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APPROXIMATION OF SMALLEST EIGENVALUE AND ITS CORRESPONDING EIGENVECTOR BY POWER METHOD

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ABSTRACT

Power method is normally used to determine the largest eigenvalue (in magnitude) and the corresponding eigenvector of the system $AX = \lambda X$. In this study, we examine power method for computing the smallest eigenvalue and its corresponding eigenvector of real square matrices. Our work is based on choosing of initial vector in power method for acceleration purpose. Finally, we illustrate the method with example and results discussed.

Keywords: *Dominant eigenvalue, power method, Adjoint of a square matrix, Inverse matrix in terms of Adjoint matrix.*

1. INTRODUCTION

We study the problem of calculating the eigenvalues and eigenvectors. If only a few eigenvalues are to be calculated, then the numerical method will be different than if all eigenvalues are required. Eigenvalues and eigenvectors play an important part in the applications of linear algebra. The naive method of finding the eigenvalues of a matrix involves finding the roots of the characteristic polynomial of the matrix. In industrial sized matrices, however, this method is not feasible, and the eigenvalues must be obtained by other means. Fortunately, there exist several other techniques for finding eigenvalues and eigenvectors of a matrix, some of which fall under the realm of iterative methods. These methods work by repeatedly refining approximations to the eigenvectors or eigenvalues, and can be terminated whenever the approximations reach a suitable degree of accuracy. Iterative methods form the basis of much of modern day eigenvalue computation.

The general problem of finding all eigenvalues and eigenvectors of a non-symmetric matrix A can be quite unstable with respect to perturbations in the coefficients of A, and this makes more difficult the design of general methods and computer programs. The eigenvalues of a symmetric matrix A are quite stable with respect to perturbations in A. The eigenvalues of a matrix are usually calculated first, and they are used in calculating the eigenvectors, if these are desired. The main exception to this rule is the power method described in this paper, which is useful in calculating a single dominant eigenvalue of a matrix. For obtaining eigenvalues and eigenvectors for low order matrices, 2×2 and 3×3. This involved firstly solving the characteristic equation $\det(A-\lambda I)=0$ for a given $n\times n$ matrix A. This is an nth order polynomial equation and, even for n as low as 3, solving it is not always straightforward. For large n even obtaining the characteristic equation may be different. Consequently, in this paper we give a brief introduction to alternative method, essentially numerical in nature, of obtaining eigenvalues and perhaps eigenvectors. Algebraic procedures for determining eigenvalues and eigenvectors are impractical for most matrices of large order. Instead, numerical methods that are efficient and stable when programmed on high-speed computers have been developed for this purpose. Such methods are iterative, and, in the ideal case, converge to the eigenvalues and eigenvectors of interest. In this paper, we outline power method, and summarize derivations, procedures and advantages. The method to be examined is the power method.

In section 2 of this paper, we have discussed some basic concepts regarding eigenvalues and eigenvectors with example required to understand the concepts that are discussed. In section 3, we have presented power method with example for approximating smallest eigenvalue and its corresponding eigenvector of the real square matrix A. Finally, in section 4, we summarized some concluding remarks that are used in practice.

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For the purposes of this paper, we restrict our attention to real-valued, square matrices with a full set of real eigenvalues.

2. PRELIMINARIES

In this section, we recall some basic concepts which would be used in the sequel.

Definition 2.1: The minor of an element of a determinant of order greater than one is the determinant of next lower order obtained by deleting the row and the column of the given determinant in which the element occurs. The minor of the element a_{ij} in the determinant $|A|$ is denoted by M_{ij}.

Definition 2.2: The cofactor of an element of a determinant of order greater than one is the coefficient of that element in the expansion of the determinant. The cofactor of the element a_{ij} in $|A|$ is denoted by A_{ij}.

The cofactor of an element a_{ij} in $|A|$ can be determined in terms of its minor as $A_{ij} = (-1)^{i+j} M_{ij}$.

Definition 2.3: Corresponding to a square matrix $A = \left[a_{ij}\right]_{n\times n}$, we form a matrix $B = \left[A_{ij}\right]_{n\times n}$, where A_{ij} is the cofactor of a_{ij} in $|A|$. Then B^T (transpose of B) is called the Adjoint Matrix or Adjugate Matrix of the Matrix A which is denoted by $AdjA$.

Definition 2.4: A square matrix A is invertible if and only if A is non-singular.

Definition 2.5: Let $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ be the eigenvalues of an $n\times n$ matrix A. λ_1 is called the dominant eigenvalue of A if $|\lambda_1| > |\lambda_i|, i = 2, \ldots, n$. The eigenvectors corresponding to λ_1 are called dominant eigenvectors of A.

Definition 2.6: Eigenvectors corresponding to distinct eigenvalues are linearly independent.
However, two or more linearly independent eigenvectors may correspond to the same eigenvalue.

Definition 2.7: Let $A^{(1)}, A^{(2)}, A^{(3)}, \ldots\ldots$ be a sequence of matrices in $R^{m\times n}$. We say that the sequence of matrices converges to a matrix $A \in R^{m\times n}$ if the sequence $A_{i,j}^{(k)}$ of real numbers converges to $A_{i,j}$ for every pair $1 \le i \le m, 1 \le j \le n$, as k approaches infinity. That is, a sequence of matrices converges if the sequences given by each entry of the matrix all converge.

2.8. EIGENVALUES AND EIGENVECTORS

Consider the equation $AX = \lambda X$ (1)

Here, A is an $n\times n$ matrix, λ is a scalar and X is a non-zero vector. The solution of (1) requires the solution of λ. The scalar λ (real or complex) is called the eigenvalue or 'latent root' or 'characteristic value' of A. X is called the corresponding eigenvector or 'characteristic vector' of the matrix A. The eigenvalues of a matrix are of great importance in physical problems. They occur in the analysis of stability and in the equations of vibrations in structures or electrical circuits. The stability of an aircraft is determined by the location of the eigenvalues of a certain matrix in the complex plane.

If $A = \left[a_{ij}\right]_{n\times n}$, then (1) can be written as

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13}\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23}\ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2} & a_{n3}\ldots\ldots\ldots a_{nn}-\lambda \end{bmatrix} X = 0 \tag{2}$$

$$\text{Or}\left[A-\lambda I\right]X = 0 \tag{3}$$

This is a set of n linear homogeneous equations. It will have a non-trivial solution if and only if $\left|A-\lambda I\right| = 0;$ that is, if and only if

$$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13}\ldots\ldots\ldots\ldots a_{1n} \\ a_{21} & a_{22}-\lambda & a_{23}\ldots\ldots\ldots\ldots a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1} & a_{n2} & a_{n3}\ldots\ldots\ldots a_{nn}-\lambda \end{vmatrix} = 0 \tag{4}$$

The determinant is a polynomial of degree $\le n$ in λ. The polynomial is called the characteristic polynomial of matrix A. It is usually denoted by $P(\lambda)$. The roots of this polynomial are the eigenvalues (or the latent roots or the characteristic values) of the matrix A. If the values of λ are $\lambda_1, \lambda_2, \ldots\ldots\ldots, \lambda_n$, which may not all be distinct, then the eigenvectors of the matrix A are given by

$$AX_1 = \lambda_1 X_1, AX_2 = \lambda_2 X_2, \ldots\ldots, AX_n = \lambda_n X_n \tag{5}$$

Hence the determination of eigenvalues of a matrix A is nothing but solving an algebraic equation of degree n.

2.9. TWO IMPORTANT PROPERTIES OF EIGENVALUES AND EIGENVECTORS

Property 1: If X is an eigenvector of A corresponding to the eigenvalue λ and A is invertible, then X is an eigenvector of A^{-1} corresponding to its eigenvalue $1/\lambda$.

Property 2: If A is a non-singular matrix, then eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A.

Proof: Let λ be an eigenvalue of A and X be a corresponding eigenvector. Then

$$AX = \lambda X$$

$$\Rightarrow X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$$

$$\Rightarrow \frac{1}{\lambda}X = A^{-1}X \quad (\because A \text{ is non-singular} \Rightarrow \lambda \neq 0)$$

$$\Rightarrow A^{-1}X = \frac{1}{\lambda}X$$

$$\Rightarrow \frac{1}{\lambda} \text{ is an eigenvalue of } A^{-1} \text{ and } X \text{ is a corresponding eigenvector.}$$

Conversely, suppose that k is an eigenvalue of A^{-1}. Since A is non-singular $\Rightarrow A^{-1}$ is non-singular and $(A^{-1})^{-1} = A$, therefore it follows from the first part of this property that $1/k$ is an eigenvalue of A. Thus each eigenvalue of A^{-1} is equal to the reciprocal of some eigenvalue of A. Hence the eigenvalues of A^{-1} are nothing but the reciprocals of the eigenvalues of A.

Example 1: Let us now consider the matrix $A = \begin{bmatrix} -2 & -12 \\ 1 & 5 \end{bmatrix}$ to find the eigenvalues and the corresponding eigenvectors by direct method i.e. by algebraic procedures for verifying the above two properties.

Solution: The characteristic equation is

$$P(\lambda) = \left|A-\lambda I\right| = 0$$

$$\Rightarrow \begin{vmatrix} -2-\lambda & -12 \\ 1 & 5-\lambda \end{vmatrix} = 0$$

Which gives $\lambda_1 = 1$ and $\lambda_2 = 2$.

The corresponding eigenvectors are obtained thus:

(i) For $\lambda_1 = 1$

Let the eigenvector be $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -12 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which gives the equations $-2x_1 - 12x_2 = x_1$ and $x_1 + 5x_2 = x_2$

which gives $x_1 = -4x_2$

Hence the eigenvector for $\lambda_1 = 1$ is $X_1 = [-4x_2, x_2]^T$. Since x_2 is arbitrary, we can take $x_2 = 1$ and hence the eigenvector is $X_1 = [-4,1]^T$.

(ii) For $\lambda_2 = 2$

Let the eigenvector be $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have

$$A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 2\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} -2 & -12 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix}$$

which gives the equations $-2x_1 - 12x_2 = 2x_1$ and $x_1 + 5x_2 = 2x_2$, which gives $x_1 = -3x_2$.

Hence the eigenvector for $\lambda_2 = 2$ is $X_2 = [-3x_2, x_2]^T$. Since x_2 is arbitrary, we can take $x_2 = 1$ and hence the eigenvector is $X_2 = [-3,1]^T$.

Thus, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$ and the corresponding eigenvectors are $X_1 = [-4,1]^T$ and $X_2 = [-3,1]^T$ respectively.

Now, to find eigenvalues and eigenvectors of the matrix A^{-1}, we need to find A^{-1} and for that we proceed as follows:

Clearly, the matrix A is non-singular.

$$AdjA = \begin{bmatrix} 5 & 12 \\ -1 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{AdjA}{|A|} = \frac{1}{2}\begin{bmatrix} 5 & 12 \\ -1 & -2 \end{bmatrix} \qquad (\because |A| = 2)$$

$$= \begin{bmatrix} 5/2 & 6 \\ -1/2 & -1 \end{bmatrix}$$

The characteristic equation of the matrix A^{-1} is

$$P(\lambda) = \left|A^{-1} - \lambda I\right| = 0$$

$$\Rightarrow \begin{vmatrix} \frac{5}{2} - \lambda & 6 \\ -\frac{1}{2} & -1-\lambda \end{vmatrix} = 0$$

which gives $\lambda_1 = 1$ and $\lambda_2 = 1/2$.

The corresponding eigenvectors are obtained thus:
(i) For $\lambda_1 = 1$

Let the eigenvector be $X_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have

$$A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 1\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5/2 & 6 \\ -1/2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

which gives the equations $\frac{5}{2}x_1 + 6x_2 = x_1$ and $\frac{-1}{2}x_1 - x_2 = x_2$, which gives $x_1 = -4x_2$.

Hence the eigenvector for $\lambda_1 = 1$ is $X_1 = \left[-4x_2, x_2\right]^T$. Since x_2 is arbitrary, we can take $x_2 = 1$ and hence the eigenvector is $X_1 = \left[-4,1\right]^T$.

(ii) For $\lambda_2 = 1/2$

Let the eigenvector be $X_2 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Then we have

$$A^{-1}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 5/2 & 6 \\ -1/2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2}x_1 \\ \frac{1}{2}x_2 \end{bmatrix}$$

which gives the equations $\frac{5}{2}x_1 + 6x_2 = \frac{1}{2}x_1$ and $\frac{-1}{2}x_1 - x_2 = \frac{1}{2}x_2$, which gives $x_1 = -3x_2$.

Hence the eigenvector for $\lambda_2 = 1/2$ is $X_2 = \left[-3x_2, x_2\right]^T$. Since x_2 is arbitrary, we can take $x_2 = 1$ and hence the eigenvector is $X_2 = \left[-3,1\right]^T$.

From the above example we have seen that if X is an eigenvector of A corresponding to the eigenvalue λ and A is invertible, then X is an eigenvector of A^{-1} corresponding to its eigenvalue $1/\lambda$. Also, we have seen that the eigenvalues of A^{-1} are the reciprocals of the eigenvalues of A (obviously, the matrix A is non-singular i.e. $|A| \neq 0$

3. THE POWER METHOD (ITERATIVE METHOD)

This method is used for eigenvalue problems where very few roots of the characteristic equation are to be found. Let all the eigenvalues be distinct. An arbitrary vector $Y^{(0)}$ can be expressed as

$$Y^{(0)} = a_1X_1 + a_2X_2 + ...+ a_nX_n$$

$$= \sum_{r=1}^{n} a_r X_r \quad ....(1)$$

To find the numerically largest or dominant eigenvalue and its associate eigenvector, we start with an arbitrary vector $Y^{(0)}$. The vector is multiplied successively by the matrix A. A convenient choice for $Y^{(0)}$ is $\left[1,0\right]^T$ or $\left[1,0,0\right]^T$. It can also be taken as $\left[0,1\right]^T$, $\left[1,1\right]^T$, $\left[1,-1\right]^T$ or any other vector of the correct size. It must be noted that all iterative procedures require an initial estimate of the quantity sought to be taken.
Multiplying the equation (1) by A, we get

$$Y^{(1)} = AY^{(0)} = \sum_{r=1}^{n} a_r AX_r = \sum_{r=1}^{n} a_r \lambda_r X_r$$

Multiplying by A again and letting $Y^{(2)} = AY^{(1)}$, we get

$$Y^{(2)} = A\sum_{r=1}^{n} a_r \lambda_r X_r = \sum_{r=1}^{n} a_r \lambda_r AX_r = \sum_{r=1}^{n} a_r \lambda_r^2 X_r$$

Proceeding like this, we get at the mth iteration

$$Y^{(m)} = \sum_{r=1}^{n} a_r \lambda_r^m X_r$$

$$= a_1\lambda_1^m X_1 + a_2\lambda_2^m X_2 + ...+ a_n\lambda_n^m X_n$$

Suppose λ_1 is the largest eigenvalue. Then,

$$Y^{(m)} = \lambda_1^m[a_1X_1 + a_2(\lambda_2/\lambda_1)^m X_2 + ...+ a_n(\lambda_n/\lambda_1)^m X_n]$$

The values $(\lambda_i/\lambda_1)^m$ $(i \neq 1)$ tend to zero as $m \to \infty$ and hence all the terms become negligible except the first term. Therefore, $Y^{(m)} \to \lambda_1^m a_1 X_1$, a scalar multiple of X_1, as $m \to \infty$. Also, $Y^{(m+1)} \to a_1\lambda_1^{m+1}X_1$ for large m. Therefore, taking the ratio of the magnitudes of $Y^{(m+1)}$ and $Y^{(m)}$, we get $\frac{\left|Y^{(m+1)}\right|}{\left|Y^{(m)}\right|} \to \lambda_1$ for large m, the required largest eigenvalue. It is clear that the rate of convergence depends on the ratio of the moduli of the two largest eigenvalues. When this ratio is nearly unity, the convergence is very poor. To avoid this, the following procedure is adopted:

(i) The arbitrary vector $Y^{(0)}$ is selected such that the largest element of this vector is unity; i.e. the vector $Y^{(0)}$ is put into the normalized form with the largest element unity.
(ii) The normalized vector is multiplied by A.
(iii) The new vector is normalized by dividing each element by the largest element. Let this largest element be l_m.
(iv) The process is repeated until the values of l_m and l_{m+1} differ by some prescribed small value. The value of l_m gives the value of the largest eigenvalue and the vector $Y^{(m)}$ is the eigenvector corresponding to l_m.

3.1. SMALLEST EIGENVALUE AND ITS CORRESPONDING EIGENVECTOR BY POWER METHOD

We have already stated that the eigenvalues of A^{-1}, if A is non-singular, are the reciprocals of the eigenvalues of A. Therefore, the smallest eigenvalue of A is the largest eigenvalue of A^{-1}. Hence we can use the power method to determine the smallest eigenvalue of A by working with A^{-1} instead of A. This procedure is illustrated in example 2.

Example 2: Let us now consider the same matrix of example 1 i.e. $A=\begin{bmatrix}-2 & -12\\ 1 & 5\end{bmatrix}$ to approximate the smallest eigenvalue and its corresponding eigenvector by applying power method to A^{-1} instead of A.

Solution: Here, $A=\begin{bmatrix}-2 & -12\\ 1 & 5\end{bmatrix}$. We know that from example 1, $A^{-1}=\begin{bmatrix}5/2 & 6\\ -1/2 & -1\end{bmatrix}$.

Let us find the largest eigenvalue of A^{-1} by power method. We begin with an initial approximation $\xi_0=[1,-1]^T$.

$$Z_1=A^{-1}\xi_0=\begin{bmatrix}-3.5\\ 0.5\end{bmatrix},\quad \alpha_1=0.5,\quad \xi_1=\begin{bmatrix}-7\\ 1\end{bmatrix}$$

$$Z_2=A^{-1}\xi_1=\begin{bmatrix}-11.5\\ 2.5\end{bmatrix},\quad \alpha_2=2.5,\quad \xi_2=\begin{bmatrix}-4.6\\ 1.0\end{bmatrix}$$

$$Z_3=A^{-1}\xi_2=\begin{bmatrix}-5.5\\ 1.3\end{bmatrix},\quad \alpha_3=1.3,\quad \xi_3=\begin{bmatrix}-4.230\\ 1.000\end{bmatrix}$$

$$Z_4=A^{-1}\xi_3=\begin{bmatrix}-4.575\\ 1.115\end{bmatrix},\quad \alpha_4=1.115,\quad \xi_4=\begin{bmatrix}-4.103\\ 1.000\end{bmatrix}$$

$$Z_5=A^{-1}\xi_4=\begin{bmatrix}-4.257\\ 1.051\end{bmatrix},\quad \alpha_5=1.051,\quad \xi_5=\begin{bmatrix}-4.050\\ 1.000\end{bmatrix}$$

$$Z_6=A^{-1}\xi_5=\begin{bmatrix}-4.125\\ 1.025\end{bmatrix},\quad \alpha_6=1.025,\quad \xi_6=\begin{bmatrix}-4.024\\ 1.000\end{bmatrix}$$

$$Z_7=A^{-1}\xi_6=\begin{bmatrix}-4.060\\ 1.012\end{bmatrix},\quad \alpha_7=1.012,\quad \xi_7=\begin{bmatrix}-4.011\\ 1.000\end{bmatrix}$$

$$Z_8=A^{-1}\xi_7=\begin{bmatrix}-4.027\\ 1.005\end{bmatrix},\quad \alpha_8=1.005,\quad \xi_8=\begin{bmatrix}-4.006\\ 1.000\end{bmatrix}$$

All these computations show that $\alpha_1,\alpha_2,........$ converges to 1, which is the largest eigenvalue of A^{-1} and $\xi_0,\xi_1,\xi_2,..........$ converges to $X=[-4,1]^T$ is the corresponding eigenvector. Since the eigenvalues of A are the reciprocals to those of A^{-1}, the smallest eigenvalue of A is 1. This is the same as the result we obtained earlier (in example 1 by direct method i.e. by algebraic procedures). We have got the corresponding eigenvector also the same as the one obtained earlier (in example 1 by direct method i.e. by algebraic procedures).

4. CONCLUSION

In this paper, we have studied power method to approximate the smallest eigenvalue and its corresponding eigenvector of real-valued square matrices. Here, we used the new initial vector for the power method. Mainly, in this paper we have seen that with examples 1 and 2, if we apply the power method to A^{-1}, we will get the approximate largest eigenvalue of A^{-1} and its corresponding eigenvector and consequently we will get the approximate smallest eigenvalue of A with the same eigenvector as if X is an eigenvector of A corresponding to the eigenvalue λ and A is invertible, then X is an eigenvector of A^{-1} corresponding to its eigenvalue $1/\lambda$. This approximate smallest eigenvalue and its corresponding eigenvector appear to be approaching the exact smallest eigenvalue and its corresponding eigenvector as we have obtained earlier in example 1 by direct method i.e. by algebraic procedures.

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