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# A PENALTY FUNCTION APPROACH FOR SOLVING INEQUALITY CONSTRAINED OPTIMIZATION PROBLEMS

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***ABSTRACT***

*In this paper, a computational approach based on an exterior penalty function MPQI method is given for solving a class of continuous inequality constrained optimization problems. The essential steps of the partial quadratic interpolation technique and its modified are given. The numerical algorithm and the flowchart of the penalty function method combined with the Modified Partial Quadratic Interpolation Technique are given. For illustration, three examples are solved using the proposed method. From the solutions obtained, we observe that the values of their object functions are amongst the smallest when compared with those obtained by other existing methods available in the literature. More importantly, our method finds solution which satisfies the continuous inequality constraints.*

***Keywords:*** *Constrained optimization, Exterior penalty function, modified partial quadratic interpolation technique.*

## 1. INTRODUCTION

In recent years, there has been a resurgence of interest in penalty methods [1-5] because of their ability to handle degenerate problems and inconsistent constraint linearization. Exact penalty methods have been used successfully to solve mathematical programs with complementarity constraints (MPCCs) [6-7], a class of problems that do not satisfy the Mangasarian-Fromovitz constraint qualification at any feasible point. They are also used in nonlinear programming algorithms to ensure the feasibility of subproblems and to improve the robustness of the iteration [8-9].

Penalty methods have undergone three stages of development since their introduction in the 1950s. They were first seen as vehicles for solving constrained optimization problems by means of unconstrained optimization techniques. This approach has not proved to be effective, except for special classes of applications. In the second stage, the penalty problem is replaced by a sequence of linearly constrained subproblems. These formulations, which are related to the sequential quadratic programming approach, are much more effective than the unconstrained approach but they leave open the question of how to choose the penalty parameter. In the most recent stage of development, penalty methods adjust the penalty parameter at every iteration so as to achieve a prescribed level of linear feasibility. The choice of the penalty parameter then ceases to be a heuristic and becomes an integral part of the step computation.

The general form of a minimization problem with inequality and equality constraints is as follows [10]: Find $x \in \Re^n$ such that minimize $f(x)$ subject to :

$$g_i(x) \le 0, \quad i = 1, \cdots, m, \tag{1}$$

and

$$h_j(x) = 0, \quad j = 1, \cdots, p. \tag{2}$$

Penalty function methods are the most popular constraint handling methods among users. Two main branches of penalty methods have been proposed in the literature: Exterior and Interior which is also called the barrier method.

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The widely used form of $\Psi(x, r_k)$ in the exterior penalty method [11] is:

$$\Psi = \Psi(x, r_k) = f(x) + r_k \left\{ \sum \langle g_i(x) \rangle^2 + \sum \left( h_j(x) \right)^2 \right\},\ r_k \to \infty \tag{3}$$

where

$$\langle g_i(x) \rangle = \max\{0\ ,\ g_i(x)\}, \quad i = 1, \cdots, m, \tag{4}$$

and $r_k$ is a parameter, which is modified at the beginning of each round of optimization. Each optimization round is defined here as a complete optimization of $\Psi(x, r_k)$ for a fixed value of $r_k$ until the convergence is achieved. The optimum point $x^*$, at the end of each round serves as the starting point, $x_1$ of the next round of optimization with a larger $r_k$.

The original form of the interior penalty function $\Psi_{in}(x, r_k)$ is as follows [12-13]:

$$\Psi_{in}(x, r_k) = f(x) + \left\{ \sum \frac{1}{g_i(x)} + \sum \frac{1}{h_j(x)} \right\},\ r_k \to 0, \tag{5}$$

where $r_k$ reduces from a high value to zero gradually. Rao [6, 7] has proposed the following function for selecting $r_k$ at the start of the optimization procedure:

$$r_k = (0.1 \sim 1) \times \frac{f(x_1)}{-\sum \frac{1}{g(x_1)}}, \tag{6}$$

where $x_1$ is the initial point in the feasible region. The optimization procedure is similar to the exterior penalty function method except that $r_k$ reduces to 0 gradually. Here, the reduction follows [8]

$$r_{k+1} = \lambda \times r_k \tag{7}$$

where $\lambda$ is a coefficient less than 1.

In this paper, a computational approach based on an exterior penalty function MPQI method is given for solving a class of continuous inequality constrained optimization problems. The essential steps of the partial quadratic interpolation technique and its modified are given. The numerical algorithm and the flowchart of the penalty function method combined with the Modified Partial Quadratic Interpolation Technique are given. For illustration, three examples are solved using the proposed method. From the solutions obtained, we observe that the values of their object functions are amongst the smallest when compared with those obtained by other existing methods available in the literature. More importantly, our method finds solution which satisfies the continuous inequality constraints.

## 2. PARTIAL QUADRATIC INTERPOLATION TECHNIQUE (PQI)

The essential steps to apply the partial quadratic interpolation technique [14] can be summarized as follows:

1) Choose some starting point $\xi^0 \in \Re_m$ and $\nu = 1$.

2) Approximate the function $F(x)$ about $\xi^r$ in the quadratic form

$$F(x) = a + [B_m(x^r)]^{Tr}[x - x^r] + \frac{1}{2}[x - x^r]^{Tr}[A_m(x^r)][x - x^r] \tag{8}$$

where $[A_m]$ and $[A_m]$ represent the gradient vector and the Hessian matrix of the function $F(x)$ respectively. To compute particular values for $a,\ A_m, B_m$ we choose a set of interpolation points as follows:

i) $m$ points $[x_{i+}^r], i = 1(1)m$

$$[x_{i+}^r] = (x_1^r,\ x_2^r, \cdots,\ x_{i-1}^r, x_i^r + l_i, x_{i+1}^r, \cdots, x_m^r) \tag{9}$$

ii) $m$ points $[x_{i-1}^r], i = 1(1)m$

$$[x_{i+}^r] = (x_1^r,\ x_2^r, \cdots,\ x_{i-1}^r, x_i^r - l_i, x_{i+1}^r, \cdots, x_m^r) \tag{10}$$

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iii) $\frac{m(m-1)}{2}$ points $[x_{ij}^r], i=1(1)m-1, \; j=i+1(1)m$ where

$$[x_{ij}^r]=(x_1^r,\, x_2^r,\cdots,\, x_{i-1}^r, x_i^r+l_i, x_{i+1}^r,\cdots, x_j^r+l_j,\cdots, x_m^r) \tag{11}$$

Using these interpolation points it can be shown that

$$a= F(x^r) \tag{12}$$

and the elements $b_i$ , $a_{ij}$ ,of $B_m$ , $A_m$ respectively are given by

$$b_i=\frac{F(x_{i+}^r)-F(x_{i-}^r)}{2\, l_i},\;\; a_{ii}=\frac{F(x_{i+}^r)-2F(x^r)+F(x_{i-}^r)}{l_i^2} \tag{13}$$

$$a_{ij}=\frac{F(x_{ij}^r)-F(x_{i+}^r)-F(x_{j+}^r)+F(x^r)}{l_i\; l_j} \tag{14}$$

$$a_{ij}=\frac{F(x_{ij}^r)-F(x_{i+}^r)-F(x_{j+}^r)+F(x^r)}{l_i\; l_j} \tag{15}$$

The $l_i$ are a set of constants which determine the accuracy of the interpolation.

3) Extract the symmetric positive definite matrix $[A_q]$ from the symmetric matrix $[A_m]$ using Choliski's method, $q \le m$, by cancelling certain rows and columns. Essentially we write $A_m=[S_m][S_m]^{Tr}$. From this we have

$$s_{11}^2 = a_{11}$$

If $a_{11} \le 0$ then we eliminate the first row and column in each of $[A_m],[S_m]$ *and* $[S_m]^{Tr}$ and perform the calculation on the $[A_{m-1}],[S_{m-1}]$ *and* $[S_{m-1}]^{Tr}$. If $a_{11} \ge 0$ then we have

$$s_{11}^2 = \sqrt{a_{11}},\;\; s_{1j} = \frac{a_{1j}}{s_{11}},\quad j=2(1)m. \tag{16}$$

Let us now suppose that we have operated on the first $j-1$ columns of $[S]$, i.e. We have either calculated the elements or eliminated them. The operation on the $j^{th}$ column gives

$$s_{jj}^2=a_{jj}-\sum_{i\in k1,\, i\le j} s_{ij}^2. \tag{17}$$

where $k1$ is the set of indices of rows and columns not eliminated.

If $s_{jj} \le 0$ we eliminate the $j^{th}$ columns and rows $[A_q]$ and $[S_q]$ where $[A_q]$ and $[S_q]$ are the current reduced matrices derived to date from $[A_m]$ and $[S_m]$, $q \le m$.

Otherwise we take

$$s_{jj}=\sqrt{a_{jj}-\sum_{i\in k1,\, i\le j} s_{ij}^2}\;,\;\; s_{ij}=\frac{[a_{ij}-\sum_{i\in k1,\, i\le j} s_{ij}^2]}{s_{jj}} \tag{18}$$

This process is repeated for each column until we finally obtain the reduced matrix $[A_q]$ given by

$$[A_q]=[S_q][S_q]^{Tr} \tag{19}$$

4) Solve the system of the linear equations

$$[A_q][\Delta x_i]=[B_q] \tag{20}$$

where $B_q$ is the reduced form of gradient vector corresponding to $A_q$.

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5) Compute a new point $x^{v+1}$ from

$$x_i^{v+1} = \begin{cases} x_i^{v+1} + \beta \, \Delta x_i & \text{for} \quad x_i \in \Re_q \\ x_i^v & \text{for} \quad x_i \in \Re_q \end{cases}$$

where $\beta$ is a parameter which takes values $1, \frac{1}{2}, \frac{1}{4}, \cdots$ and we use the first value of $\beta$ which satisfies $F(x^{v+1}) < F(x^v)$. If $\beta$ becomes too small without satisfying this condition, the calculation can be restarted with a finer approximation of the matrices $[A_m]$ and $[B_m]$, i.e. smaller values $l_i$.

**3. MODIFIED PQI TECHNIQUE (MPQI) [15]**

In PQI technique we set $t = 1, \frac{1}{2}, \frac{1}{4}, \cdots$ and we take the first value of $t$ which satisfies the condition

$f(x^{r+1}) < f(x^r)$, $x^r \in \Re_n$.

However, the value of $t$ taken by this way may not be the optimal value of $t$. Since there is a great possibility that the optimal value $t^*$ lies between these values, i.e. between $t = 1, \frac{1}{2}, \frac{1}{4}, \cdots$ to get the optimal step size $t^*$ we suggest the following modification:

Let us approximate $f(t) = f(x^r + t\delta x^r)$ by a polynomial of second degree $p_2(t)$ over the interval [0,1] as following:

$$f(t) = p_2(t) = \begin{bmatrix} 1 & \frac{t}{h} & \left(\frac{t}{h}\right)^2 \end{bmatrix} \begin{bmatrix} L_2 \end{bmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} \tag{21}$$

where $h$ is the interval of the interpolation, $f_{2i} = f(u^r + i\, \delta u^r)$, $i = 0, \frac{1}{2}, 1$ and $L_2$ is the Lagrange matrix where

$$L_2 = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 \\ -3 & 4 & -1 \\ 1 & -2 & 1 \end{pmatrix} \tag{22}$$

from (21) and (22) we have

$$p(t) = \begin{bmatrix} 1 & \frac{t}{h} & \left(\frac{t}{h}\right)^2 \end{bmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -\frac{2}{3} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} f_0 \\ f_1 \\ f_2 \end{pmatrix} \tag{23}$$

then, $p'(t) = \frac{1}{h}(-\frac{2}{3}\ f_0 + 2\ f_1 - \frac{1}{2}\ f_2) + \frac{2t^*}{h^2}(\frac{1}{2}\ f_0 - 4\ f_1 + f_2) = 0$ and taking $h = \frac{1}{2}$ we obtain

$$t^* = \frac{3\ f_0 - 4\ f_1 + f_2}{4\ (f_0 - 2\ f_1 + f_2)} \tag{24}$$

**4. PENALTY MODIFIED PQI TECHNIQUE (PMPQI)**

The following numerical algorithm and the flowchart of the penalty function method combined with the Modified Partial Quadratic Interpolation Technique (Figure 1) are as follows:

1. Given $g_i(x) \le 0,\ i = 1, \cdots, m,$ and $h_j(x) = 0,\ \ j = 1, \cdots, p$, $It = 0,\ \varepsilon' > 0,\ r_{It} > 0,\ \varepsilon > 0.$
2. Initialize $x^{(It)} \in \Re_n$ arbitrarily.

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3. If the optimality conditions are satisfied at $x^{(It)}$ , then stop.
4. Compute $\Psi(x^{(It)}, r_{It}) = \min_{x\geq 0} \Psi(x, r_{It})$ and minimize $x^{(It)}$ using MPQI technique and

   $r_{It+1} = 10 \times r_{It}$ .
5. If $\left\| x^{(It)} - x^{(It-1)} \right\| < \varepsilon$ or $\left|\Psi(x^{(It)}, r_{It}) - \Psi(x^{It-1}, r_{It-1})\right| < \varepsilon$ ; then stop.

   Else $It = It + 1$ and go to step 3.

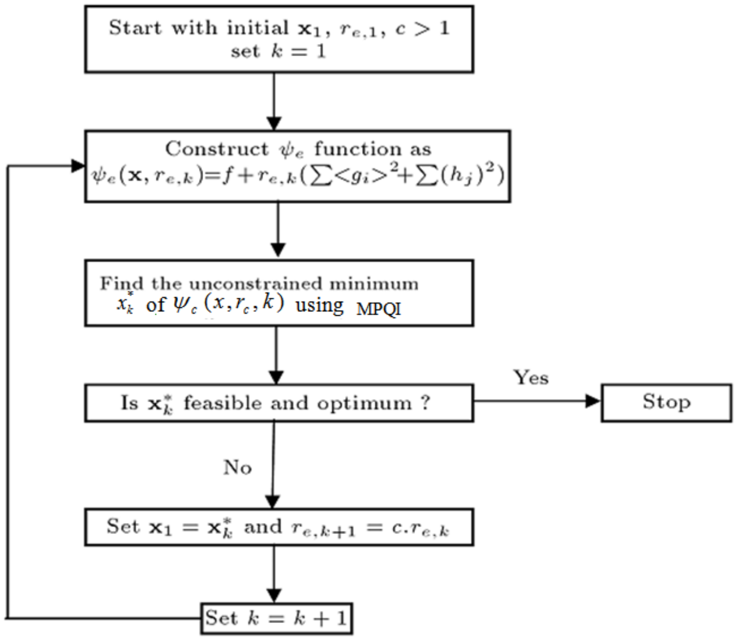


**Figure - 1:** Penalty Modified Partial Quadratic Interpolation Method

## 5. NUMERICAL EXAMPLES

We resent here some numerical examples to minimize the unconstrained and constrained optimization problems which use the modified partial quadratic interpolation technique (MPQI) and combined exterior penalty function-modified partial quadratic interpolation technique (PMPQI) and compare with other methods.

### A) Unconstrained Examples

**P1) Brown and Dennis function [16]:** This example gives the local minimum for the function

Minimize $f(X) = \sum_{k=1}^{20} \left\{ (x_1 + t_k\ x_2 - e^{t_k}) + (x_3 + x_4 \sin(t_k) - \cos(t_k))^2 \right\}^2$

where $t_k = 0.2k$ with $f(X^*) = 85822.2$

**P2) Extended Powell singular function [16]:** In this case we obtain the local minimum for the function :

Minimize $f(X) = (x_1 + 10\ x_2)^2 + 5(x_3 - x_4)^2 + (x_2 - 2\ x_3)^4 + 10\ (x_1 - x_4)^4$

where $f(X^*) = 0.65 \times 10^{-13}$ at $X^* = (0.18 \times 10^{-5},\ 0.61 \times 10^{-5}, 0.60 \times 10^{-5})$

**P3) Rosenbrock function [16]:** This example gives the local minimum for the function

Minimize $f(X) = 100\ (x_2 - x_1^2)^2 + (1.0 - x_1)^2$

with $f(X^*) = 0.0$ at $X^* = (1,1)$ .

In Table 1 we compare the numerical results obtained, for various starting points, by applying other methods (Armijo's quadratic method) [16], ARMBIS [17], Fletcher-Reeves [12]]) and the modified partial quadratic interpolation technique (MPPQI). The first column in Table 1 contains the problem number and the next two columns of each method contain the total iterations (IT) and the total number of function and gradient evaluations (NFG) of each method.

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Table 1
Numerical results for Armijo, ARMBIS , Fletcher R. and MPQI

| P. NO. | Initial values | Armijo | | ARMBIS | | Fletcher R. | | MPQI | |
|---|---|---|---|---|---|---|---|---|---|
| | | IT | NFG | IT | NFG | IT | NFG | IT | NFG |
| P1 | (0,0,0,0) | 54 | 511 | 10 | 98 | 20 | 363 | 7 | 182 |
| | (1,2,4,8) | 72 | 709 | 15 | 146 | 30 | 523 | 8 | 364 |
| | (-5,-1,1,5) | 72 | 708 | 13 | 126 | 33 | 565 | 9 | 312 |
| P2 | (3,-1,0,1) | 1197 | 16515 | 710 | 6799 | 8660 | 122575 | 17 | 417 |
| | (1,-1,1,-1) | 1184 | 16316 | 598 | 5712 | 610 | 86593 | 18 | 443 |
| | (0,1,2,3) | 1262 | 17505 | 635 | 6194 | 9943 | 140655 | 18 | 443 |
| P3 | (-1.2,1) | 794 | 8901 | 957 | 5303 | 193 | 3211 | 32 | 482 |
| | (-1.2,-1) | 745 | 8283 | 485 | 2800 | 39 | 615 | 30 | 333 |
| | (0,-1.2) | 778 | 8697 | 484 | 2807 | 31 | 469 | 30 | 397 |

**B) Constrained Examples [3]**

**P4) Minimize** $f(X) = 0.4\,x_1 + 0.5x_2$

**Subject to** $0.3\,x_1 + 0.1x_2 \geq 2.7,\ 0.5\,x_1 + 0.5x_2 = 6.0,\ \ x_i \geq 0,\ i = \overline{1,2}$

**P5) Minimize** $f(X) = 4.0\,x_1 + 3.0x_2$

**Subject to** $2.0\,x_1 + 3.0x_2 \geq 6.0, 4.0\,x_1 + x_2 \geq 4.0,\ \ x_i \geq 0,\ i = \overline{1,2}$

**P6) Minimize** $f(X) = 3.0\,x_1 + 8.0\,x_2$

**Subject to** $3.0\,x_1 + 4.0x_2 \leq 20.0,\ \ x_1 + 3.0x_2 \geq 12.0,\ \ x_i \geq 0,\ i = \overline{1,2}$

Table 2
Numerical results for polynomial penalty method (Algorithm 1), Karmarkar's Algorithm and PMPQI

| P. NO. | Algorithm 1 ($\rho = 2$) | | Karmarkar's Algorithm | | PMPQI | |
|---|---|---|---|---|---|---|
| | Total It | Time (Secs.) | Total It | Time (Secs.) | Total It | Time (Secs.) |
| P4 | 9 | 3.9 | 19 | 3.7 | 4 | 1.4634 |
| P5 | 9 | 5.8 | 19 | 3.7 | 4 | 1.5622 |
| P6 | 10 | 8.7 | 18 | 3.8 | 6 | 1.9125 |

Table 2 reports the results computational for Polynomial penalty method (Algorithm 1) (PPMA1), Karmarkar's Algorithm [13] and PMPQI technique. The first column in Table 1 contains the problem number and the next two columns of each method contain the total iterations and the times (in seconds) of each method.

## 6. CONCLUSIONS

In this paper, a computational approach based on an exterior penalty function MPQI method is given for solving a class of continuous inequality constrained optimization problems. The essential steps of the partial quadratic interpolation technique and its modified are given. The numerical algorithm and the flowchart of the penalty function method combined with the Modified Partial Quadratic Interpolation Technique are given. For illustration, three examples are solved using the proposed method. From the solutions obtained, we observe that the values of their object functions are amongst the smallest when compared with those obtained by other existing methods available in the literature. More importantly, our method finds solution which satisfies the continuous inequality constraints.

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