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SOME REMARKS ON D-QUASI CONTRACTION ON CONE SYMMETRIC SPACE

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ABSTRACT:

The present note introduces and proves some fixed point theorem of D-quasi contraction on cone symmetric space.

***Keywords**: Fixed point, D-quasi contraction, contraction, cone metric space, symmetric space.*

***2000 Mathematics Subject Classification:** 47H10, 54H25.*

1. INTRODUCTION AND PRELIMINARIES

Huang and Zhang [2] obtained a generalisation of metric space by introducing the concept of cone metric space. They used an ordered Banach Space in place of set of real numbers in metric space. They also obtained some fixed point theorems in this space for mappings satisfying various types of contractive conditions. In this work we introduce the concept of fixed point theorem of D-quasi contraction on cone symmetric space.

Let E be a real Banach Space. A subset P of E is called a cone if
(i) P is closed, nonempty and $P \neq \{0\}$
(ii) $a, b \in R$, $a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$
(iii) $P \cap (-P) = \{0\}$

Given a cone $P \subset E$, we define the partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$. We write $x < y$ to denote that $x \leq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in \text{int } P$ (interior of P).

There are two kinds of cone. They are normal cone and non-normal cones. A cone $P \subset E$ is normal if there is a number $K > 0$ such that for all $x, y \in P$, $0 \leq x \leq y \Rightarrow \|x\| \leq K\|y\|$. In other words if $x_n \leq y_n \leq z_n$ and $\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x$ imply $\lim_{n\to\infty} y_n = x$.

The least positive number K satisfying $\|x\| \leq K\|y\|$ is called the normal constant of P. It is clear that $K \geq 1$.

Definition [1]: Let $((X, d)$ be metric space. A map $f : X \to X$ with the property that for some constant $\lambda \in (0,1)$ and for every $x, y \in X$,
$d(fx, fy) \leq \lambda.\max\{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}$ is called a quasi- contraction.

Definition 1.1[2]: Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies
i) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
iii) $d(x, y) \leq d(x, z) + d(z, y)$ For all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.2[3]: Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satisfies
i) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
ii) $d(x, y) = d(y, x)$ for all $x, y \in X$

Then d is called a cone symmetric on X and (X, d) is called a cone symmetric space.

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Definition 1.3 [3]: Let (*X*, *d*) be a cone symmetric space. We define a mapping $D: X \times X \to R$ such that $D(x, y) = \|d(x, y)\|$ for all $x, y \in X$. Then D is called a symmetric metric on *X* if it satisfies the following conditions

i) $0 < D(x, y)$ for all $x, y \in X$ and $D(x, y) = 0$ if and only if $x = y$;
ii) $D(x, y) = D(y, x)$ For all $x, y \in X$.

Then the ordered pair (*X*, *D*) is called the symmetric space associated with the cone symmetric space (*X*, *d*).

Remark 1.4 [3]: The symmetric space (*X*, *D*) is almost a metric space associated with the cone symmetric space (*X*, *d*) if the cone (*X*, *d*) is a normal.

Definition 1.5 [3]: Let (X, d) be cone metric space. A map $f: X \to X$ with the property that for some constant $\lambda \in (0,1)$ and for every $x, y \in X$,
$D(fx, fy) \le \lambda.\max\{D(x, y), D(x, fx), D(y, fy), D(x, fy), D(y, fx)\}$ is called a D-quasi- contraction.

Note: If $f: X \to X$ and $n \in N$, we set

$$O(x;n) = \{x, fx, f^2x, ..., f^nx\} \text{ And } O(x;\infty) = \{x, fx, f^2x, ...\}$$

2. MAIN RESULTS

Lemma 2.1: Let (*X*, *D*) be a symmetric space associated with cone metric space (*X*, *d*) and let P be normal cone with normal constant $K \ge 1$. Let $f: X \to X$ be a D-quasi contraction. Then, there exists $n_0 \in N$ such that for every $n > n_0$,

$$\partial(O(x;n)) = \max\{\|D(x, f^lx)\|, \|D(f^ix, f^jx)\| : 1 \le l \le n, 1 \le i, j \le n_0\}$$

and

$$\partial(O(x;\infty)) \le \max\{\frac{K}{1 - K^2\lambda^{n_0}}\|D(x, f^{n_0+1}x)\|, \lambda K\partial(O(x;n_0)), \|D(x, f^lx)\| : 1 \le l \le n_0\}.$$

Proof: Let $n_0 \in N$ be such that $\max\{\lambda^{n_0}K, \lambda^{n_0}K^2\} < 1$. We choose $i, j \in N$ such that $n_0 < i < j \le n$. There exists $s_1 \in C(f, f^{i-1}x, f^{j-1})$ such that $D(f^ix, f^jx) \le \lambda.s_1$.

Moreover, there exists $s_2 \in \{D(a,b) : a, b \in O(x;n)\}$ such that $s_1 \le \lambda s_2$.

Hence, $D(f^ix, f^jx) \le \lambda s_2$.

Then we can conclude that $D(f^ix, f^jx) \le \lambda^{n_0}s_{n_0}$ for some $s_{n_0} \in \{D(a,b) : a, b \in O(x;n)\}$.

Now, $\|D(f^ix, f^jx)\| \le \lambda^{n_0}.\|s_{n_0}\| < \|s_{n_0}\|K\partial(O(x;n_0))$.

This proves that $\partial(O(x;n)) = \max\{\|D(x, f^lx)\|, \|D(f^ix, f^jx)\| : 1 \le l \le n, 1 \le i, j \le n_0\}$

For $1 \le i < j \le n_0$, $D(f^ix, f^jx) \le \lambda s_1$

For some $s_1 \in \{D(a,b) : a, b \in O(x;n_0)\}$. Hence, $\|D(f^ix, f^jx)\| \le \lambda K\partial(O(x;n_0))$.

So, if $\partial(O(x;n)) = \|D(f^ix, f^jx)\|$, for some $1 \le i < j \le n_0$, then $\partial(O(x;n)) \le \lambda K\partial(O(x;n_0)$

Also for some $D(x, f^lx) \le K(D(x, f^{n_0+1}x) + D(f^{n_0+1}x, f^lx))$

This implies that $\left\|D(x, f^{l}x)\right\| \le K\left\|D(x, f^{n_0+1}x)\right\| + \lambda^{n_0}K^2\partial(O(x;n))$.

If $\partial(O(x;n)) = \left\|D(x, f^{l}x)\right\|$ for some $n_0 < l \le n$, then $\partial(O(x;n)) \le \dfrac{K}{1-\lambda^{n_0}K^2}\left\|D(x, f^{n_0+1}x)\right\|$.

This completes the proof.

Theorem 2.2: Let **(***X*, *D*) be a symmetric space associated with cone metric space **(***X*, *d*) and let P be normal cone with normal constant $K \ge 1$. Suppose that $f : X \to X$ be a D-quasi contraction. Then f has a unique fixed point in X and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.

Proof: Let x be an arbitrary point of X. We shall prove that $\{f^n x\}$ is a Cauchy sequence. Then
$D(f^n x, f^{n-1}x) \le \lambda s_{n,n-1}$ where $s_{n,n-1} \in \{D(f^n x, f^{n-1}x), D(f^n x, f^{n-2}x), D(f^{n-1}x, f^{n-2}x)\}$.

Moreover, $D(f^n x, f^{n-2}x) \le \lambda s_{n,n-2,}$ for
$s_{n,n-2} \in \{D(f^n x, f^{n-1}x), D(f^n x, f^{n-3}x), D(f^{n-1}x, f^{n-2}x), D(f^{n-1}x, f^{n-3}x), D(f^{n-2}x, f^{n-3}x)\}$,

And $D(f^{n-1}x, f^{n-2}x) \le \lambda s_{n-1,n-2,}$

Where $s_{n-1,n-2} \in \{D(f^{n-1}x, f^{n-2}x), D(f^{n-1}x, f^{n-3}x), D(f^{n-2}x, f^{n-3}x), D(f^n x, f^{n-1}x)\}$.

So, $D(f^n x, f^{n-1}x) \le \lambda^2 s^{(2)}_{n,n-1}$,

Where $s^{(2)}_{n,n-1}$ is an element of $\{D(f^n x, f^{n-1}x), D(f^n x, f^{n-2}x), D(f^n x, f^{n-3}x), D(f^{n-1}x, f^{n-2}x), D(f^{n-1}x, f^{n-3}x), D(f^{n-2}x, f^{n-3}x)\}$.

We continue in this way and after $n-1$ steps we get
$D(f^n x, f^{n-1}x) \le \lambda^{n-1}s^{(n-1)}_{n,n-1}$, where $\lambda^{n-1}s^{(n-1)}_{n,n-1}$ is an element of $U^{n-1}_{j=0}U^{n}_{i=j+1}\{D(f^{n-j}x, f^{n-i}x)\}$.

Now, for $m > n$, we have

$$
\begin{aligned}
D(f^n x, f^m x) &\le D(f^n x, f^{n+1}x) + D(f^{n+1}x, f^{n+2}x) + ,\ldots, + D(f^{m-1}x, f^m x)\\
&= \sum_{k=0}^{m-n-1} D(f^{n+k}x, f^{n+k+1}x)\\
&\le \sum_{k=0}^{m-n-1} \lambda^{n+k} S^{(n+k)}_{n+k+1,n+k}.
\end{aligned}
$$

Thus
$\left\|D(f^n x, f^m x)\right\| \le K\lambda^n \left\|\sum_{k=0}^{m-n-1}\lambda^k S^{(n+k)}_{n+k+1,n+k}\right\| \le K\dfrac{\lambda^n}{1-\lambda}.\partial(O(x,\infty))$, and $\{f^n x\}$ is a Cauchy sequence. Thus, there exists $y \in X$ such that $y \in X \lim_{n\to\infty} f^n x = y$.

Now for each $n \in N$, there exists
$s_n \in \{D(f^n x, y), D(f^{n+1}x, f^n x), D(fy, f^n x), D(fy, y)\}$ such that
$D(y, fy) \le K(D(y, f^{n+1}x) + D(f^{n+1}x, fy)) \le KD(y, f^{n+1}x) + \lambda s_n$.

We know that the elements of the sequence $\{s_n\}$ are of the form $D(f^n x, y), D(f^{n+1}x, f^n x), D(fy, f^n x)$ or $D(fy, y)$. We now consider the sub sequences $\{s_{n,i}\}, i = 1,2,...,5$ of the sequence $\{s_n\}$, such that for all elements of the sequence $\{s_{n,i}\}, i = 1,2,...,5$, are of the form $D(f^n x, y), D(f^{n+1}x, f^n x), D(fy, f^n x)$ and $D(fy, y)$, respectively. It is clear that $\lim_{n\to\infty} s_{n,i} = 0, i = 1,2,3$ and $\lim_{n\to\infty} s_{n,i} = D(y, fy), i = 4,5$ .Then $D(y, fy) = 0$ i.e. $fy = y$.

In order to prove the uniqueness of the fixed point, let us suppose that there exists $z \in X$ such that $fz = z$. Then $D(z, y) = D(fz, fy) \le \lambda D(z, y)$ and so $y = z$.

Corollary 2.3: Let **(***X*, *D***)** be a symmetric space associated with cone metric space **(***X*, *d***)** and let P be normal cone with normal constant $K \ge 1$. Suppose that the mapping $f : X \to X$ satisfies the contractive condition $D(fx, fy) \le \lambda D(x, y)$ for all $x, y \in X$ where $\lambda \in [0,1)$ is a constant. Then f has unique fixed point in X and for any $x \in X$, the iterative sequence $\{f^n x\}$ converges to the fixed point.

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