x\geq\alpha;\ \alpha,\theta>0\\ 0\quad, & \text{otherwise}\end{cases} \tag{2}$$

So, the reliability function is:

$$R(t)=\left(\frac{\alpha}{t}\right)^{\theta} \tag{3}$$

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In this paper, for the simplification we'll assume that $\alpha = 1$

2. MAXIMUM LIKELIHOOD ESTIMATOR

Given $x_1, x_2,\dots, x_n$ a random sample of size n from Pareto distribution, we consider estimation using Maximum likelihood method as follows:

$$L(x_1, \dots, x_n|\theta) = \prod_{i=1}^{n} f(x_n; \theta)$$

$$L(x_1, \dots, x_n|\theta) = \theta^n e^{-(\theta+1)\sum \ln x}$$

The Log- likelihood function is given by

$$\ln L(x_1, \dots, x_n|\theta) = n \ln\theta - (\theta+1)\sum_{i=1}^{n} \ln x_i$$

Differentiating the log likelihood with respect to θ:

$$\frac{\partial[\ln L(x_1, \dots, x_n|\theta)]}{\partial\theta} = \frac{n}{\theta} - \sum_{i=1}^{n} \ln x_i$$

Hence, the MLE of θ is:

$$\hat{\theta}_{ML} = \frac{n}{\sum_{i=1}^{n} \ln x_i}$$

$$\hat{\theta}_{ML} = \frac{n}{T}, \text{ where } T = \sum_{i=1}^{n} \ln x_i \tag{4}$$

Using the invariant property, the MLE$\hat{R}_{ML}(t)$ for R(t) may be obtained by replacing θ by its MLE$\hat{\theta}_{ML}$ in (3) [6]

$$\hat{R}_{ML}(t) = \left(\frac{1}{t}\right)^{\hat{\theta}_{ML}} \tag{5}$$

3. BAYES ESTIMATOR UNDERSQUARED-LOG ERROR LOSS FUNCTION

Bayes estimators for the shape parameter θ and Reliability function were considered under squared-log error loss function with non-Informative prior which represented by Jeffrey prior and informative loss function represented by Exponential prior where the Squared-log error loss function is of the form:

$$L(\hat{\theta}, \theta) = \left(\ln\hat{\theta} - \ln\theta\right)^2$$

Which is balanced with $Lim_{L(\hat{\theta},\theta)\to\infty}$ as $\hat{\theta} \to 0$ or ∞. A balanced loss function takes both error of estimation and goodness of fit into account but the unbalanced loss function only considers error of estimation. This loss function is convex for $\frac{\hat{\theta}}{\theta} \leq e$ and concave otherwise, but its risk function has a unique minimum with respect to $\hat{\theta}$. [3]

According to the above mentioned loss functions, we drive the corresponding Bayes' estimators for θ using Risk function $R(\hat{\theta} - \theta)$which minimizes the posterior risk

$$R(\hat{\theta} - \theta) = E\left[L(\hat{\theta}, \theta)\right] = \int_0^\infty \left(\ln\hat{\theta} - \ln\theta\right)^2 h(\theta|x_1 \dots\dots\dots x_n)d\theta$$

$$= (\ln\hat{\theta})^2 - 2\left(\ln\hat{\theta}\right) E(\ln\theta|x) + E((\ln\theta)^2|x)$$

$$\frac{\partial Rsik}{\partial\hat{\theta}} = 2\left(\ln\hat{\theta}\right)\frac{1}{\hat{\theta}} - \frac{2}{\hat{\theta}}E((\ln\theta)|x)$$

By letting,

$$\frac{\partial R \ \left(\hat{\theta} - \theta\right)}{\partial\hat{\theta}} = 0$$

The Bayes estimator for the parameter θ of Pareto distribution under the squared-log error loss function is:

$$\hat{\theta} = Exp[E(\ln\theta|x)] \tag{6}$$

According to the Squared-Log error loss function, the corresponding Bayes' estimator for the reliability function will be:

$$\hat{R}(t) = \mathrm{Exp}[E(\ln R(t)|t)] \tag{7}$$

$$E(\ln R(t)|t) = \int_0^\infty \ln R(t)\; h(\theta|t)d\theta$$

We have $R(t) = \left(\frac{1}{t}\right)^\theta$

Hence,

$$E[\ln R(t)] = \ln\left(\frac{1}{t}\right)E[\theta] \tag{8}$$

Substituting (8) in (7), we get:

$$\hat{R}(t) = \mathrm{Exp}\left[\ln\left(\frac{1}{t}\right)E[\theta]\right] \tag{9}$$

4. PRIOR AND POSTERIOR DISTRIBUTIONS

In this paper, we consider informative as well as non-informative prior density for θ in order to get better understanding of our Bayesian analysis as follows:

(i) Bayes Estimator Using Jeffery Prior Information

Let us assume that θ has non-informative prior density defined by using Jeffrey prior information $g_1(\theta)$ which given by:

$$g_1(\theta) \propto \sqrt{I(\theta)}$$

where $I(\theta)$ represents Fisher information which defined as follows:

$$I(\theta) = -nE\left(\frac{\partial^2 \ln f}{\partial\theta^2}\right)$$

$$g_1(\theta) = C\sqrt{-nE\left(\frac{\partial^2 \ln f}{\partial\theta^2}\right)} \tag{10}$$

$$\ln f(x;\theta) = \ln\theta - (\theta+1)\ln x$$

$$\frac{\partial \ln f}{\partial\theta} = \frac{1}{\theta} - \ln x$$

$$\frac{\partial^2 \ln f}{\partial\theta^2} = -\frac{1}{\theta^2}$$

$$E\left(\frac{\partial^2 \ln f}{\partial\theta^2}\right) = -\frac{1}{\theta^2}$$

After substitution in (10) we find that:

$$g_1(\theta) = \frac{c}{\theta}\sqrt{n}$$

So, the posterior distribution for θ using Jeffery prior is:

$$h_1(\theta|x) = \frac{L(x_1,\dots,x_n|\theta)g_1(\theta)}{\int_0^\infty L(x_1,\dots,x_n|\theta)g_1(\theta)d\theta}$$

$$= \frac{\theta^n e^{-(\theta+1)\sum \ln x}\,\frac{c}{\theta}\sqrt{n}}{\int_0^\infty \theta^n e^{-(\theta+1)\sum \ln x}\,\frac{c}{\theta}\sqrt{n}\,d\theta}$$

$$= \frac{T^n\theta^{n-1}e^{-\theta T}}{\Gamma(n)} \tag{11}$$

This posterior density is recognized as the density of the gamma distribution:

$$\theta \sim \text{Gamma}\left(n, \sum_{i=1}^{n} \ln x_i\right), \text{with:}$$

$$E(\theta) = \frac{n}{\sum_{i=1}^{n} \ln x_i}, \qquad ver(\theta) = \frac{n}{\left(\sum_{i=1}^{n} \ln x_i\right)^2}$$

Now,

$$E(\ln\theta|x) = \frac{T^n}{\Gamma(n)} \int_0^\infty \ln\theta\, \theta^{n-1} e^{-\theta T} d\theta$$

Let $y = \theta T$

Hence,

$$E(\ln\theta|x) = \frac{T^n}{\Gamma(n)} \int_0^\infty \ln\left(\frac{y}{T}\right)\left(\frac{y}{T}\right)^{n-1} e^{-y} \frac{dy}{T}$$
$$= \frac{T^n}{\Gamma(n) T^n} \int_0^\infty [\ln y - \ln T] y^{n-1} e^{-y} dy$$
$$= \int_0^\infty \frac{\ln y\, y^{n-1} e^{-y}}{\Gamma(n)} dy - \frac{\ln T}{\Gamma(n)} \int_0^\infty y^{n-1} e^{-y} dy$$

$$E(\ln\theta|x) = \varphi(n) - \ln T \qquad (12)$$

Where, $\varphi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$ is the digamma function [5]

Substituting (12) in (6), we get

$$\hat{\theta}_J = \text{Exp}[\varphi(n) - \ln T] \qquad (13)$$

Now, using (9) to estimate Reliability function we reach to:

$$\hat{R}_J(t) = \text{Exp}\left[\frac{n}{T} \ln\left(\frac{1}{t}\right)\right] \qquad (14)$$

We can notice that $\hat{R}_J(t)$is equivalent to the Maximum Likelihood Estimator for R(t).

(ii) Posterior Distribution Using Exponential Prior Distribution

Assuming that θ has informative prior as Exponential prior, which takes the following form:

$$g_2(\theta) = \frac{1}{\lambda} e^{-\frac{\theta}{\lambda}}, \quad \theta, \lambda > 0$$

So, the posterior distribution for the parameter θ given the data $(x_1, x_2, \dots x_n)$ is:

$$h_2(\theta|\underline{X}) = \frac{\pi_{i=1}^{n} f(x_i|\theta) g_2(\theta)}{\int_0^\infty \pi_{i=1}^{n} f(x_i|\theta) g_2(\theta) d\theta}$$

Then the posterior distribution became as follows:

$$h_2(\theta|t) = \frac{\left[T + \frac{1}{\lambda}\right]^{n+1} \theta^n e^{-\theta\left[T + \frac{1}{\lambda}\right]}}{\Gamma(n+1)} \qquad (15)$$

This posterior density is recognized as the density of the gamma distribution

where: $\theta \sim \text{Gamma}\left(n+1, \frac{1}{\lambda} + \sum_{i=1}^{n} \ln x_i\right)$, With:

$$E(\theta) = \frac{n+1}{\frac{1}{\lambda} + \sum_{i=1}^{n} \ln x_i}, \qquad ver(\theta) = \frac{n+1}{\left(\frac{1}{\lambda} + \sum_{i=1}^{n} \ln x_i\right)^2}$$

The Bayes estimator under Squared-Log error loss function will be:

$$\hat{\theta}_E = \text{Exp}\left[\int_0^\infty \ln\theta\, h_2(\theta|t) d\theta\right]$$

$$= \int_0^\infty \ln\theta \frac{\left[T+\frac{1}{\lambda}\right]^{n+1} \theta^n e^{-\theta\left[T+\frac{1}{\lambda}\right]}}{\Gamma(n+1)} d\theta \tag{16}$$

Let $y = \theta\left[T+\frac{1}{\lambda}\right]$

Substituting in (16), we have:

$$E(\ln\theta|x) = \frac{\left[T+\frac{1}{\lambda}\right]^{n+1}}{\Gamma(n+1)} \int_0^\infty \ln\left[\frac{y}{\left[T+\frac{1}{\lambda}\right]}\right]\left[\frac{y}{T+\frac{1}{\lambda}}\right]^n e^{-y} \frac{dy}{\left[T+\frac{1}{\lambda}\right]}$$

By simplification, we get:

$$E(\ln\theta|x) = \varphi(n+1) + \ln\left[T+\frac{1}{\lambda}\right]$$

$$\hat{\theta}_E = \text{ExP}\left[\varphi(n+1) + \ln\left[T+\frac{1}{\lambda}\right]\right] \tag{17}$$

Now, the corresponding Bayes estimator for $\hat{R}_E(t)$ with posterior distribution (15), come out as:

$$\hat{R}_E(t) = \text{ExP}\left[\frac{(n+1)\ln(\frac{1}{t})}{\left[T+\frac{1}{\lambda}\right]}\right]$$

5. SIMULATION RESULTS

In our simulation study, we generated I = 2500 samples of sizes n = 20, 50, and100 from Pareto type I distribution to represent small, moderate and large sample size with the shape parameter θ =0.5, 1.5, 2.5 and taking t = 1.5, 3. We chose two values of λ for the Exponential prior (λ=0.5, 3).

In this section, Monte – Carlo simulation study is performed to compare the methods of estimation by using mean square Errors (MSE's) as an index for precision to compare the efficiency of each of estimators,

where: $MSE(\hat{\theta}) = \frac{\sum_{i=1}^{I}\left(\hat{\theta}_i - \theta\right)^2}{I}$

The results were summarized and tabulated in the following tables for each estimator and for all sample sizes.

6- NUMERICAL VALUES OF ESTIMATOR ($\hat{\theta}$)

The expectations and MSE's for **θ** are schedule in tables (1, 2, and 3) according to the sequence of tables as follows:

Table - 1: Expected Values and MSE's of the Parameter of Pareto Distribution with $\theta = 0.5$

N	Criteria	Bayes(Jeffery) $\hat{\theta}_J$	$\hat{\theta}_E$ λ=0.5	$\hat{\theta}_E$ λ=3
20	Exp.(θ)	0.516100	0.513880	0.537547
	MSE	0.015625	0.013726	0.017743
50	Exp.(θ)	0.504876	0.504564	0.513291
	MSE	0.005465	0.005226	0.005760
100	Exp.(θ)	0.502409	0.502345	0.506610
	MSE	0.002633	0.002578	0.002706

Table - 2: Expected Values and MSE's of the Parameter of Pareto Distribution with $\theta = 1.5$

N	Criteria	Bayes (Jeffery) $\hat{\theta}_J$	$\hat{\theta}_E$ λ=0.5	$\hat{\theta}_E$ λ=3
20	Exp.(θ)	1.548297	1.395296	1.583429
	MSE	0.140624	0.091596	0.143131
50	Exp.(θ)	1.514626	1.454416	1.529289
	MSE	0.049183	0.041884	0.049711
100	Exp.(θ)	1.507228	1.477195	1.514685
	MSE	0.023701	0.021876	0.023851

Table - 3: Expected Values and MSE's of the Parameter of Pareto Distribution with $\theta = 2.5$

N	Criteria	Bayes (Jeffery) $\hat{\theta}_J$	$\hat{\theta}_E$ λ=0.5	$\hat{\theta}_E$ λ=3
20	Exp.(θ)	2.580500	2.125352	2.592182
	MSE	0.390623	0.295328	0.359498
50	Exp.(θ)	2.524381	2.332812	2.531424
	MSE	0.136620	0.122543	0.132948
100	Exp.(θ)	2.512047	2.414210	2.515959
	MSE	0.065835	0.062163	0.065016

7. DISCUSSION

From tables (1, 2, 3) when θ=0.5, 1.5, 2.5, the simulation results show that $\hat{\theta}_E$ with **λ=0.5** was the best in performance, followed by $\hat{\theta}_J$ (which equivalent to $\hat{R}_{ML}(t)$) for different size of samples. and we can notice that MSE's increases with increases of λ (λ =3). Finally for all sample sizes, an obvious increase in MSE is observed with the increase of the shape parameter values.

8. NUMERICAL VALUES OF ESTIMATOR $\hat{R}(t)$

The numerical results are schedule in tables (4, 5, 6, 7, 8, and 9) according to the sequence of tables as follows:

Table - 4: Expected Values and MSE's of the Reliability Function of Pareto Distribution with $\theta = 0.5,\ t = 1.5,\ (R(t)_{t=1.5} = 0.816497)$

n	Criteria	Bayes (Jeffery) $\hat{R}_J(t)$	$\hat{R}_E(t)$ λ=0.5	$\hat{R}_E(t)$ λ=3
20	Exp. R(t)	0.807908	0.808736	0.801013
	MSE	0.001732	0.001530	0.001967
50	Exp. R(t)	0.813546	0.813665	0.810787
	MSE	0.000602	0.000577	0.006373
100	Exp. R(t)	0.814982	0.815011	0.813604
	MSE	0.000290	0.000284	0.000299

Table - 5: Expected Values and MSE's of the Reliability Function of Pareto Distribution with $\theta = 0.5, t = 3,\ (R(t)_{t=3} = 0.5773503)$

n	Criteria	Bayes (Jeffery) $\hat{R}_J(t)$	$\hat{R}_E(t)$ λ=0.5	$\hat{R}_E(t)$ λ=3
20	Exp. R(t)	0.564311	0.565504	0.551550
	MSE	0.005720	0.005117	0.006282
50	Exp. R(t)	0.572896	0.573075	0.567680
	MSE	0.002122	0.002035	0.002211
100	Exp. R(t)	0.575028	0.575074	0.572403
	MSE	0.001041	0.001020	0.001064

Table - 6: Expected Values and MSE's of the Reliability Function of Pareto Distribution with $\theta = 1.5,\ t = 1.5,\ (R(t)_{t=1.5} = 0.5443319)$

n	Criteria	Bayes (Jeffery) $\hat{R}_J(t)$	$\hat{R}_E(t)$ **λ=0.5**	$\hat{R}_E(t)$ **λ=3**
20	**Exp. R(t)**	0.531299	0.563970	0.523920
	MSE	0.006138	0.004498	0.006157
50	**Exp. R(t)**	0.539809	0.553024	0.536717
	MSE	0.002299	0.002036	0.002306
100	**Exp. R(t)**	0.542009	0.548608	0.540388
	MSE	0.001131	0.001062	0.001134

Table - 7: Expected values and MSE's of the Reliability Function of Pareto Distribution with $\theta = 1.5, t = 3,\ (R(t)_{t=3} = 0.1924501)$

n	Criteria	Bayes (Jeffery) $\hat{R}_J(t)$	$\hat{R}_E(t)$ **λ=0.5**	$\hat{R}_E(t)$ **λ=3**
20	**Exp. R(t)**	0.188909	0.100107	0.181803
	MSE	0.004813	0.002969	0.004547
50	**Exp. R(t)**	0.191610	0.079374	0.188568
	MSE	0.001980	0.000888	0.001931
100	**Exp. R(t)**	0.191916	0.071787	0.190364
	MSE	0.001003	0.000380	0.000991

Table - 8: Expected Values and MSE's of the Reliability Function of Pareto Distribution with $\theta = 2.5\ , t = 1.5,\ (R(t)_{t=1.5} = 0.3628883)$

n	Criteria	Bayes (Jeffery) $\hat{R}_J(t)$	$\hat{R}_E(t)$ **λ=0.5**	$\hat{R}_E(t)$ **λ=3**
20	**Exp. R(t)**	0.352656	0.419037	0.350530
	MSE	0.006952	0.007480	0.006452
50	**Exp. R(t)**	0.359528	0.387663	0.358448
	MSE	0.002742	0.002868	0.002661
100	**Exp. R(t)**	0.361080	0.375469	0.360505
	MSE	0.001371	0.001400	0.001351

Table - 9: Expected Values and MSE's of the Reliability Function of Pareto Distribution with $\theta = 2.5, t = 3,\ (R(t)_{t=3} = 0.0071278)$

n	Criteria	Bayes (Jeffery) $\hat{R}_J(t)$	$\hat{R}_E(t)$ **λ=0.5**	$\hat{R}_E(t)$ **λ=3**
20	**Exp. R(t)**	0.066859	0.100107	0.065278
	MSE	0.001538	0.002969	0.001385
50	**Exp. R(t)**	0.065596	0.079374	0.064983
	MSE	0.000621	0.000888	0.000594
100	**Exp. R(t)**	0.064821	0.071787	0.064521
	MSE	0.000312	0.000380	0.000305

9. DISCUSSION

From tables (4, 5, 6, 7) when $\theta = 0.5\ , 1.5$ and $t = 1.5\ , 3$ the simulation results shows that with all sample sizes,$\hat{R}_E(t)$ with **λ=0.5** was better in performance than each of $\hat{R}_J(t)$(which equivalent to MLE) and $\hat{R}_E(t)$ with **λ=3**. Also tables (4, 5, 6, and 7) shows that the results of estimators (included MLE) are closed especially with large sample sizes.

Tables (8, 9) showing that, with a large value of $\theta, (\theta = 2.5)$ MSE's is decreases with increasing of λ $(\lambda = 3)$ for the Bayes estimator with exponential prior so, we can say that $\widehat{R}_E(t)$ with **λ=3**is better than each of the estimator with Jeffery prior (MLE) and $\widehat{R}_E(t)$ with **λ=0.5**.

In general, we conclude that in situations involving estimation of parameter Reliability function of Pareto type I distribution under Squared-Log error loss function, using exponential prior with small value of λ $(\lambda = 0.5)$ is more appropriate than using Jeffery prior (or MLE) when t,θ are small relatively (t=1.5, $\theta = 0.5$). Otherwise using exponential prior with large value of λ $(\lambda = 3)$ is better than using Jeffery prior (or MLE).

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