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A NEW REFINEMENT OF GENERALIZED GAUSS-SEIDEL METHOD FOR SOLVING SYSTEM OF LINEAR EQUATIONS

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ABSTRACT

In this paper refinement of generalized Gauss-Seidel(RGGS)method for solving system of linear equations is presented and its convergence is discussed. Few numerical examples are considered to show the efficiency of the refinement of generalized Gauss-Seidel method over generalized Gauss-Seidel method.

Keywords: *Generalized Gauss-Seidel method (GGS), Banded matrix, Row strictly diagonally dominant matrix, convergence.*

INTRODUCTION

The linear system problem is

$$Ax = b \tag{1}$$

Where A is an n×n nonsingular matrix, b is an n-vector and x is an n-vector to be found out.

As it is discussed by Ibrahim B. Kalambi [2] and Davod K. Salkuyeh [3] that Gauss-Seidel method is easier method to use for determination of the n-dimensional solution vector x of (1) but little slow to converge. Davod K. Salkuyeh [3] introduced generalized Gauss-Seidel method which is more efficient than conventional Gauss-Seidel method. Again V.B. Kumar Vatti and Genanew Gonfa[4] developed the method called the refinement of Generalized Jacobi method and mentioned that this refinement method is faster than Generalized Jacobi method
.

Preliminary notes: Consider the linear system of equations (1) and splitting made by Davod K. Salkuyeh [3] as

$$A=T_m + E_m + F_m \tag{2}$$

Where $T_m = (a_{ij})$ be a banded matrix with band length 2m+1 is defined as,

$$t_{ij} = \begin{Bmatrix} a_{ij}, |j-i| \le m \\ 0, otherwise. \end{Bmatrix}$$

Where E_m and F_m are strictly lower and strictly upper triangular parts of A-T_m respectively and they are defined as follows:

$$T_m = \begin{pmatrix} a_{1,1\ldots\ldots} & a_{1,m+1} & \cdots\cdots\cdots \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ a_{m+1,1} & \vdots & a_{n-m,n} \\ \vdots & \vdots & \vdots \\ \vdots & a_{n,n-m} & a_{n,n} \end{pmatrix}, \quad E_m = \begin{pmatrix} a_{m+2,1} & \cdots & \cdots\ldots \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n-m-1} \end{pmatrix}$$

$$F_m = \begin{pmatrix} a_{1,m+2} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ \cdots\cdots. & \cdots & a_{n-m-1,n} \end{pmatrix}$$

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Then the generalized Gauss-Seidel method for solving equation (1) is defined as,

$x^{(k+1)}=-(T_m + E_m)^{-1}F_m\, x^{(k)} + (T_m + E_m)^{-1}b$ …………………; k = 0,1,2,… (3)

Where $B_{GGS}^{m} = -(T_m + E_m)^{-1}F_m$ is called generalized Gauss-Seidel iteration matrix and $b_{GGS}^{m} = (T_m + E_m)^{-1}b$ is called the generalized Gauss-Seidel iteration vector.

REFINEMENT OF GENERALIZED GAUSS-SEIDEL METHOD (RGGS)

Let $x^{(1)}$ be an initial approximation for solution of the linear system (1) and $b_i^{(1)} = \sum_{j=1}^{n} a_{ij}\, x_j^{(1)}$; i=1,2,3,………,n.

After solution of k-steps of (3) ,we have $x^{(k+1)}=(x_1^{(k+1)}, x_2^{(k+2)}, \dots\dots\dots, x_n^{(k+1)})$. So, we again have
$b_i^{(k+1)} = \sum_{j=1}^{n} a_{ij}\, x_j^{(k+1)}; i = 1,2, \dots\dots\dots, n.$

Now, we refine this obtained solution as Here we again assume $\bar{x}^{(k+1)} = (\bar{x}_1^{(k+1)}, \bar{x}_2^{(k+1)}, \dots\dots\dots\dots, \bar{x}_n^{(k+1)})$, be a good approximation for solution of the linear system (1) i.e. $\bar{x}^{(k+1)} \to x$, where x is the exact solution for (1) and $b_i = \sum_{j=1}^{n} a_{ij}\, \bar{x}_j^{(k+1)}$; i=1,2, ……….,n. Since all $\bar{x}_j^{(k+1)}$ are unknown, so we solve this problem by using $x_j^{(k+1)}$; $j \neq i$.

Since $b_i^{(k+1)} \to b_{i.}$, so $b_i = b_i^{(k+1)} - a_{ii}x_i^{(k+1)} + a_{ii}\bar{x}_i^{(k+1)}$

$$or, \bar{x}_i^{(k+1)} = x_i^{(k+1)} + \frac{1}{a_{ii}}\left(b_i - b_i^{(k+1)}\right).$$

REFINEMENT OF GENERALIZED GAUSS-SEIDEL METHOD IN MATRIX FORM

Ax = b

or,$(T_m + E_m + F_m)x = b$

or, $(T_m + E_m)x = b - F_m x$

or, $(T_m + E_m)x = b + (T_m + E_m - A)x$ $[since\ A = T_m + E_m + F_m]$

or, $(T_m + E_m)x = (b - Ax) + (T_m + E_m)x$

or, $x=x+(T_m + E_m)^{-1}(b - Ax)$

So, the refinement formula in matrix form as,

$\bar{x}^{(k+1)} = x^{(k+1)} + (T_m + E_m)^{-1}(b - Ax^{(k+1)})$ (4)

Where $x^{(k+1)}$ in the R.H.S. is given in (3).

Now, (4) takes the form,
$\bar{x}^{(k+1)} = -(T_m + E_m)^{-1}F_m x^{(k)} + (T_m + E_m)^{-1}b + (T_m + E_m)^{-1}[b - A\{-(T_m + E_m)^{-1}F_m x^{(k)} + (T_m + E_m)^{-1}b\}]$

$=\{(T_m + E_m)^{-1}F_m\}^2 x^{(k)} + \{I - (T_m + E_m)^{-1}F_m\}(T_m + E_m)^{-1}b.$

Where $\bar{B}_{GGS}^{m} = \{(T_m + E_m)^{-1}F_m\}^2$ is called the refinement of Gauss-Seidel iteration matrix and $\bar{b}_{GGS}^{m} = \{I - (T_m + E_m)^{-1}F_m\}(T_m + E_m)^{-1}b$ is called the refinement of generalized Gauss -Seidel vector.

CONDITION ON THE CONVERGENCE OF RGGS METHOD

Definition: An n×n matrix $A=(a_{ij})$ is row strictly diagonally dominant (SDD) if $|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|$; i=1,2,……….,n.

Definition: $A\in R^{m\times n}$ has lower bandwidth p if a_{ij}=0 for $i > j + p$ and upper bandwidth q if a_{ij}=0 for $j > i + q.$ *For* e.g. $A=\begin{pmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 4 & 1 \end{pmatrix}$ has lower bandwidth 1 and upper bandwidth 1 and (1+1+1=3) is called the bandwidth of the matrix A. Also a band matrix is a sparse matrix whose nonzero entries are confined to a diagonal band, comprising the main diagonal and zero or more diagonals on either side.

Theorem: Let A be SDD matrix, then the GGS method converges for any arbitrary choice of the initial approximation $x^{(0)}$.

Proof: See Davod K. Salkuyeh [3].

Theorem: If A is SDD matrix then the RGGS method converges for any arbitrary choice of the initial approximation $x^{(0)}$.

Proof: Let x be the real solution of (1). Since A is SDD, so the GGS method is convergent and so let $x^{(k+1)} \to x(exact\ solution)$.

Then, $\bar{x}^{(k+1)} = x^{(k+1)} + (T_m + E_m)^{-1}\left(b - Ax^{(k+1)}\right)$

Or, $\bar{x}^{(k+1)} - x = (x^{(k+1)} - x) + (T_m + E_m)^{-1}\left(b - Ax^{(k+1)}\right)$

Or, $\left\|\bar{x}^{(k+1)} - x\right\| \le \left\|(x^{(k+1)} - x)\right\| + \|(T_m + E_m)^{-1}\|\left(b - Ax^{(k+1)}\right)$

$\to \|x - x\| + \|(T_m + E_m)^{-1}\|\|b - Ax\|$

$= 0 + \|(T_m + E_m)^{-1}\|\|b - b\|$

$= 0+0$

$= 0$

Or, $\bar{x}^{(k+1)} \to x$ and hence the RGGS method is convergent.

Theorem: If A is SDD matrix, then $\|B_{GGS}^{m}\|_{\propto} < 1$.

Proof:- $\|B_{GGS}^{m}\|_{\propto} = \|-(T_m + E_m)^{-1}F_m\|_{\propto}$

$\le \|-(T_m + E_m)^{-1}\|_{\propto}\|F_m\|_{\propto}$

$= \|(T_m + E_m)^{-1}\|_{\propto}\|F_m\|_{\propto}$

$= \frac{\|F_m\|_{\propto}}{\|T_m + E_m\|_{\propto}} < 1$

or, $\|B_{GGS}^{m}\|_{\propto} < 1$.

Theorem: If A is SDD matrix, then $\|\bar{B}_{GGS}^{m}\|_{\propto} \le \|B_{GGS}^{m}\|_{\propto} < 1$.

Proof:- $\|\bar{B}_{GGS}^{m}\|_{\propto} = \|\{(T_m + E_m)^{-1}F_m\}^2\|_{\propto}$

$= \|(T_m + E_m)^{-1}F_m\|_{\propto}^2 = \|-(T_m + E_m)^{-1}F_m\|_{\propto}^2$

$= \|B_{GGS}^{m}\|_{\propto}^2$

$\le \|B_{GGS}^{m}\|_{\propto}$. $[\,since\ \|B_{GGS}^{m}\|_{\propto} < 1\,]$

i.e. $\|\bar{B}_{GGS}^{m}\|_{\propto} \le \|B_{GGS}^{m}\|_{\propto} < 1$.

Theorem: When generalized Gauss-Seidel and refinement of generalized Gauss-Seidel method converge, then refinement of generalized Gauss-Seidel method converges faster than generalized Gauss-Seidel method.

Proof:- We have , the iterative matrix of refinement of generalized Gauss-Seidel is square of the generalized Gauss-Seidel iterative matrix i.e.

$$\bar{B}_{GGS}^{m} = (B_{GGS}^{m})^2 .$$

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It can be easily realize that,

$$\rho(\bar{B}_{GGS}^{m}) = [\rho(B_{GGS}^{m})]^2; \rho \text{ denotes the spectral radius.}$$

Since, generalized Gauss-Seidel method converges i.e.

$$\rho(B_{GGS}^{m}) < 1 \text{ and so } \rho(\bar{B}_{GGS}^{m}) < \rho(B_{GGS}^{m}).$$

Hence, when GGS and RGGS method converge, then refinement of GGS method converges faster than the GGS method.

COMPARISON OF NUMERICAL RESULTS

Consider the following system of equations considered by F. Naeimi Dafchahi [1] as,

$4x_1-x_2-x_4=1$

$-x_1+4x_2-x_3-x_5=0$

$-x_2+4x_3-x_6=0$

$-x_1+4x_4-x_5=0$

$-x_2-x_4+4x_5-x_6=0$

$-x_3-x_5+4x_6=0$

The solution of the above system is obtained and tabulated below by using the GGS and refinement of GGS method taking initial approximation for x's as all zeroes.

Table-1: Generalized Gauss-Seidel method when m= 1

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	$x_5^{(k)}$	$x_6^{(k)}$
0	0	0	0	0	0	0
1	0.267857	0.071429	0.017857	0.077168	0.040816	0.014668
2	0.281250	0.082031	0.023438	0.078125	0.042969	0.016602
………..	…………..	……………	……………	…………..	……………	……………
8	0.294823	0.093166	0.028157	0.086127	0.049689	0.019461
9	0.294823	0.093167	0.028157	0.086128	0.049689	0.019462

Table-2: Refinement of generalized Gauss-Seidel method when m=1

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$	$x_5^{(k)}$	$x_6^{(k)}$
0	0	0	0	0	0	0
1	0.281250	0.082031	0.023438	0.078125	0.042969	0.016602
2	0.294594	0.092972	0.028074	0.085990	0.049571	0.019411
3	0.294794	0.093142	0.028146	0.086110	0.049674	0.019455
4	0.294823	0.093167	0.028157	0.086128	0.049689	0.019462
5	0.294824	0.093168	0.028157	0.086128	0.049689	0.019462

CONCLUSIONS

In this paper, we developed a refinement for generalized Gauss-Seidel method for the solution of system of linear equations .This method in compare with generalized Gauss-Seidel method is much faster as shown above and its error in any level is less than the generalized Gauss-Seidel method.

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