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A NOTE ON ANALYTIC FUNCTIONS WITH VARYING ARGUMENTS

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ABSTRACT

In the present paper, we define the subclasses $V(A,B,a,c)$ and $K(A,B,a,c)$ of analytic functions by using $L(a,c)$. For functions belonging to these classes, we obtain co-efficient estimates, distortion bounds and many more properties.

Keywords and phares: Univalent functions, Komato operator, Carlson – Shaffer linear operator.

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INTRODUCTION:

Let A denote the class of all analytic functions of the form

$$f(z) = z + \sum_{m=2}^{\infty} a_m z^m$$

defined in the unit disc $U = \{z : |z| < 1\}$. Let N denote the subclass of A consisting of functions normalized by $f(0) = 0$ *and* $f'(0) = 1$ which are univalent in U.

Silverman [5] defined the class $V(\theta_m)$ as the class of all functions in N such that $\arg a_m = \theta_m$ for all m. If further there exists a real number β such that $\theta_m + (m-1)\beta = \pi (\mathrm{mod}\ 2\pi)$, then f is said to be in the class $V(\theta_m, \beta)$. The union of $V(\theta_m, \beta)$ taken over all possible sequences $\{\theta_m\}$ and all possible real numbers β is denoted by V.

The class A is closed under convolution or Hadamard product

$$(f * g)(z) = z + \sum_{m=2}^{\infty} a_m b_m z^m, \qquad z \in U,$$

where f is given by (1.1) and $g(z) = z + \sum_{m=2}^{\infty} b_m z^m$.

Let $\Phi(a,c;z) = z + \sum_{m=2}^{\infty} \frac{(a)_{m-1}}{(c)_{m-1}} z^m, \quad c \neq 0, -1, -2, \cdots,$

Where $(a)_m$ is the Pochhammer symbol defined interms of Gamma functions by,

$$(a)_m = \frac{\Gamma(a+m)}{\Gamma(a)} = \begin{cases} 1, & for\ m = 0 \\ a(a+1)(a+2)\cdots(a+m-1), & for\ m \in \mathrm{N} \end{cases}$$

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Further, fo $f \in A,$ a linear operator on A called Carlson – Shaffer operator [2] defined by

$$L(a,c)f(z) = \Phi(a,c;z) * f(z) = z + \sum_{m=2}^{\infty} \frac{(a)_{m-1}}{(c)_{m-1}} a_m z^m$$

Here $*$ stands for the hadamard product of two power series as given by (1.2).

If $a = 0,-1,-2,\cdots$, then $L(a,c)f$ is a polynomial. If $a \neq 0,-1,-2,\cdots$, application of the root test shows that the infinite series for $L(a,c)f$ has the same radius of convergence as that for f. Also, $L(a,c)f$ has a continuous inverse $L(a,c)f$ and is a one to one mapping on A onto itself. This convolution operator provides a convenient representation of differentiation.

$L(1,1)f = f(z)$, $L(2,1)f = zf'$. In fact, the Ruscheweyh derivatives of f are $L(n+1,1)f$, $n = 0,1,2,\cdots$. Now we define the class $V(A,B,a,c)$ consisting of functions $f \in V$, such that

$$\frac{L(a+1,c)f(z)}{L(a,c)f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \quad -1 \le A < B \le 1, \quad a,c \neq 0,-1,-2,\cdots \textit{ and } z \in U.$$

Here $\omega(z)$ is analytic,

$\omega(0) = 0$ *and* $|\omega(z)| < 1$, $z \in U$.

Let $K(A,B,a,c)$ denote the class of functions $f \in V$ such that $zf' \in V(A,B,a,c)$.

MAIN RESULTS:

THEOREM 2.1: Let function $f \in V$ is in $V(A,B,a,c)$ if and only if

$$\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} D_m |a_m| \le (B-A),$$

where

$D_m = [(B+1)(a+m-1)+(A+1)a]$, $-1 \le A < B \le 1$, $a,c \in R \backslash Z_0^-$.

Proof: Since $f \in V(A,B,a,c)$. Then

$$\frac{L(a+1,c)f(z)}{L(a,c)f(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \quad -1 \le A < B \le 1, \quad a,c \neq 0,-1,-2,\cdots \textit{ and } z \in U.$$

From this we get,

$$\omega(z) = \frac{L(a,c)f(z) - L(a\_1,c)f(z)}{BL(a+1,c)f(z) - AL(a,c)f(z)} \quad \textit{and} \quad |\omega(z)| < 1$$

Implies

$$|\omega(z)| = \left| \frac{\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [a-(a+m-1)] a_m z^{m-1}}{(B-A) + \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [B(a+m-1) - Aa] a_m z^{m-1}} \right| < 1.$$

Since $f \in V$, f lies in $V(\theta_m, \beta)$ for sequence $\{\theta_m\}$ and there exists real a number β, such that $\theta_m + (m-1)\beta = \pi (\bmod 2\pi)$.

Setting $z = re^{i\beta}$, we get

$$R \left| \frac{\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [a-(a+m-1)] a_m r^{m-1} e^{i(\theta_m + \overline{(m-1)\beta})}}{(B-A) + \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [B(a+m-1) - Aa] a_m r^{m-1} e^{i(\theta_m + \overline{(m-1)\beta})}} \right| < 1.$$

$$\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [a-(a+m-1)] |a_m| r^{m-1}$$

$$< (B-A) + \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [B(a+m-1) - Aa] |a_m| r^{m-1}$$

$$\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [(B+1)(a+m-1) - (A+1)a] |a_m| r^{m-1} < (B-A).$$

Hence $$\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} D_m |a_m| r^{m-1} \le (B-A).$$

Letting $r \to 1$, we get (2.1)

Conversely, suppose $f \in V$ and satisfies (2.1). In view of (2.4), which is implied by (2.1), since $r^{m-1} < 1$, we have

$$\left| \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [a-(a+m-1)] |a_m| z^{m-1} \right|$$

$$\le \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [a-(a+m-1)] |a_m| r^{m-1}$$

$$< (B-A) - \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [B(a+m-1) - Aa] |a_m| r^{m-1}$$

$$\leq (B-A) - \sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} [Aa - B(a+m-1)] |a_m| z^{m-1}$$

Which gives (2.2) and hence it follows that $f \in V(A,B,a,c)$.

Corollary 2.2: If $f \in V$ is in $V(A,B,a,c)$, then

$$|a_m| \leq \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)} \frac{(B-A)}{D_m}, \quad for\ m \geq 2, -1 \leq A < B \leq 1, a,c \in R \setminus Z_0^-.$$

The equality holds for the function f given by

$$f(z) = z + \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)} \frac{(B-A)}{D_m} e^{i\theta_m} z^m, \quad z \in U.$$

For parametric values $a = n+1, \quad c = 1,$ we get the following result proved by Padmanabhan and Jayamala [3] as corollaries to the above Theorem.

Corollary 2.3: Let $f \in V$. Then $f \in V_n(A,B)$ if and only if

$$\sum_{m=2}^{\infty} \frac{(n+m-1)!}{(n+1)!(m-1)!} C_m |a_m| \leq (B-A),$$

where $C_m = (B+1)(n+m) - (A+1)(n+1)$.

The equality holds for the functions f is given by,

$$f(z) = z + \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)} \frac{(B-A)}{C_m} e^{i\theta_m} z^m, z \in U.$$

THEOREM 2.4: Let function $f \in V$ is in $K(A,B,a,c)$ if and only if

$$\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} m D_m |a_m| \leq (B-A),$$

where

$$D_m = [(B+1)(a+m-1) + (A+1)a], \quad -1 \leq A < B \leq 1, a,c \in R \setminus Z_0^-.$$

Now we examine the Extreme points of the class $V(A,B,a,c)$.

THEOREM 2.5: Let $f \in V(A,B,a,c)$ with $\arg a_m = \theta_m$ where $[\theta_m + (m-1)\beta] \equiv \pi (\text{mod}\, 2\pi)$. Define $f_1(z) = z$ and

$$f(z) = z + \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)} \frac{(B-A)}{D_m} e^{i\theta_m} z^m, \quad m = 2,3,\cdots, -1 \leq A < B \leq 1, a,c \in R \setminus Z_0^-, z \in U.$$

$f \in V(A,B,a,c)$ if and only if f can be expressed as

$$f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z) \text{ where } \mu_m \geq 0 \text{ and } \sum_{m=1}^{\infty} \mu_m = 1.$$

Proof: If $f(z) = \sum_{m=1}^{\infty} \mu_m f_m(z)$ with $\sum_{m=1}^{\infty} \mu_m = 1$, $\mu_m \geq 0$, then

$$\sum_{m=2}^{\infty} \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} D_m \mu_m \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)} \frac{(B-A)}{D_m}$$

$$= \sum_{m=2}^{\infty} \mu_m (B-A) = (1-\mu_1)(B-A) \leq (B-A).$$

Hence $f \in V(A,B,a,c)$.

Conversely, let $f(z) = z + \sum_{m=2}^{\infty} a_m z^m \in V(A,B,a,c)$, define

$$\mu_m = \frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)} \frac{|a_m| D_m}{(B-A)}, \quad m = 2,3,\cdots$$

and define $\mu_1 = 1 - \sum_{m=2}^{\infty} \mu_m$. From Theorem2.1, $\sum_{m=2}^{\infty} \mu_m \leq 1$ and so $\mu_1 \geq 0$. Since

$$\mu_m f_m(z) = \mu_m f + a_m z^m,$$

$$\sum_{m=1}^{\infty} \mu_m f_m(z) = z + \sum_{m=2}^{\infty} a_m z^m = f(z).$$

THEOREM 2.6: Define $f_1(z) = z$ and

$$f_m(z) = z + \frac{\Gamma(c+m-1)\Gamma(a+1)}{\Gamma(a+m-1)\Gamma(c)} \frac{(B-A)}{D_m} z^m, \qquad m = 2,3,\cdots,$$

$$-1 \leq A < B \leq 1, a,c \in R \setminus Z_0^-, z \in U.$$

Then $f \in K(A,B,a,c)$ if and only if f can be expressed as $f(z)=\sum_{m=1}^{\infty}\mu_m f_m(z)$ where $\mu_m \geq 0$ and $\sum_{m=1}^{\infty}\mu_m = 1$.

THEOREM 2.7: The class $V(A,B,a,c)$ is closed under convex linear combination.

Proof: Let $f, g \in V(A,B,a,c)$ and let

$$f(z)=z+\sum_{m=2}^{\infty}a_m z^m, \qquad g(z)=z+\sum_{m=2}^{\infty}b_m z^m.$$

For η such that $0 \leq \eta < 1$, it is suffices to show that the function defined by $h(z)=(1-\eta)f(z)+\eta g(z)$, $z \in U$ belongs to $V(A,B,a,c)$. Now $h(z)=z+\sum_{m=2}^{\infty}\left[(1-\eta)a_m+\eta b_m\right]z^m$. Applying Theorem 2.1, to $f, g \in V(A,B,a,c)$, we have

$$\sum_{m=2}^{\infty}\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}D_m\left[(1-\eta)a_m-\eta b_m\right]$$

$$=\sum_{m=2}^{\infty}\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}D_m a_m+\eta\sum_{m=2}^{\infty}\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}D_m b_m$$

$$\leq (1-\eta)(B-A)+\eta(B-A) = (B-A).$$

This implies that $h \in V(A,B,a,c)$.

Corollary 2.8: If $f_1(z), f_2(z)$ are in $V(A,B,a,c)$ then the function defined by

$g(z)=\frac{1}{2}\left[f_1(z)+f_2(z)\right]$ is also in $V(A,B,a,c)$.

THEOREM 2.9: The class $K(A,B,a,c)$ is closed under convex linear combination.

THEOREM 2.10: Let for $j=1,2,\cdots,m,\ f_j(z)=z+\sum_{m=2}^{\infty}a_{m,j}z^m \in V(A,B,a,c)$ and $0<\lambda_j<1$ such that $\sum_{j=1}^{m}\lambda_j=1$, then the function $F(z)$ defined by $F(z)=\sum_{j=1}^{m}\lambda_j f_f(z)$ is also in $V(A,B,a,c)$.

Proof: For each $j \in \{1,2,3,\cdots,m\}$ we obtain

$$\sum_{m=2}^{\infty}\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}D_m|a_m| < B-A.$$

Since $F(z)=\sum_{j=1}^{m}\lambda_j(z-\sum_{m=2}^{\infty}a_{m,j}z^m)$

$$= z-\sum_{m=2}^{\infty}\left(\sum_{j=1}^{m}\lambda_j a_{m,j}\right)z^m.$$

$$\sum_{m=2}^{\infty}\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}D_m\left[\sum_{j=1}^{m}\lambda_j a_{m,j}\right]$$

$$=\sum_{j=1}^{m}\lambda_j\left[\sum_{m=2}^{\infty}\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}D_m\right]$$

$$<\sum_{j=1}^{m}\lambda_j\,(B-A) < (B-A).$$

Therefore $F(z) \in V(A,B,a,c)$.

THEOREM 2.11: Let $f(z) \in V(A,B,a,c)$. Komato operator of f is defined by

$$k(z)=\int_0^1\frac{(c+1)^{\gamma}}{\Gamma(\gamma)}t^c\left(\log\frac{1}{t}\right)^{\gamma-1}\frac{f(tz)}{t}dt,\ c>-1,\ \gamma\geq 0$$ then $k(z) \in V(A,B,a,c)$.

Proof: We have $\int_0^1 t^c\left(\log\frac{1}{t}\right)^{\gamma-1}dt=\frac{\Gamma(\gamma)}{(c+1)^{\gamma}}$

$$\int_0^1 t^{m+c-1}\left(\log\frac{1}{t}\right)^{\gamma-1}dt=\frac{\Gamma(\gamma)}{(c+1)^{\gamma}},\quad m=2,3,\cdots,.$$

$$k(z)=\frac{(c+1)^{\gamma}}{\Gamma(\gamma)}\left[\int_0^1 t^c\left(\log\frac{1}{t}\right)^{\gamma-1}z\,dt+\sum_{m=2}^{\infty}z^m\int_0^1 a_m t^{m+c-1}\left(\log\frac{1}{t}\right)^{\gamma-1}dt\right]$$

$$= z+\sum_{m=2}^{\infty}\left(\frac{c+1}{c+m}\right)^{\gamma}a_m z^m.$$

Since $f \in V(A,B,a,c)$ and since $\left(\frac{c+1}{c+m}\right)^{\gamma}<1$, we have

$$\sum_{m=2}^{\infty}\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}\left[(1+A)-m(1+B)\right]\left(\frac{c+1}{c+m}\right)^{\gamma}a_m<B-A.$$

In the next theorem we will find the distortion bound for $L(a,c)f(z)$.

THEOREM 2.12: If $f\in V(A,B,a,c)$, then

$$|z|-\frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2\le|L(a,c)f(z)|\le|z|+\frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2.$$

Proof: Let $f(z)\in V(A,B,a,c)$. Using Theorem 2.1, z

$$\sum_{m=2}^{\infty}a_m\le\frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}.$$

Therefore

$$|L(a,c)f(z)|\le|z|+|z|^2\sum_{m=2}^{\infty}\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}a_m<|z|+\frac{(B-A)\Gamma(c+1)}{D_2\Gamma(2)}|z|^2$$

and

$$|L(a,c)f(z)|\ge|z|-|z|^2\sum_{m=2}^{\infty}\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}a_m>|z|-\frac{(B-A)\Gamma(c+1)}{D_2\Gamma(2)}|z|^2.$$

REMARK 2.13: For parametric values of $a=1,\ c=1$ *and* $a=2,\ c=1$ we get the upper and lower bounds for $|f(z)|$ *and* $|f'(z)|$ respectively.

$$|z|-\frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2\le|f(z)|\le|z|+\frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|^2 \text{ and}$$

$$|z|-\frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|\le|f'(z)|\le|z|+\frac{(B-A)\Gamma(c+1)}{D_2\Gamma(c)}|z|.$$

THEOREM 2.14: Let $f\in V(A,B,a,c)$. Then for every $0\le\delta<1$ the function

$$H_\delta(=(1-\delta)f(z)+\delta\int_0^z\frac{f(t)}{t}dt\in V(A,B,a,c).$$

Proof: We have $H_\delta(z)=z+\sum_{m=2}^{\infty}\left(1+\frac{\delta}{m}-\delta\right)a_mz^m.$

Since $\left(1+\frac{\delta}{m}-\delta\right)<1,\ m\ge2,$ so by Theorem 2.1,

$$\sum_{m=2}^{\infty}\left(1+\frac{\delta}{m}-\delta\right)D_ma_m\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}$$

$$<\sum_{m=2}^{\infty}D_ma_m\frac{\Gamma(a+m-1)\Gamma(c)}{\Gamma(c+m-1)\Gamma(a+1)}<B-A.$$

Therefore $H_\delta\in V(A,B,a,c)$.

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