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SOME FIXED POINT THEOREMS FOR A PAIR OF EXPANSION TYPE MAPPINGS IN FUZZY METRIC SPACES

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ABSTRACT

The purpose of this paper is to prove two common fixed point theorems for a pair of nonsurjective mappings satisfying different expansion type conditions in fuzzy metric spaces. Our results improve and generalize the results of Kumar, Chugh and Vats [10]; and Pathak, Kang and Ryu [15].

Key Words and phrases: *fuzzy metric spaces, non-surjective mapping, expansion mapping, fixed point.*

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1. INTRODUCTION:

The concept of fuzzy sets was introduced initially by Zadeh [24] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and its applications.Especially, Deng [2], Erceg [3], Kaleva and Seikkala [7], Kramosil and Michalek [9] have introduced the concept of fuzzy metric space in different ways. Grabiec [5] followed Kramosil and Michalek [9] and obtained the fuzzy version of Banach contraction principle. Moreover, it appears that the study of Kramosil and Michalek [9] of fuzzy metric spaces proves the way for developing smoothing machinery in the field of fixed point theorems, in particular for the study of contractive type maps.

In 1984, Wong, Li, Gao and Iseki [23] proved some fixed point theorems on expansion mappings in metric space, which correspond to some contractive mappings in [16]. Later, using expansion type conditions, several fixed point theorems have been proved for surjective mappings (see [8], [17], [22]) and for nonsurjective mappings (see [15], [21]) in metric spaces. Subsequently, there are a number of generalizations of these results in different settings such as probabilistic metric space ([1], [11], [14]); fuzzy metric space [18]; and intuitionistic fuzzy metric space [12].

In [18], Saini, Vishal and Singh proved some fixed point theorems for surjective expansion mappings in fuzzy metric space. The purpose of this paper is to prove two common fixed point theorems for a pair of nonsurjective mappings using different expansion type condition in fuzzy metric space. Our results improve and generalize the results of Kumar, Chugh and Vats [10] and Pathak, Kang and Ryu [15].

2. PRELIMINARIES:

Definition: 2.1 [24] Let X be any set**.** A fuzzy set A in X is a function with domain X and values in $[0, 1]$.

Definition: 2.2 [19] A binary operation $*:[0,1]\times[0,1]\to[0,1]$ is called a continuous t-norm if $([0,1],*)$ is an abelian topological monoid with the unit 1 such that $a*b\le c*d$ whenever $a\le c$ and $b\le d$ for all $a,b,c,d\in[0,1]$.
Example of t-norms are $a*b=ab$ and $a*b=min\{a,b\}$.

Definition: 2.3 [9]. The triplet $(X,M,*)$ is called a fuzzy metric space (FM-Space) if *X* is an arbitrary set, $*$ is a continuous t-norm and *M* is a fuzzy set in $X^2\times[0,\infty)$ satisfying the following conditions: for all $x,y,z\in X$ and $t,s>0$.

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(1) $M(x, y, 0) = 0$;

(2) $M(x, y, t) = 1$ iff $x = y$;

(3) $M(x, y, t) = M(y, x, t)$;

(4) $M(x, y, t) * M(y, z, s) \le M(x, z, t+s)$;

(5) $M(x, y, .) : [0, \infty) \to [0,1]$ is left continuous.

Then *M* is called a fuzzy metric on *X*. The function $M(x, y, t)$ denote the degree of nearness between *x* and *y* with respect to *t*.

Definition: 2.4. A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called Cauchy if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1$ for every $t > 0$ and each $p > 0$. $(X, M, *)$ is complete if every Cauchy sequence in *X* converges in *X*.

Definition: 2.5. A sequence $\{x_n\}$ in X is convergent to $x \in X$ if

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \text{ for each } t > 0.$$

Example: 2.6. [4] Let (X, d) be a metric space. Define $a * b = ab$ (or $a * b = \min\{a, b\}$) and for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + d(x, y)}$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric *M* induced by the metric *d* the standard fuzzy metric.

Definition: 2.7. [6, 20]. Self-mappings *S* and *T* of a fuzzy metric space $(X, M, *)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if

$$Sp = Tp \text{ some } p \in X \text{ then } STp = TSp.$$

Lemma: 2.8 [13]. Let $\{y_n\}$ be a sequence in a FM-Space $(X, M, *)$. If there exists a number $k \in (0,1)$ such that

$M(y_{n+2}, y_{n+1}, kt) \ge M(y_{n+1}, y_n, t)$;

for all $t > 0$ and $n = 1,2,3...$ then $\{y_n\}$ is a Cauchy sequence in *X*.

3. MAIN RESULTS:

Theorem: 3.1. Let $(X, M, *)$ be a fuzzy metric space with $(a * b) = \min(a, b)$ for all $a, b \in [0,1]$. Let *S* and *T* be weakly compatible self mappings of *X* satisfying the following conditions:

$$T(X) \subseteq S(X); \tag{3.1}$$

$$(M(Su, Sv, ht))^2 \le M(Su, Tu, t) M(Sv, Tv, t) \tag{3.2}$$

for each $u, v \in X$, $h > 1$ and for all $t > 0$.

If one of the subspaces *S(X)* or *T(X)* is complete, then *S* and *T* have a unique common fixed point.

Proof: Let $u_0 \in X$. Since $T(X) \subseteq S(X)$; choose $u_1 \in X$ such that $Su_1 = Tu_0$. In general, choose u_{n+1} such that $Su_{n+1} = Tu_n$.

By (3.2), for $h > 1$, we have

$$(M(Su_n, Su_{n+1}, ht))^2 \le M(Su_n, Tu_n, t)M(Su_{n+1}, Tu_{n+1}, t)$$
$$\le M(Su_n, Su_{n+1}, t)M(Su_{n+1}, Su_{n+2}, t).$$

Hence,

$$M(Su_n, Su_{n+1}, ht) \le M(Su_{n+1}, Su_{n+2}, t).$$

Similarly,

$$M(Su_{n+1}, Su_{n+2}, ht) \le M(Su_{n+2}, Su_{n+3}, t).$$

From Lemma 2.8, $\{Su_n\}$ is a Cauchy sequence. Since *S(X)* is complete, $\{Su_n\}$ has a limit in *S(X).* Call it *z*. Hence there exists a point *p* in *X* such that $Sp = z$. Consequently, the subsequence $\{Tu_n\}$ also converges to *z*.
From (3.2), we have

$$(M(Sp, Su_n, ht))^2 \le M(Sp, Tp, t)M(Su_n, Tu_n, t).$$

Taking limit as $n \to \infty$, we have
$(M(z, z, ht))^2 \le M(z, Tp, t)M(z, z,, t)$ implying $Tp = z$.

Therefore, $Sp = Tp = z$. Since *S* and *T* are weakly compatible, therefore, $TSp = STp$ i.e. $Tz = Sz$. Now, we show that *z* is a fixed point of *S* and *T*. From (3.2),

$$(M(Sz, Su_n, ht))^2 \le M(Sz, Tz, t)M(Su_n, Tu_n, t).$$

Taking limit as $n \to \infty$, we have
$(M(Sz, z, ht))^2 \le M(Sz, Tz, t)M(z, z, t)$, which implies that $Sz = z$.

Hence *z* is a common fixed point of *S* and *T*. To prove the uniqueness of *z* as a common fixed point of *S* and *T*, let $y(\neq z)$ be another fixed point. From (3.2),

$$(M(z, y, ht))^2 = (M(Sz, Sy, ht))^2 \le (M(Sz, Tz, t)M(Sy, Ty, t)$$
$$\le (M(z, z, t)M(y, y, t)$$

This implies that $y = z$ and hence *z* is a unique common fixed point of *S* and *T*.

Remark: 3.2 Theorem 3.1 is an improvement of Kumar, Chugh and Vats [10, Theorem 4.1] in the sense that we have taken completeness of one of the subspaces, not the whole space.

Theorem: 3.3 Let $(X, M, *)$ be a fuzzy metric space with $(a * b) = \min(a, b)$ for all $a, b \in [0,1]$. Further *S* and *T* be continuous self mappings of *X* satisfying the following conditions:

$$S(X) \subseteq S^2(X),\ S(X) \subseteq ST(X), \tag{3.3}$$

$$M(S^2u, TSv, ht) \le \min\{M(Su, S^2u, t),\ M(TSv, Sv, t),\ M(Su, Sv, t)\} \tag{3.4}$$

for each $u, v \in X$, $h > 1$ and for all $t > 0$.
If the subspace *S(X)* is complete, then *S* or *T* has a fixed point; or *S* and *T* have a common fixed point.

Proof: Let $u_0 \in X$. Since $S(X) \subseteq S^2(X)$ and $S(X) \subseteq TS(X)$, choose a point $u_1 \in X$ such that $S^2u_1 = Su_0 = v_0$ and for this point u_1, there exists a point $u_2 \in X$ such that $TSu_2 = Su_1 = v_1$. Continuing in this way, we obtain a sequence $\{v_n\}$ in *S(X)* as follows:

$S^2u_{2n+1} = Su_{2n} = v_{2n}$ and $TSu_{2n+2} = Su_{2n+1} = v_{2n+1}$.

Now, if $v_{2n} = v_{2n+1}$ for any *n*, one has that v_{2n} is a fixed point of *S* from the definition of $\{v_n\}$. It then follows that also $v_{2n+1} = v_{2n+2}$, which implies that v_{2n} is also a fixed point of *T*.

Suppose that $v_{2n} \neq v_{2n+1}$, then by (3.4), for $h > 1$, we have

$$\begin{aligned}M(v_{2n}, v_{2n+1}, ht) &= M(S^2u_{2n+1}, TSu_{2n+2}, ht)\\ &\leq \min\{M(Su_{2n+1}, S^2u_{2n+1}, t),\ M(TSu_{2n+2}, Su_{2n+2}, t),\ M(Su_{2n+1}, Su_{2n+2}, t)\}\\ &\leq \min\{M(v_{2n+1}, v_{2n}, t),\ M(v_{2n+1}, v_{2n+2}, t),\ M(v_{2n+1}, v_{2n+2}, t)\}\\ &\leq M(v_{2n+1}, v_{2n+2}, t).\end{aligned}$$

Similarly, we have

$$M(v_{2n+2}, v_{2n+1}, ht) \leq M(v_{2n+3}, v_{2n+2}, t).$$

In general, for any *n* and for $h > 1$, by (3.4), we have

$$M(v_n, v_{n+1}, ht) \leq M(v_{n+1}, v_{n+2}, t).$$

By Lemma 2.8, $\{v_n\}$ is a Cauchy sequence and it converges to some point *z* in *S(X)*. Consequently the subsequences $\{v_{2n}\}$, $\{v_{2n+1}\}$ and $\{v_{2n+2}\}$ converge to *z*. By continuity of *S* and *T*,

$$S^2u_{2n+1} = Su_{2n} = v_{2n} \to Sz \text{ and } TSu_{2n+2} = Su_{2n+1} = v_{2n+1} \to Tz, \text{ as } n \to \infty.$$

Thus *S* and *T* have a common fixed point.

Remark: 3.4 Theorem 3.3 is a fuzzy version of the result of Pathak, Kang and Ryu [15, Corollary 2.3].

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