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PSEUDO PROJECTIVELY FLAT ALMOST PSEUDO RICCI-SYMMETRIC MANIFOLDS

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ABSTRACT
The object of the present paper is to study pseudo projectively flat almost pseudo Ricci symmetric manifolds.

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1. INTRODUCTION

As an extended class of pseudo Ricci symmetric manifolds, very recently M.C.Chaki and T. Kawaguchi [1] introduced the notation of almost pseudo Ricci-symmetric manifolds. A Riemannian manifold (M^n, g) is called an almost pseudo Ricci-symmetric manifold if its Ricci tensor S of type $(0, 2)$ is not identically zero and satisfies a relation

$$(D_X S)(Y, Z) = \{A(X) + B(X)\}S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X), \tag{1}$$

where D denotes the operator of covariant differentiation with respect to the Riemannian metric g and A, B are nowhere vanishing 1-forms such that $g(X, \rho) = A(X)$ and $g(X, \mu) = B(X)$ for all X, ρ and μ are called the basic vector fields of the manifold.

The one form A and B are called the associated $1-$forms and $n-$dimensional manifold of this kind is denoted by $A(PRS)_n$.

If, in particular $B = A$, then (1) reduces to

$$(D_X S)(Y, Z) = 2A(X)S(Y, Z) + A(Y)S(X, Z) + A(Z)S(Y, X) \tag{2}$$

which represents a pseudo Ricci-symmetric manifold [2]. In [1], Chaki and Kawaguchi also studied conformally flat $A(PRS)_n$. Recently Shaikh and Hui [5] studied the properties of quasi-conformally flat almost pseudo Ricci-symmetric manifold. In [6], Prasad defined and studied a tensor field $\tilde{P}$ of type $(1, 3)$ which is the generalisation of Weyl projective curvature tensor, called pseudo-projective curvature tensor. The present paper deals with a study of pseudo-projectively flat $A(PRS)_n$.

The paper is organized as follows. Section 2 concerned with preliminaries. Section 3 devoted to the study of pseudo-projectively flat $A(PRS)_n$ and proved that the vector fields μ and ξ are co-directional. It is shown that in a Pseudo-projectively flat $A(PRS)_n$ the integral curves of the generator λ defined by $g(X, \lambda) = T(X)$ are geodesic and the vector field λ is a unit proper con-circular vector field. Also it is shown that in this manifold the Ricci tensor is Codazzi type and the vector field λ is a unit parallel vector field.

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2. PRELIMINARIES

Let Q be the symmetric endomorphism of the tangent space at any point of the manifold corresponding to the Ricci tensor S, *i.e.*, $S(X,Y) = g(QX,Y)$ for all vector fields X,Y and $\{e_i\}$, $i = 1,2,3 \dots\dots n$ be an orthonormal basis of tangent space at any point of the manifold . Then by setting $Y = Z = e_i$ in (1) and then taking summation over i, $1 \le i \le n$, we obtain

$$dr(X) = r\{A(X) + B(X)\} + 2A(QX) \tag{3}$$

where r is the scalar curvature of the manifold.

Again from (1), we get

$$(D_X S)(Y,Z) - (D_Y S)(X,Z) = B(X)S(Y,Z) - B(Y)S(X,Z) \tag{4}$$

Setting $Y = Z = e_i$ in (4) and then taking summation over i, for $1 \le i \le n$, we obtain

$$dr(X) = 2rB(X) - 2B(QX) \tag{5}$$

If the scalar curvature r is constant, then

$$dr(X) = 0, \text{ for all } X. \tag{6}$$

By virtue of (6), (5) yields,

$$rB(X) = B(QX) \tag{7}$$

i.e., $S(X,\mu) = rg(X,\mu)$ (8)

Proposition1: In an $A(PRS)_n$ of constant scalar curvature, r is an eigen value of the Ricci tensor S corresponding to the eigen vector μ.

The pseudo-projective curvature tensor $\tilde{P}$ of type (1, 3) is defined by [6]

$$\tilde{P}(X,Y)Z = -(n-1)bP(X,Y)Z + \{a + (n-1)b\}C(X,Y)Z \tag{9}$$

where a and b are arbitrary constants not simultaneously zero and P, C are respectively Weyl projective and concircular curvature tensors. It bridges the gap between the Weyl projective and concircular curvature tensors. Its tensorial relation is given by

$$\tilde{P}(X,Y)Z = a\, R(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y] - \frac{r}{n}\left[\frac{a}{n-1} + b\right][g(Y,Z)X - g(X,Z)Y] \tag{10}$$

$$\left(div\tilde{P}\right)(X,Y)Z = a(div\, R)(X,Y)Z + b\{(D_X S)(Y,Z) - (D_Y S)(X,Z)\} - \frac{1}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)drX - g(X,Z)drY] \tag{11}$$

where div denotes divergence. Again it is known that in a Riemannian manifold, we have

$$(divR)(X,Y)Z = \{(D_X S)(Y,Z) - (D_Y S)(X,Z)\}.$$

Consequently by the virtue of above equation (11) takes the form

$$\left(div\tilde{P}\right)(X,Y)Z = (a+b)\{(D_X S)(Y,Z) - (D_Y S)(X,Z)\} - \frac{1}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)drX - g(X,Z)drY] \tag{12}$$

3. PSEUDO PROJECTIVELY FLAT $A(PRS)_n$

Let us consider a pseudo projectively flat $A(PRS)_n$, then we have

$$\left(div\tilde{P}\right)(X,Y)Z = 0. \tag{13}$$

and hence (12) yields

$$(a+b)\{(D_X S)(Y,Z) - (D_Y S)(X,Z)\} = -\frac{1}{n}\left[\frac{a}{n-1} + 2b\right][g(Y,Z)dr(X) - g(X,Z)dr(Y)] \tag{14}$$

By virtue of (3) and (5), it follows from (14) that,

$$(a+b)\{B(X)S(Y,Z) - B(Y)S(X,Z)\} = 2\left[\frac{a+(n-1)b}{n(n-1)}\right] [r\{g(Y,Z)B(X) - g(X,Z)B(Y)\} - \{g(Y,Z)B(QX) - g(X,Z)B(QY)\} \tag{15}$$

Provided that $a + b \ne 0$. putting $Z = \mu$ in (15), we obtain

$$B(X)B(QY) - B(Y)B(QX) = 0. \tag{16}$$

Provided that $a + b \ne 0$ and $(n+1)a - (n-1)b \ne 0$.

Let $B(QX) = g(QX, \mu) = P(X) = g(X, \xi)$, for all X.

Then from (16), we get,

$$B(X)P(Y) = B(Y)P(X), \tag{17}$$

which shows that the vector fields μ and ξ are co- directional. Hence we can state the following

Theorem 3.1: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, with $a + b \neq 0$ and $(n + 1)a - (n - 1)b \neq 0$, the vector fields μ and ξ are co- directional.

If $a + b = 0$ and $(n + 1)a - (n - 1)b \neq 0$, then using (5) in (14), it can be easily shown that the relation (17) holds. Hence we can state the following

Corollary 3.1: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, with $a + b = 0$ and $(n + 1)a - (n - 1)b \neq 0$, the vector fields μ and ξ are co- directional.

Again if $a + b \neq 0$ and $(n + 1)a - (n - 1)b = 0$, then using (4) in (14), it can be easily shown that the relation (17) holds. Hence we can state the following

Corollary 3.2: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, with $a + b \neq 0$ and $(n + 1)a - (n - 1)b = 0$, the vector fields μ and ξ are co- directional.

It may be noted that in a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, with $a + b = 0$ and $(n + 1)a - (n - 1)b = 0$ can not hold simultaneously as a and b are not simultaneously zero.

Again setting $Y = Z = e_i$ in (15) and then taking summation over i, $1 \leq i \leq n$, then we obtain

$$B(QX) = rB(X), \tag{18}$$

Provided that $a - b \neq 0$, i.e.

$$S(X, \mu) = rg(X, \mu) \tag{19}$$

Hence we can state the following

Theorem 3.2: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, with $a + b \neq 0$ and $a - b \neq 0$, r is an eigen value of the Ricci tensor S corresponding to the eigen vector μ.

If $a + b = 0$, then it follows from (15) that (19) holds provided that $a + (n - 1)b \neq 0$.

Hence, we can state the following

Corollary 3.3: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, with $a + b = 0$ and $a - (n - 1)b \neq 0$, r is an eigen value of the Ricci tensor S corresponding to the eigen vector μ.

Also for $a + b \neq 0$ and $a + (n - 1)b = 0$, then we can state the following

Corollary 3.4: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, with $a + b \neq 0$ and $a + (n - 1)b = 0$, r is an eigen value of the Ricci tensor S corresponding to the eigen vector μ.

In view of (18), (15) yields

$$B(X)S(Y, Z) = B(Y)S(X, Z). \tag{20}$$

Setting $X = \mu$ in (20), we get

$$S(Y, Z) = \frac{1}{B(\mu)} B(Y)B(QZ). \tag{21}$$

In view of (18), (21) yields

$$S(Y, Z) = rT(Y)T(Z), \tag{22}$$

where $T(X) = g(X, \lambda) = \frac{1}{\sqrt{B(\mu)}} B(X)$, λ being a unit vector field associated with the nowhere vanishing 1-form T. From (22), it follows that if $r = 0$, then $S(Y, Z) = 0$, which is inadmissible by the definition of $A(PRS)_n$. Hence we can state the following

Theorem 3.3: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, with $a + b \neq 0$ and $a - b \neq 0$, the scalar curvature cannot vanish and the Ricci tensor is of the form (22).

Again from (22), we have

$$(D_X S)(Y, Z) = dr(X)T(Y)T(Z) + r\{(D_X T)(Y)T(Z) + (D_X T)(Z)T(Y)\}. \tag{23}$$

Using (23) in (14) we obtain

$$(a + b)[\{dr(X)T(Y)T(Z) - dr(Y)T(Z)\}r\big((D_X T)(Y)T(Z) + (D_X T)(Z)T(Y) - (D_Y T)(X)T(Z) - (D_Y T)(Z)T(X)\big)]$$
$$= \frac{1}{n}\left[\frac{a}{n-1} + 2b\right][g(Y, Z)drX - g(X, Z)drY]. \tag{24}$$

Setting $Y = Z = e_i$ in (24) and then taking summation over, $1 \leq i \leq n$, then we obtain

$$(a + b)\left[dr(\lambda)T(X) + r\{(D_\lambda T)(X) + T(X)\sum_{i=1}^{n-1}\big(D_{e_i}T\big)(e_i)\}\right] = \left\{\frac{(n-1)a + b}{n}\right\}dr(X). \tag{25}$$

Again putting $Y = Z = \lambda$ in (24), we get

$$r(a + b)(D_\lambda T)(X) = \left\{\frac{(n^2 - n - 1)a + (n-1)^2 b}{n(n-1)}\right\}\{dr(X) - T(X)dr(\lambda)\} \tag{26}$$

Using (26) in (25), we get

$$r(a + b)T(X)\textstyle\sum_{i=1}^{n-1}\big(D_{e_i}T\big)(e_i) + E\{(n-2)dr(X) + dr(\lambda)T(X)\} = 0, \tag{27}$$

where $E = \frac{a+(n-1)b}{n(n-1)}$. Substituting $X = \lambda$ in (27), we get

$$r(a + b)\sum_{i=1}^{n-1}\big(D_{e_i}T\big)(e_i) = -(n-1)Edr(\lambda), \tag{28}$$

From (27) and (28), we have

$$dr(X) = dr(\lambda)T(X) \tag{29}$$

Provided that $a + (n-1)b \neq 0$. Again putting $Z = \lambda$ in (24) and then using (29), we get

$$r(a + b)\{(D_X T)(Y) - (D_Y T)(X)\} = 0,$$

which implies that

$$(D_X T)(Y) - (D_Y T)(X) = 0, \tag{30}$$

because $r \neq 0$ and $a + b \neq 0$. The relation (30) implies that the 1-form T is closed.

In view of (29), it follows from (26) that

$$(D_\lambda T)(X) = 0, \tag{31}$$

provided $a + b \neq 0$, which implies that $D_\lambda \lambda = 0$. Hence we can state the following

Theorem 3.4: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, with $a + b \neq 0$ and $a - b \neq 0$ and $a + (n-1)b \neq 0$, the integral curve of the generator λ are geodesics.

Also setting $Y = \lambda$ in (24), we obtain by virtue of (29) and (31) that

$$(D_X T)(Z) = \frac{E}{r(a+b)}dr(\lambda)\{T(X)T(Z) - g(X, Z)\}, \tag{32}$$

provided that $a + b \neq 0$.

Let us now consider a non zero scalar function $f = \frac{E}{r(a+b)}dr(\lambda)$, where the scalar curvature r is non constant. Then we have

$$D_X f = \frac{E}{r^2(a+b)}\{dr(\lambda)dr(X) - rd^2 r(\lambda, X)\} \tag{33}$$

From (29) it follows that

$$d^2 r(X, Y) = d^2 r(\lambda, Y)T(X) + dr(\lambda)(D_Y T)(X). \tag{34}$$

Again in a Riemannian manifold, the second covariant derivative of any function $h \in \mathbb{C}^\infty(M)$ is defined by

$$d^2 h(X, Y) = X(Yh) - (D_X Y)(h),$$

for all $X, Y \in \mathcal{X}(M)$, which implies that
$d^2h(X,Y) = d^2h(Y,X),$

for all $X, Y \in \mathcal{X}(M)$, and hence (34) implies that
$$d^2r(\lambda,Y)T(X) = d^2r(\lambda,X)T(Y), \tag{35}$$

replacing Y by λ in (35) we have
$$d^2r(\lambda,X) = d^2r(\lambda,\lambda)T(X) = \psi T(X) \tag{36}$$

where $\psi = d^2r(\lambda,\lambda)$ is a scalar function. Using (29) and (36) in (33), we obtain
$$D_X f = \sigma T(X), \tag{37}$$
where $\sigma = \frac{E}{r^2(a+b)}\{r\psi - \left(dr(\lambda)\right)^2\}$ is a non zero scalar.

We now consider a 1-form ω, given by
$$\omega(X) = \frac{E}{r(a-b)} dr(\lambda)T(X) = fT(X).$$

Then by the virtue of (30), (37) and above relation, we have
$d\omega(X,Y) = 0.$

Hence 1- form ω is closed, therefore (32) can be written as
$$(D_X T)(Z) = -fg(X,Z) + \omega(X)T(Z), \tag{38}$$

which implies that the vector field λ corresponding to the 1-form T defined by $g(X,\lambda) = T(X)$ is a proper concircular vector field [4], [7].

Hence we can state the following

Theorem 3.5: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$ of non constant scalar curvature with $a + b \neq 0$ and $a - b \neq 0$ and $a + (n-1)b \neq 0$, the vector field λ is a unit proper concircular vector field.

If in particular, $a + b = 0$, then from (14), we get
$dr(X)g(Y,Z) - dr(Y)g(X,Z) = 0,$

which yields,
$$dr(X) = 0, \tag{39}$$
for all X, provided that $a + (n-1)b \neq 0$. This means that the scalar curvature of the pseudo-projectively flat $A(PRS)_n$ is constant.

Hence we can state the following

Theorem 3.6: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$, the scalar curvature r is constant provided $a + b = 0$ and $a + (n-1)b \neq 0$.

Again if $a + b \neq 0$ and $a + (n-1)b = 0$, then from (14), we get

$$(D_X S)(Y,Z) = (D_Y S)(X,Z) \tag{40}$$

for all X, Y, Z, which means that the Ricci tensor is of Codazzi type [3] and hence
$$dr(X) = 0, \tag{41}$$

for all X. hence (24) takes the form
$(D_X T)(Y)T(Z) - (D_Y T)\big(X + T(Y)\big)T(Z)(D_X T)(Z) - T(X)(D_Y T)(Z) = 0.$

Putting $Z = \lambda$ in (42), we get (30). Also for $Y = \lambda$ (30) implies that
$$(D_\lambda T)(X) = 0,$$
for all X. Using this relation we obtain from (42) (for $Y = \lambda$) that $(D_X T)(Z) = 0$, for all X, Z. This implies that $g(Z, D_X\lambda) = 0.$

For all X, Z. Since g is non degenerate, the last relation yields $D_X\lambda = 0$, for all X, which means that λ is a parallel vector field. Hence we can state the following

Theorem 3.7: In a pseudo-projectively flat $A(PRS)_n$ $(n > 2)$ with $a + (n-1)b = 0$ and provided $a + b \neq 0$, the Ricci tensor is of Codazzi type and the vector field λ is a unit parallel vector field.

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