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ANALYTICAL EXPRESSION OF DENSITY OF A POPULATION IN COMMUNITY ECOLOGY FROM LOTKA-VOLTERRA MODEL: HOMOTOPY PERTURBATION APPROACH

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ABSTRACT

A mathematical model of Lotka-Volterra equations, in population ecology is analysed. This model explain the population density relate to interspecific competitions and carrying capacity. It is devoted to the description of prey–predator or host–parasitoid system dynamics. This paper presents an approximate analytical solution of the general Lotka-Volterra equations. Analytical expressions for density of a species, product concentration and corresponding population density have been derived for all values of parameters using Homotopy perturbation method. The analytical results are compared with our numerical results and are found to be in good agreement.

Keywords: *Nonlinear differential equations; Lotka-Volterra equation; Community ecology; Homotopy perturbation method; Numerical simulation.*

1. INTRODUCTION

Population models are often used to guide conservation management decisions. Sensitivity analysis of such models can be useful in setting research priorities, by highlighting those parameters that have the most influence on population growth rate. Understanding the processes which drive population growth and dynamics are crucial for conservation managers to make sound ecological decisions for species conservation (Sinclair and Krebs, 2002; Bowden et al., 2003). Many populations may grow to a maximum density which is set by the interplay of resource availability and per capita resource requirements. Resource availability is determined in part by the kind of interactions occurring. All populations are limited in some way. They may be limited intrinsically, by competition for resources within the population, or extrinsically, by a competitor, a predator, a disease, or an abiotic disturbance (Pearl, 1927; Turchin, 1999). Populations of many kinds of organisms, including algae, bacteria, insects, plants, and humans, occasionally escape their limitations and grow to a larger size, sometimes at a fast rate (Delong and Hanson, 2011). Relaxation of both intrinsic and extrinsic regulating factors may stimulate such increases. The nature of interactions among populations (e.g. host plants and herbivores, mutualists, hosts and parasites/parasitoids, predators and prey, competitors) is variable in many ways (for example, Polis, 1984; Bronstein, 1994). Thus, two populations that might normally be mutualists may become a host-parasite system in some circum-stances (Bronstein, 1994).

The Interaction strength–the dynamic consequences of interactions between species–can only be determined by conducting appropriate perturbation experiments, in which certain interactions within communities are 'isolated' for further study (Connell, 1983; Mac Nally, 1983; Schoener, 1983). It is crucial to realize that such experiments are the only ways in which to estimate quantities needed to characterize the intensity and nature of the dynamics of interactions (Paine, 1992; Schoener, 1993). Mac Nally (2000) has obtained the mass balance equation in community ecology. In population science research, the Lotka–Volterra model (LVM) is considered a classical dynamic model (Lotka, 1925; Volterra, 1926). The Lotka–Volterra model is one of many continuous (Gertsev and Gertseva, 2004) differential mathematical models devoted to the description of prey–predator or host–parasitoid system dynamics. Lotka-Volterra equations have played a significant role in the development of theoretical ecology, and although there have been many heated debates about whether they are realistic or appropriate for many problems (Goel *et al*., 1971; Gopalsamy, 1992; Kuang, 1993; Olek, 1994; Ahmad and Lazer, 1995; Zeeman, 1995; Takeuchi, 1996; Ahmad, 1999; Saito, 2001; Zu and Takeuchi, 2012, Capone *et al*., 2013). It has been broadly used to explain dynamic phenomena in population ecology and other life science fields (Pielou, 1969; Krivan, 1997; Redheffer, 2001; Tonnang *et al*., 2009). The Lotka-Volterra

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equations (Pielou, 1969) which describe the population dynamics of prey-predator species have been the subject of several recent papers (Vayenas and Pavlou 2001; Aziz *et al*., 2012; Shao, 2012; Zu and Takeuchi, 2012; Capone *et al.*, 2013; Hou, 2012; Hou *et al*., 2013; Huang *et al*., 2013). Understanding limiting factors affecting population growth for imperilled species is crucial for conservation and management (Rooney *et al.*, 2004).

A fundamental question in ecology is what determines the density of a population. To our knowledge no rigorous analytical solutions for the density population have been reported. The purpose of this paper is to derive the analytical expressions of Lotka-Volterra model using Homotopy perturbation method for all values of parameters.

2. MATHEMATICAL FORMULATION OF THE PROBLEM

The most commonly used mathematical model for interspecific competition is the classical Lotka–Volterra dynamic equations (Mac Nally, 2000).

$$\frac{dx}{dt} = \frac{r_1 x}{k_1}\left(k_1 - x - \alpha_{12} y\right) \tag{1}$$

$$\frac{dy}{dt} = \frac{r_2 y}{k_2}\left(k_2 - y - \alpha_{21} x\right) \tag{2}$$

where *x*, *y* denote the density of population, r_1, r_2 represents the growth rate of population in the absence of the competitor, k_i is the carrying capacity of the population and α is the competition coefficients, which relate the relative competitiveness of a heterospecific competitor to that of a conspecific competitor. The initial conditions are

$$\text{At } t = 0,\ x = l \tag{3}$$

$$\text{At } t = 0, y = m \tag{4}$$

3. ANALYTICAL SOLUTION OF EQUATION USING HOMOTOPY PERTURBATION METHOD

The Homotopy perturbation method (HPM) was proposed by Ji-Huan He in 1999 (He, 1999, 2000, 2003, 2004a, 2004b, 2005a, 2005b, 2006). In this method, the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution. Using the homotopy technique from topology, a homotopy is constructed with an embedding parameter $p \in [0,1]$ which is considered as a "small parameter". The approximations obtained by the homotopy perturbation method are uniformly valid not only for small parameters, but also for very large parameters. Considerable research has been recently conducted in applying this method to a wide class of linear and nonlinear equations and also to nonlinear oscillator problems, a comparison of the homotopy perturbation method (HPM) and homotopy analysis method (HAM) was made, revealing that the former is more powerful than the latter. Application of the homotopy perturbation method to various integral equations has become a hot topic (Ghorbani and Saberi-Nadjafi, 2008) and references there in).

Recently, many authors have applied the HPM to various problems and demonstrated the efficiency of the HPM for handling non-linear structures and solving various physics and engineering problems. This method is a combination of homotopy in topology and classic perturbation techniques. Ji Huan He used the HPM to solve the Lighthill equation, the Duffing equation and the Blasius equation .The idea has been used to solve non-linear boundary values problems, integral equations and many other problems. The HPM is unique in its applicability, accuracy and efficiently. The HPM uses the imbedding parameter r as a small parameter and only a few iterations are needed to search for an asymptotic solution. Using the HPM (see Appendix A), we can obtain the following solution to the eqns. (1)-(4) are as follows:

$$x(t) = le^{r_1 t} + e^{r_1 t}\left[\frac{l^2}{k_1}\left(1 - e^{r_1 t}\right) + \frac{r_1 \alpha_{12} ml}{k_1 r_2}\left(1 - e^{r_2 t}\right)\right] \tag{5}$$

$$y(t) = me^{r_2 t} + e^{r_2 t}\left[\frac{m^2}{k_2}\left(1 - e^{r_2 t}\right) + \frac{r_2 \alpha_{21} ml}{k_2 r_1}\left(1 - e^{r_1 t}\right)\right] \tag{6}$$

when t is small, the above equations becomes

$$x(t) = l + lr_1 t\left[1 - \frac{l}{k_1} - \frac{\alpha_{12} m}{k_1}\right] \tag{7}$$

$$y(t) = m + mr_2 t\left[1 - \frac{m}{k_2} - \frac{\alpha_{21} l}{k_2}\right] \tag{8}$$

4. RESULTS AND DISCUSSION

The eqns. (5) and (6) represent the most general new approximate analytical expressions for the population x and y, for the possible values of parameters r_i, k_i, α_{ij}. In Fig. 1 analytical expressions of the density of population x versus time (t) is plotted for the various values of growth rate of population $(r_2 = 0.8)$, competition coefficients (0.01) at the fixed carrying capacity of population ($k_1 = 10$ and $k_2 = 5$). From this it is inferred that the rate of density of population x increases from the initial value of density $(x(0) = 10)$. The analytical expressions of the density of population y versus time (t) is plotted in Fig.2 for the various values of growth rate of population $(r_1 = 0.1)$, competition coefficients $(\alpha_{12} = 0.1, \alpha_{21} = 1)$ at the fixed carrying capacity of population ($k_1 = 100$ and $k_2 = 50$). From this it is inferred that the rate of density of population y increases from the initial value of density $(y(0) = 5)$. From these figures it is inferred that the value of the concentration of population x and y increases from the initial value of density $x(0) = l$ and $y(0) = m$ respectively when the value of parameters r_1 and r_2 increases. The populations of x and y are interdependent and it is one of the many ways of species interaction i.e., Consumer-resource interactions, interactions in which individuals of one species consumes individuals of another species. Examples of consumer-resource interactions include predator-prey interactions and herbivore-plant interactions. These consumer-resource interactions affect the species involved in different ways, the resource species is negatively impacted while the consumer species is positively impacted.

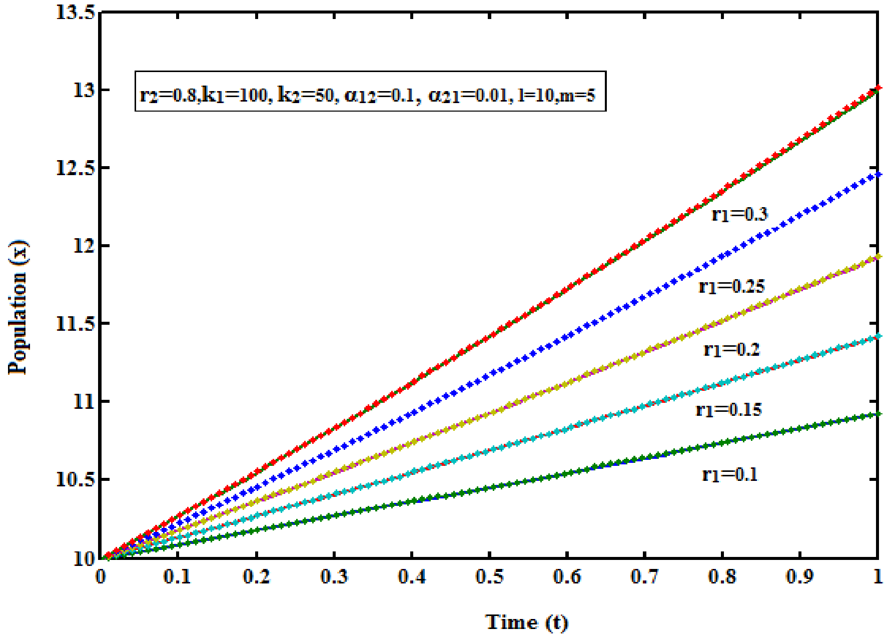

Fig. - 1: Density of population (x) versus time (t) for various values of r_1 and some fixed values of other parameters (r_2 = 0.8, k_1=100, k_2 = 50, α_{12}=0.1, α_{21}=0.01, l= 10, m=5). The key to the graph: stacked line represents eqn. (5) and dotted line represents the numerical simulation.

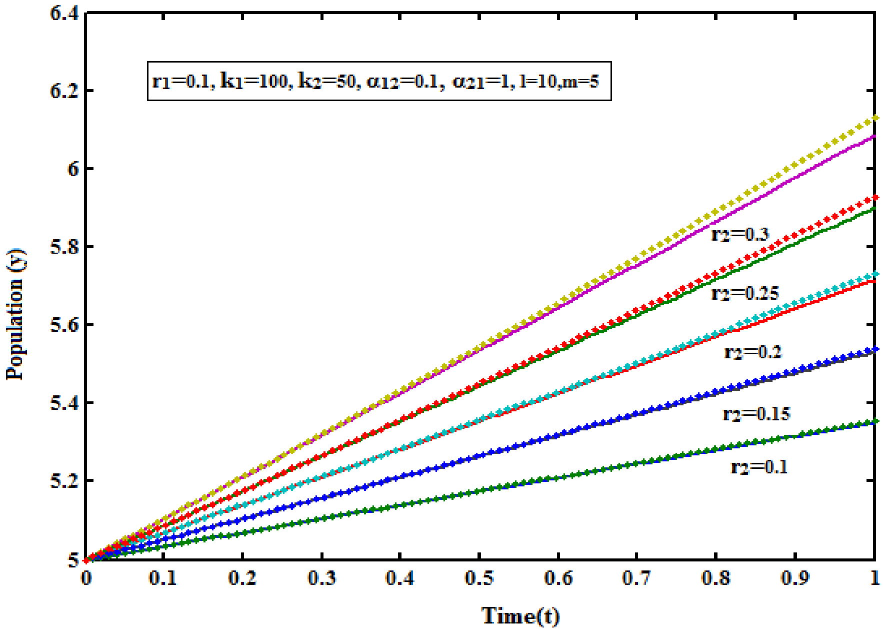

Fig. - 2: Density of population (*y*) versus time (t) for various values of r_2 and some fixed values of other parameters (r_1 = 0.1, k_1=100, k_2 = 50, α_{12}=0.1, α_{21}=1, l= 10, m=5). The key to the graph: tacked line represents eqn. (6) and dotted line represents the numerical simulation.

5. CONCLUSION

The Lotka-Volterra equations (Pielou, 1969) which describe the population dynamics of prey-predator species are discussed. The approximate analytical solutions to the non-linear reaction equations are derived by using the Homotopy perturbation method. A simple, straight forward and a new method of estimating the density of populations is presented. This solution procedure can be easily extended to all kinds of system of coupled non-linear equation with various complex boundary conditions in population density in community ecology.

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APPENDIX-A

Basic concept of the He's Homotopy perturbation method (HPM)

To explain this method, let us consider the following function:

$$D_o(u) - f(r) = 0, \ \mathrm{r} \in \Omega \tag{A.1}$$

with the boundary conditions of

$$B_o(u, \ \frac{\partial u}{\partial n}) = 0, \ \mathrm{r} \in \Gamma \tag{A.2}$$

where D_o is a general differential operator, B_o is a boundary operator, $f(r)$ is a known analytical function and Γ is the boundary of the domain Ω. In general, the operator D_o can be divided into a linear part L and a non-linear part N. Eq. (A1) can therefore be written as

$$L(u) + N(u) - f(r) = 0 \tag{A.3}$$

By the Homotopy technique, we construct a Homotopy $v(r, p) : \Omega \times [0,1] \to \Re$ that satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[D_o(v) - f(r)] = 0. \tag{A.4}$$

$$H(v, p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0. \tag{A.5}$$

where $p \in [0, \ 1]$ is an embedding parameter, and u_0 is an initial approximation of the eqn.(B1) that satisfies the boundary conditions. From the eqns. (A.4) and (A.5) we have

$$H(v,0) = L(v) - L(u_0) = 0 \tag{A.6}$$

$$H(v,1) = D_o(v) - f(r) = 0 \tag{A.7}$$

When p=0, the eqns. (A.4) and (A.5) become linear equations. When p =1, they become non-linear equations. The process of changing p from zero to unity is that of $L(v) - L(u_0) = 0$ to $D_o(v) - f(r) = 0$. We first use the embedding parameter p as a "small parameter" and assume that the solutions of the eqns.(A.4) and (A.5) can be written as a power series in p :

$$v = v_0 + pv_1 + p^2 v_2 + ... \tag{A.8}$$

Setting $p = 1$ results in the approximate solution of the eqna. (A.1):

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + ... \tag{A.9}$$

This is the basic idea of the HPM.

APPENDIX-B

Solution of the nonlinear the differential eqns. (1)-(4) using the Homotopy perturbation method

In this appendix, we indicate how the eqns. (5)-(8) are derived in this paper. We construct the Homotopy are as follows:

$$(1 - p)\left[\frac{dx}{dt} - r_1 x\right] + p\left[\frac{dx}{dt} - r_1 x + \frac{r_1 x^2}{k_1} - \frac{r_1 \alpha_{12} xy}{k_1}\right] = 0 \tag{B.1}$$

$$(1 - p)\left[\frac{dy}{dt} - r_2 y\right] + p\left[\frac{dy}{dt} - r_2 y + \frac{r_2 y^2}{k_2} - \frac{r_2 \alpha_{21} xy}{k_2}\right] = 0 \tag{B.2}$$

The initial approximations are as follows:

$$x_0(0) = l,\ y_0(0) = m \tag{B.3}$$

$$x_i(0) = 0, y_i(0) = 0,\ \text{for } i = 1,2,3..... \tag{B.4}$$

The approximate solution of the eqns. (B.1) and (B.2) is

$$x = x_0 + px_1 + p^2 x_2 + ... \tag{B.5}$$

$$y = y_0 + py_1 + p^2 y_2 + ... \tag{B.6}$$

Substituting the eqns. (B.5) and (B.6) into the eqns. (B.1) and (B.2) and comparing the coefficients of like powers of *p*. we obtain the following differential equation

$$p^0 : \frac{dx_0}{dt} - r_1 x_0 = 0 \tag{B.7}$$

$$p^1 : \frac{dx_1}{dt} - r_1 x_1 + \frac{r_1 x_0^{\ 2}}{k_1} + \frac{r_1 \alpha_{12} x_0 y_0}{k_1} = 0 \tag{B.8}$$

and

$$p^0 : \frac{dy_0}{dt} - r_2 y_0 = 0 \tag{B.9}$$

$$p^1 : \frac{dy_1}{dt} - r_2 y_1 + \frac{r_2 y_0^{\ 2}}{k_2} + \frac{r_2 \alpha_{21} x_0 y_0}{k_2} = 0 \tag{B.10}$$

Solving the eqns. (B.7) - (B.10) and using the boundary conditions (B.3) and (B.4) we can obtain the following results:

$$x_0 = le^{r_1 t} \tag{B.11}$$

$$x_1 = e^{r_1 t}\left[\frac{l^2}{k_1}\left(1 - e^{r_1 t}\right) + \frac{r_1 \alpha_{12} ml}{k_1 r_2}\left(1 - e^{r_2 t}\right)\right] \tag{B.12}$$

$$y_0 = me^{r_2 t} \tag{B.13}$$

$$y_1 = e^{r_2 t}\left[\frac{m^2}{k_2}\left(1 - e^{r_2 t}\right) + \frac{r_2 \alpha_{21} ml}{k_2 r_1}\left(1 - e^{r_1 t}\right)\right] \tag{B.14}$$

According to HPM we conclude that

$$x(t) = \lim_{p \to 0} x(t) = x_0 + x_1 \tag{B.15}$$

$$y(t) = \lim_{p \to 0} y(t) = y_0 + y_1 \tag{B.16}$$

Substituting the eqns. (B.11)-(B.12) into the eqn. (B.15) and (B.13)-(B.14) into the eqn. (B.16) we can obtain the solution in the text.

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