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ON PRIME IDEAL CHARACTERIZATION OF QUASI-COMPLEMENTED ALMOST DISTRIBUTIVE LATTICES

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ABSTRACT

Some necessary and sufficient conditions for an Almost Distributive Lattice (ADL) to become a quasi-complemented ADL in topological and algebraic terms are proved. Characterization of quasi-complemented Almost Distributive Lattice in terms of prime ideals and minimal prime ideals are established.

Key words: *Almost Distributive Lattice, ∗-ADL, prime ideal, minimal prime ideal, quasi-complemented ADLs.*

AMS Subject classification (2000): *06D99, 06D15.*

1. INTRODUCTION

The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy U M and Rao G C, as a common abstraction of existing lattice theoretic and ring theoretic generalizations of Boolean algebra. The class of ADLs with pseudo-complementation was introduced in [4], and it was observed that an ADL can have more than one pseudo-complementation unlike in the case distributive lattice. The concept of quasi-complemented Almost Distributive Lattices was introduced in [3] and proved that a uniquely quasi-complemented ADL is a pseudo-complemented ADL. Also, it was proved that an ADL L is quasi-complemented ADL if and only if every prime ideal in L is maximal.

In this paper, we characterize the quasi-complemented ADLs are both algebraically and topologically in terms of their prime ideals and minimal prime ideals with hull-kernel topology and dual hull-kernel topology.

2. PRELIMINARIES

In this section, we recall the definitions and certain properties Almost Distributive Lattice are taken from [4].

Definition: 2.1 An algebra of type $(L,\vee,\wedge,0$ of type (2, 2, 0) is called an Almost Distributive Lattice (ADL), if it satisfies the following axioms:

(1) $(a\vee b)\wedge c=(a\wedge b)\vee(b\wedge c)$
(2) $a\wedge(b\vee c)=(a\wedge b)\vee(a\wedge c)$
(3) $(a\vee b)\wedge b=b$
(4) $(a\vee b)\wedge a=a$
(5) $a\vee(a\wedge b)=a$
(6) $0\wedge a=0$ $for\ all\ a,b,c\in L$

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A non-empty subset I of an ADL L is called an ideal (filter) of L if $a \vee b \in I$ $(a \wedge b \in I)$ and, $a \wedge x \in I$ $(x \wedge a \in I)$ for any $a, b \in I$ and $x \in L$. If I is an ideal of L and $a, b \in L$, then $a \wedge b \in I \Leftrightarrow b \wedge a \in I$. The set I(L) of all ideals of L is a complete distributive lattice under set inclusion with the least element (0] and the greatest element L in which, for any I, J, I $\cap J$ is the infimum of I, J and the supremum is given by $I \vee J = \{i \vee j / i \in I, j \in J\}$. Let S be a non-empty subset of an ADL L. Then the set (S] = $\left\{ (\bigvee_{i=1}^{n}) \wedge x / x \in L, s_i \in S \text{ and } n \in Z^+ \right\}$ is the smallest ideal of L containing S.

For any $a \in L, (a] = \{a \wedge x / x \in L\}$ is the principal ideal generated by a. Similarly, $a \in L, (a] = \{a \wedge x / x \in L\}$ is the principal filter generated by a. The set PI (L) of all principal ideals of L is a sub lattice of I (L). A proper ideal P of L is said to be a prime if for any $x, y \in L$, $x \wedge y \in P$ implies either $x \in P$ or $y \in P$. A proper ideal M of L is said to be a maximal ideal if it is not contained in any proper ideal of L. For any subset S of an ADL L, we define $S^* = \{x \in L / x \wedge a = 0$, for all $a \in I\}$. Then S^* is an ideal and is called the annihilator ideal of S. For $x \in L$, $(x]^*$ is called an annulet of L.

An element $a \in L$ is called a dense if $(a]^* = (0]$ and set of all dense elements in L is denoted by D. Then D is a filter, whenever D is non-empty. An ADL L with 0 is called a $*$-ADL if to each $x \in L$, there exists $y \in L$ such that $[x]^{**} = [y]^*$. An ADL L with 0 is a $*$-ADL if and only if to each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y \in D$. Every $*$-ADL possesses a dense element. For any $x, y \in L$, define $x \leq y$ if and only if x = $x \wedge y$ or equivalently, $x \vee y$ = y, then $\leq$ is a partial ordering on L, in which 0 is the least element. An element m is maximal in $(L, \leq)$ if and only if m $\wedge$ x = x, for all $x \in L$. We don't know, so far, whether $\vee$ is associative in an ADL or not. In this paper L denotes ADL with maximal element in which $\vee$ is associative.

Lemma: 2.2 The set I(L) of all ideals of L is complete pseudo-complemented distributive lattice with least element {0}, and greatest element L in which for any I, J $\in$I(L) where $I \overline{\wedge} J = I \cap J$ is infimum of I and J and the supemum is given by $I \underline{\vee} J = \{i \wedge j / i \in I \text{ and } j \in J\}$.

Lemma: 2.3 Let L be an ADL, Define a relation θ = $\left\{(x, y) \in L \times L / [x]^* = [y]^*\right\}$. Then θ is a congruence relation on L.

If L is a bounded distributive lattice and X is the set of all prime ideals of L, then the hull-kernel topology τ_h^X on X is a topology on X for which $\{X_a / a \in L\}$ is a basis, where for any $a \in L, X_a = \{p \in X / a \notin P\}$. In this topology, for any subset F of X the closure $\overline{F} = \left\{ Q \in X / \bigcap_{p \in F} P \subseteq Q \right\}$. The dual hull-kernel topology τ_d^X on X is the topology for which $\{h_X(a) / a \in L\}$ is a base where $h_X(A) = \{p \in X / A \subseteq P\}$ For any $A \subseteq L$, write $h_X(A) = \{p \in X / a \subseteq P\}$ and it can be observe that $h_X(A) = \bigcap_{a \in A} h_X(a)$.

All these concepts can be analogously defined in case of ADLs also. Let L be an ADL and S (M) denote the set of all prime ideals (minimal prime ideals) of L and also m $(I(L))$ denote the space of minimal prime ideals in I(L). The hull-kernel topology on S is denoted by τ_h^s. In this topology for any $a \in L$, the corresponding basic open sets is denoted by S_a. We write τ_h^M for topology on M induced by τ_h^s. In this topology, the basic open sets are $\{M_a/a \in L\}$ where $M_a = M \cap S_a$. The dual hull-kernel topology on S (M) is denoted by $\tau_d^s \left(\tau_d^M\right)$. In this topology, for any $a \in L$, the basic open sets are denoted by $h_s(a)(h_M(a))$. For a prime ideal P of I(L), C(P)= $\cup\{J \in I(L)/J \in P\}$, Q is a prime ideal in L, $\tau(Q) = \{J \in I(L)/J \subseteq Q\}$. It is easy to see that C(P), $\tau(Q)$ are prime ideals in L and I(L) respectively.

Theorem: 2.4 *Let P be a prime ideal of an ADL L. Then P is a minimal prime ideal if and only if for each $x \in P$, there exists $y \notin P$ such that $x \wedge y = 0$*

Lemma: 2.5 *A prime ideal P of an ADL L is minimal if and only if for each $x \in P$ implies $[x]^* \not\subset P$.*

Lemma: 2.6 *For any x, y, z ∈L, we have the following:*
[1] $M_x = h_M([x]^*)$
[2] $h_M(x) = h_M([x]^{**})$
[3] $[x]^* \subseteq [y]^* \Leftrightarrow h_M(x) \subseteq h_M(y)$
[4] $[x]^* \subseteq [y]^* \Leftrightarrow M_y \subseteq M_x$
[5] $[z]^* = [x]^* \cap [y]^* \Leftrightarrow h_M(z) = h_M(x) \cap h_M(y)$
[6] $[x]^{**} = [y]^* \Leftrightarrow h_M(x) = h_M(y)$

Lemma: 2.7[2] Let L be an ADL such that C(P)∈M, for each p∈m(I(L)). Then the mapping ϕ:m(I(L))→M defined by ϕ(P) = C(P) for each p ∈ m(I(L)) is an onto continuous closed mapping.

Theorem: 2.8 For any ideal I in L, $M_I = \bigcap_{x\in I} M_x = \bigcap_{x\in I} M_{(x]^{**}}$

Theorem: 2.9 [2] Let L be an ADL. Then the following are equivalent:
(1) *M is compact, Hausdorff and extremally disconnected space.*
(2) *The space M and m (I (L)) are homeomorphic.*
(3) $J^{**} \in A_0(L)$*, for each* $J \in I(L)$

3. ON PRIME IDEAL CHARACTERIZATION OF QUASI-COMPLEMENTED ALMOST DISTRIBUTIVE LATTICES

In this section, we characterize the quasi-complemented ADL in terms of hull - kernel topology and dual hull-kernel topology. Recall that an Almost Distributive Lattice L with 0 is called quasi-complemented if for each $x \in L$, there is an element $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is a maximal. Here y is called a quasi-complement of x.

Theorem: 3.1 *Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if for each* $x \in L$*, there exists* $y \in L$ *such that* $M_x = h_M(y)$.

Proof: Suppose L is quasi-complemented ADL. We have every quasi-complemented ADL is a ∗ ADL. Let $x \in L$. Then there exists $y \in L$ such that $[x]^* = [y]^{**}$. Therefore $h_M([x]^*) = h_M([y]^{**})$ and hence lemma 2.6, $h_M([x]^*) = h_M(y)$. Thus $M_x = h_M(y)$. Conversely, suppose that for each $x \in L$, there exists $y \in L$ such that $M_x = h_M(y)$. we shall prove that L is quasi complemented ADL. Since $M_x = h_M(y)$, $h_M([x]^*) = h_M([y]^{**})$. Hence by lemma 2.6, we get $[x]^* = [y]^{**}$. Therefore $x \wedge y = 0$ and $x \vee y$ is dense. It follows that $x \wedge y = 0$ and $x \vee y$ is a maximal. Thus L is quasi-complemented ADL.

Theorem: 3.2 *Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if M is compact in the hull-kernel topology.*

Proof: Suppose L is quasi-complemented ADL. Then for each $x \in L$ there exists $y \in L$ such that $M_x = h_M(y)$ and hence M_x is a basic closed set in M. Let $\{M_x / x \in \Delta\}$ be a family of closed sets in M with finite intersection property for some $\Delta \subseteq L$. Let F be a filter in L generated by Δ. Then for any $x_1, x_2, ...x_n \in \Delta$, $\bigcap_{i=1}^{n} M_x \neq \phi$ and hence M $M_{\wedge_{i=1}^{n} x_i} \neq \varphi$

It follows that $\wedge_{i=1}^{n} x_i \neq 0$. Therefore $0 \notin F$ and hence F is a proper filter of L. It follows that F is contained in a maximal filter say K of L. Therefore L-M is minimal prime ideal of L. Let $x \in \Delta$. Then $x \notin L - K$. Therefore $L - K \in M_x$ for all $x \in \Delta$. Hence $L - K \in \bigcap_{x\in\Delta} M_x$, we get $\bigcap_{x\in\Delta} M_x \neq \phi$. Thus M is compact in hull-kernel topology.

Conversely suppose M is compact in the hull-kernel topology on M and $x \in L$. Then $h_M(x)$ being a closed subset of M, is compact. If $p \in h_M(x)$, then $x \in P$. Hence by lemma 2.5 $h_M(x) \cap h_M([x]^*) = \phi$. So that $h_M(x) \cap \bigcap_{t\in[x]^*} h_M(t) = \phi$.

Now $\left\{h_M(x)\cap h_M(t)/t\in[x]^*\right\}$ is a class of closed sets in $h_M(x)$ having empty intersection. There exist $t_1,t_2,......t_n\in[x]^*$ such that $h_M(x)\cap h_M(t_1)\cap h_M(t_2)\cap h_M(t_3)......\cap h_M(t_n)=\phi$ Write $x'=\vee_{i=1}^{n} t_i$, then $h_M(x)\cap h_M(x')=\phi$. It follows that $M_x\cup M_{x'}=M$ and $M_x\cup M_{x'}=M_{x\wedge x'}=M_0=\phi$. Therefore $M_{x'}=h_M(x)$ and $M_x=h_M(x')$. Hence $h_M([x]^{**})=h_M(x)=M_{x'}=h_M([x']^*)$. By lemma 2.6, we get $[x]^{**}=[x']^*$. It follows that $x\wedge y=0$ and $x\vee y$ is dense. Thus $x\wedge y=0$ and $x\vee y$ is a maximal and hence L is quasi-complemented ADL.

Hence, from the above theorem 3.1 and 3.2, we have the following

Corollary: 3.3 *Let L be an ADL with maximal element m in which every dense element is maximal. Then the following are equivalent:*
(1) *L is quasi-complemented ADL*
(2) $\tau_h{}^M = \tau_d{}^M$
(3) *M is compact in hull-kernel topology.*

Next, we give an algebraic characterization of quasi-complemented ADL. First, we need the following

Lemma: 3.4 *Let L be an ADL with maximal element m. Then $M_m = M$ and $M_0=\phi$.*

Proof: Suppose m is maximal element in L. Clearly $M_m\subseteq M$. Conversely suppose $P\in M$ .Then $P\neq L$ and hence $m\notin P$. Therefore $P\in M_m$. Hence $M_m=M$ and clearly $M_0=\phi$ since every ideal contains 0.

Theorem: 3.5 Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if $B=\left\{M_x/x\in L\right\}$ is a Boolean algebra under the operations $\cup$ *and* $\cap$.

Proof: Suppose L is quasi- complemented ADL. It can be easily seen that B is a Boolean algebra under the operations $\cup$ *and* $\cap$. Conversely, suppose that B is a Boolean algebra. Let $x\in L$. Then there exist $x'\in L$ such that and $M_x\cup M_{x'}=M_m$ and hence $M_{x\wedge x'}=M_0$ and $M_{x\vee x'}=M_m$. It follows that $x\wedge x'=0$ and $x\vee x'$ is dense. Hence $x\wedge x'=0$ and $x\vee x'$ is maximal. Thus L is quasi-complemented ADL.

Recall that, the relation θ on an ADL L, defined by $(x, y)\in\theta \Leftrightarrow [x]^* = [y]^*$ is a congruence relation on L and hence $L|\theta$ of all congruence classes, is bounded distributive lattice under the induced operations on L. Now, we give another characterization of quasi-complemented ADL, if first we need the following lemma.

Lemma: 3.6 *Let L be an ADL with maximal element. Then we have the following:*
(1) *If $m_1|\theta$, $m_2|\theta$ are two maximal elements in $L|\theta$, then $m_1|\theta = m_2|\theta$*
(2) *If d is dense element in L, then $d|\theta = m|\theta$, for all maximal element m in L*

Proof: Let $t\in m_1|\theta$. Then $(t, m_1)\in\theta$ implies $[t]^* = [m_1]^* = \{0\}$. Therefore $[t]^* = [m_2]^*$ and hence $(t, m_2)\in\theta$. Thus $m_1|\theta \subseteq m_2|\theta$. Similarly $m_2|\theta\subseteq m_1|\theta$. Hence $m_1|\theta = m_2|\theta$. Obviously $d|\theta = m|\theta$, for all maximal element m in L

It can be easily see that the elements $0|\theta$ and $m|\theta$ are bounds of $L|\theta$ and hence $L|\theta$ is a bounded distributive lattice.

Theorem: 3.7 *Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if $L|\theta$ is a Boolean algebra*

Proof: Suppose L is a quasi-complemented ADL in which every dense element is maximal. Define f: $B\to L|\theta$ by $f(M_x) = x|\theta$, for each $x\in L$. Let M_x , $M_y\in B$. Then $M_x = M_y \Leftrightarrow [x]^* = [y]^* \Leftrightarrow (x, y)\in\theta \Leftrightarrow x|\theta = y|\theta \Leftrightarrow f(M_x) = f(M_y)$. Hence f is well-defined and one-one. Clearly f is an onto and hence f is a homomorphism. Also, $f(M_0) = 0|\theta$ and $f(M_m) = m|\theta$. Thus f is an isomorphism. Therefore $L|\theta$ is a Boolean algebra

Conversely, suppose that $L|\theta$ is a Boolean algebra. Let $x\in L$. Then there exists $x'|\theta\in L|\theta$ such that $x|\theta\wedge x'|\theta = 0|\theta$ and $x|\theta\vee x'|\theta = m|\theta$. It follows that $(x\wedge x')|\theta = 0|\theta$ and $(x\vee x')|\theta = m|\theta$. Therefore $[x\wedge x']^* = [0]^* = L$ and hence $x\wedge x' = 0$. Clearly $x\vee x'$ is dense. It follows that hypothesis $x\wedge x' = 0$ and $x\vee x'$ is a maximal. Hence L is quasi-complemented ADL.

Therefore from the above theorem 3.5 and 3.7, we have the following.

Corollary: 3.8 *Let L be an ADL with maximal element m in which every dense element is maximal. Then the following are equivalent:*
(i) L is quasi-complemented ADL
(ii) B = {M_x/x ∈ L} is a Boolean algebra
(iii) L|θ is a Boolean algebra

Recall that for a prime ideal P of I (L), C (P) = $\cup\{J \in I(L)/J \in P\}$ is a prime ideal in L.

Theorem: 3.9 *Let L be an ADL with maximal element m in which every dense element is maximal. Then L is quasi-complemented ADL if and only if C(P)∈ M, for each P ∈ m(I(L)).*

Proof: Suppose L is quasi-complemented ADL and $P \in m(I(L))$.Clearly C(P) is a prime ideal in L. Let $x \in C(P)$. Then $x \in J$. Therefore $(x] \subseteq J$ and hence $(x] \in P$. Since L is quasi-complemented ADL, there exists $x^{'} \in L$ such that $x \wedge x^{'} = 0$ and $x \vee x^{'}$ is maximal. But $(x] \vee (x^{'}] = (x \vee x^{'}] = L$ and $(x] \vee (x^{'}] \notin P$, since P is maximal prime ideal in I(L), it follows that $(x^{'}] \notin P$. Hence $x^{'} \notin P$. Thus for each $x \in C(P)$, Hence there exists $x^{'} \notin C(P)$ such that $x \wedge x^{'} = 0$. (by lemma 2.5), C(P) is a minimal prime ideal of L. Hence $C(P) \in M$, for each $P \in m(I(L))$. Conversely assume the condition. We have the mapping ϕ: m(I(L)) → M defined by $\phi(P) = C(P)$, for each $P \in$ m(I(L)) is onto continues and closed since by lemma 2.7. Therefore M is a compact. It follows that, by theorem 3.2, L is quasi-complemented ADL.

Recall that the set $A_0(L)$ of all annulets of an ADL L forms a distributive lattice under the binary operations ∨ and ∧ defined by $[x]^* \vee [y]^* = [x \vee y]^*$ and $[x]^* \wedge [y]^* = [x \wedge y]^*$, for any $[x]^*, [y]^* \in A_0(L)$.

Theorem: 3.10 Let L be an ADL with maximal element m in which every dense element is maximal. Then $J^{**} \in A_0(L)$, for each $J \in I(L)$ if and only if L is quasi-complemented ADL and for each $Q \in M$, there exists unique $P \in m(I(L))$ such that C(P)=Q.

Proof: Suppose $J^{**} \in A_0(L)$, for each $J \in I(L)$. Let $x \in L$. Then we have $[x]^{**} \in A_0(L)$. It follows that, there exist $x^{'} \in L$ such that $[x]^{**} = [x^{'}]^*$. Hence $x \wedge x^{'} = 0$ and $x \vee x^{'}$ is maximal. Thus L is quasi-complemented ADL. Now, let $Q \in M$ and $P_1, P_2 \in m(I(L))$ such that $C(P_1) = C(P_2) = Q$. Let $J \in P_1$. Then $J^* \in I(L)$ and $J \cap J^* = (0]$. Since $P_1 \in m(I(L)), J \in P_1, J^* \notin P_1$, (by lemma 2.5). Again, since $J^* \in I(L), J^* \in A_0(L)$. Hence there exist $y \in L$ such that $J^* = (y]^*$. Again, since $P_1 \in m(I(L)), J \in P_1, J^{**} \in P_1$. Therefore $(y]^{**} \in P_1$, and hence $(y] \in P_1$. It follows that $y \in C(P_1) = C(P_2)$ and hence $(y] \in P_2$. Therefore $[y]^* \notin P_2$, since P_2 is a minimal prime ideal. Hence $J^* \notin P_2$. Therefore $J \in P_2$ and hence $P_1 \subseteq P_2$. Hence $P_1 = P_2$, since P_1, P_2 are minimal prime ideals. Therefore for each $Q \in M$, there exist unique $P \in m(I(L))$ such that C(P)=Q.

Conversely, suppose L is Quasi-complimented ADL and for each $Q \in M$, there exist unique $P \in m(I(L))$ such that C(P)=Q. By lemma 2.7, we have there exists a mapping ϕ : m(I(L)) → M defined $\phi(P) = C(P)$ is a homeomorphism. Let $J \in I(L)$. Then $h_M(J^*)$ is a both open and closed set in M. Hence $M - h_M(J^*)$ is compact (being closed subset of compact space M is compact). Therefore $M - h_M(J^*) = \bigcup_{i=1}^{n} M_{a_i} = M_{\bigvee_{i=1}^{n} a_i}$ Now, put $y = \bigvee_{i=1}^{n} a_i$.

Then $M - h_M(J^*) = M_y = h_M([y]^*) = M_y - h_M([y]^{**})$. Hence $h_M[J]^* = h_M([y]^{**})$. It follows that $J^{**} \in A_0(L)$.

Corollary: 3.11 *Let L be an ADL with maximal element m in which dense element is maximal. Then the following are*

(1) M is compact, Hausdorff and extremally disconnected space.
(2) The space M and m (I (L)) are homeomorphic.
(3)$J^{**} \in A_0(L)$, for each $J \in I(L)$

Further any of the above conditions implies that L is quasi-complemented ADL

Theorem: 3.12 *Let L be an ADL with maximal element m in which every dense element is maximal. Then the following are equivalent:*

(1) *L is quasi-complemented ADL*
(2) $\tau_h^M = \tau_d^M$
(3) *M is compact in hull-kernel topology*
(4) *(B = {M_x / x ∈ L}, ∩, ∪) is a Boolean algebra.*
(5) *L|θ is Boolean Lattice , where θ is the congruence relation on L and defined by*
$\theta = \{(x, y) \in L \times L / [x]^* = [y]^*\}$
(6) *C (P) ∈ M, for each P ∈ m (I(L)).*

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