

STUDY OF NONLINEAR ELASTICITY PROBLEM BY ELLIPTIC REGULARIZATION WITH LAMÉ SYSTEM

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ABSTRACT

In this work, we look for the existence and uniqueness of a function  $u = u(x, t)$ ,  $x \in \Omega$ ,  $t \in [0, T]$  solution of the nonlinear boundary value problem by the elliptic regularization techniques based on theory of monotone operators.

**Keywords:** *Priori estimates, monotone operators, Lamé system, elliptic regularization.*

1. NOTATIONS AND POSITION OF THE PROBLEM:

Let  $\Omega$  an open bounded domain of  $\mathbb{R}^n$ , with regular boundary  $\Gamma$ . We denote by  $Q$  the cylinder  $\mathbb{R}_x^n \times \mathbb{R}_t$ ;  $Q = \Omega \times [0, T]$ , with boundary  $\Sigma$ .  $L$  designed Lamé system define by  $\mu\Delta + (\lambda + \mu) \nabla \text{div}$ ,  $\lambda$  and  $\mu$  are constants Lamé with  $\lambda + \mu \geq 0$ . and  $h, f$  are functions. We look for the existence and uniqueness of a function  $u = u(x, t)$ ,  $x \in \Omega$ ,  $t \in [0, T]$ , solution of the problem (P):

$$(P) \begin{cases} u'' - Lu + |u'|^{p-2}u' = f & , \text{ in } Q & (1.1.1) \\ u = 0 & , \text{ on } \Sigma & (1.1.2) \\ u(x, 0) = u(x, T) & , \forall x \in \Omega & (1.1.3) \\ u'(x, 0) = u'(x, T) & , \forall x \in \Omega & (1.1.4) \end{cases} \quad (1.1)$$

2. EXISTENCE OF THE SOLUTION:

**Theorem: 1** Assume that  $\Omega$  is bounded open of  $\mathbb{R}^n$  are given  $f$ , with  $f \in L(Q)$ . Then there exists a function  $u = w_0 + w$  satisfying (P)

$$w_0 \in H_0^1(\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \quad (1.2)$$

$$w \in L^2(0, T; H_0^1(\Omega)) \quad (1.3)$$

$$w \in L^p(Q) \quad (1.4)$$

**Proof:** we use an approach due to G. Prodi [11] we have

$$\begin{cases} u = w_0 + w \\ w_0 \text{ independent of } T \\ \int_0^T w dt = 0 \end{cases} \quad (1.5)$$

We introduce the Prodi idea (1. 5) in (1.1.1) we having

$$u'' - Lu + |u'|^{p-2}u' - f = f + Lu_0 \quad (1.6)$$

We consider the derivative of (1.6) we obtain

$$\frac{d}{dt}(u'' - Lu + |u'|^{p-2}u') = \frac{df}{dt} \quad (1.7)$$

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And

$$\begin{cases} \int_0^T u dt = 0 \\ u(T) = u(0) \\ u'(x, 0) = u'(x, T) \end{cases} \quad (1.8)$$

We deduce to (1.7)

$$u'' - Lu + |u'|^{p-2}u' - f = h_0 \text{ with } h_0 \text{ independent of } t \quad (1.9)$$

For resolve (1.7) and (1.8) we denotes.  $L = -A$ ;  $\beta(u') = |u'|^{p-2}u'$

And we define the functional space V:

$$V = \left\{ \begin{array}{l} v: v \in L^2(0, T, H_0^1(\Omega)); \quad v' \in L^2(0, T, (H_0^1(\Omega)) \cap L^p(Q)); \\ v'' \in L^2(0, T, L^2(\Omega)); \quad \int_0^T v(t) dt = 0; \quad v(T) = v(0); \quad v'(T) = v'(0) \end{array} \right. \quad (1.10)$$

The Banach structure of V is defined by

$$\|v\|_V = \|v\|_{L^2(0, T, H_0^1(\Omega))} + \|v'\|_{L^2(0, T, H_0^1(\Omega))} + \|v\|_{L^p(Q)} + \|v\|_{L^2(0, T, L^2(\Omega))}$$

We define the bilinear form:

$$b(u, v) = \int_0^T [(u'', v) + (Au, v) + (\beta(u'), v)] dt \quad (1.11)$$

The weak formulation of (1.7) and (1.8) is to find  $u \in V$  such that

$$b(u, v) = \int_0^T (f, v') dt \quad \forall v \in V \quad (1.12)$$

Following some ideas of Lions for obtain the elliptic regularization, given  $\delta > 0$  and  $u, v \in V$  we define

$$\pi_\delta(u, v) = \delta \int_0^T [(u'', v'') + (Au', v')] ds + \int_0^T (u'' + Au + \beta(u'), v') ds. \quad (1.13)$$

The application  $v \rightarrow \pi_\delta(u, v)$  is continuous on V so there existes an application  $\pi_\delta \in V'$ :

$$\pi_\delta(u, v) = (B_\delta(u), v) \quad (1.14)$$

The linear operator  $B_\delta : V \rightarrow V'$  satisfies the four properties:

$B_\delta$  is bounded in  $V'$  for all bounded set in V and is a hemi continuous and is a strictly monotonous and is coercive.

In view of these properties and as consequence of theorem of Lions [5], there exist unique a function  $u_\delta \in V$ :

$$\pi_\delta(u_\delta, v) = \int_0^T (f, v') dt \quad \forall v \in V \quad (1.15)$$

## 2.1 A PRIORI ESTIMATES I:

Explicitly the elliptic regularization (1.15) and setting  $v = u_\delta$ , we obtain:

$$\delta \int_0^T [|u''_\delta|^2 + \|u'_\delta\|^2] dt + \int_0^T [|u'_\delta|^2 + (\beta(u'_\delta), u'_\delta)] dt = \int_0^T (f, u_\delta) dt \quad (1.16)$$

Or

$$\int_0^T (\beta(u'), u') dt = \|u'\|_{L^p(Q)}^p \text{ and } \int_0^T u dt = 0 \Rightarrow \|u\|_{L^2(0, T, H_0^1(\Omega))} \leq C \|u'\|_{L^2(0, T, H_0^1(\Omega))}$$

Then

$$u'_\delta \text{ is bounded in } L^p(Q) \text{ when } \delta \rightarrow 0 \quad (1.17)$$

$$\delta \int_0^T [|u''_\delta|^2 + |u'_\delta|^2 + \|u'_\delta\|^2] dt \leq C \quad (1.18)$$

Or

$$\int_0^T u_\delta dt = 0, \text{ we have by (1.17) and (1.18) that:} \\ u_\delta \text{ is bounded in } L^p(Q) \quad (1.19)$$

$$\delta \int_0^T \|u_\delta\|^2 dt \leq C \quad (1.20)$$

## 2.2 A PRIORI ESTIMATES II:

Exchange in (1.15)  $v$  with:

$$v(t) = \int_0^T u_\delta(s) ds - \frac{1}{T} \int_0^T (T-s) u_\delta(s) ds \quad (1.21)$$

We verify that:

$$\begin{cases} \int_0^T v dt = 0 & \forall v \in V \\ v' = u_\delta \end{cases} \quad (1.22)$$

Taking into account (1.21) in (1.15) we get

$$\delta \int_0^T [(u''_\delta, u'_\delta) + (u'_\delta, u_\delta) + (Au'_\delta, u_\delta)] dt + \int_0^T [(u''_\delta, u_\delta) + (u'_\delta, u_\delta) + \|u_\delta\|^2] dt \\ + \int_0^T (\beta(u'_\delta), u'_\delta) dt = \int_0^T (f, u_\delta) dt \quad (1.23)$$

By using periodicity of  $u_\delta, u'_\delta \in V$  we obtain:

$$\int_0^T (u''_\delta, u'_\delta) dt = \int_0^T (Au'_\delta, u_\delta) dt = 0 \quad (1.24)$$

And

$$\int_0^T (u'_\delta, u_\delta) dt = (u'_\delta(T), u_\delta(T)) - (u'_\delta(0), u_\delta(0)) - \int_0^T (u'_\delta, u'_\delta) dt = - \int_0^T |u'_\delta|^2 dt \quad (1.25)$$

By (1.24) and (1.17) we have

$$\left| \int_0^T (u'_\delta, u_\delta) dt \right| \leq C \text{ when } \delta \rightarrow 0 \quad (1.26)$$

Also, from (1.17) and (1.19) we obtain:

$$\left| \int_0^T (\beta(u'_\delta), u_\delta) dt \right| \leq \|\beta(u'_\delta)\|_{L^p(Q)} \|u_\delta\|_{L^p(Q)} \leq C' \quad (1.27)$$

Combining (1.24), (1.26), (1.27) with (1.23) we deduce

$$\int_0^T \|u_\delta\|^2 dt \leq C \quad (1.28)$$

## 2.3 PASSAGE TO THE LIMIT:

From (1.17) and (1.28) that there exists a subsequence from  $(u_\delta)$ , such that

$$u_\delta \rightarrow 0 \text{ weak in } L^2(0, T, H_0^1(\Omega)) \quad (1.29)$$

$$u'_\delta \rightarrow u' \text{ weak in } L^p(Q) \quad (1.30)$$

$$\beta(u'_\delta) \rightarrow \chi \text{ weak in } L^q(Q) \quad (1.31)$$

Passage to the limit in (1.15) we obtain

$$\int_0^T [(-u', v'') + (Au, v') + (\chi, v')] dt = \int_0^T (f, v') dt \quad \forall v \in V \quad (1.32)$$

Use the convolution technical in (1.32) we have

$$\int_0^T (\chi, u' * \eta_\delta * \eta_\delta) dt = \int_0^T (f, u' * \eta_\delta * \eta_\delta) dt \quad \forall v \in V \quad (1.33)$$

When

$$\int_0^T (\chi, u'') dt = \int_0^T (f, u') dt \quad \forall v \in V \quad (1.34)$$

### 3. UNIQUENESS OF SOLUTION:

**Theorem:** Under the hypotheses of the theorem of existence, we consider two solutions  $u_1$  and  $u_2$  of the problem (P) then  $u_1 = u_2$ .

**Proof:** We subtract the equations (1.5) corresponding to  $u_1$  and  $u_2$  and sitting  $\phi = u_1 - u_2$  we have:

$$\phi'' + \phi' + A\phi + \beta(u_1') - \beta(u_2') \quad (2.1)$$

Denoting by  $(\eta_\delta)$  the regularizing sequence a similar argument by Brézis [2] we obtain

$$\phi' * \eta_\delta * \eta_\delta = \phi * \eta'_\delta * \eta_\delta \quad (2.2)$$

Hence, by using (1.2) and (1.3), we have

$$\phi = \varphi + \phi_0 : \quad \phi_0 \in V \text{ and } \varphi \in L^2(0, T, H_0^1(\Omega)) \quad (2.3)$$

From (2.2) we get

$$\phi' * \eta_\delta * \eta_\delta = \phi * \eta'_\delta * \eta_\delta = \phi' * \eta_\delta * \eta_\delta \quad (2.4)$$

Show that

$$\int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta) dt = 0$$

When

$$\int_0^T \frac{d}{dt} (\phi', \phi' * \eta_\delta * \eta_\delta) dt = \int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta) dt + \int_0^T (\phi', \phi'' * \eta_\delta * \eta_\delta) dt = 2 \int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta) dt = 0 \quad (2.5)$$

Therefore

$$\int_0^T (\phi'', \phi' * \eta_\delta * \eta_\delta) dt = \frac{1}{2} \int_0^T \frac{d}{dt} (\phi', \phi' * \eta_\delta * \eta_\delta) dt = 0 \quad (2.6)$$

$\phi'$  and  $\eta_\delta$  Periodic then we have

$$\int_0^T (\phi, \phi' * \eta_\delta * \eta_\delta) dt = \int_0^T (\phi', \phi' * \eta_\delta * \eta_\delta) dt = \int_0^T (A\phi, \phi' * \eta_\delta * \eta_\delta) dt = 0 \quad (2.7)$$

From (2.1); (2.6) and (2.7) we obtain:

$$\int_0^T (\beta(u_1') - \beta(u_2'), \phi' * \eta_\delta * \eta_\delta) dt = 0 \quad (2.8)$$

Passage to the limit in (2.8) we have

$$\int_0^T (\beta(u_1') - \beta(u_2'), u_1' - u_2') dt = 0 \quad (2.9)$$

Where

$$u_1' - u_2' = 0 \Rightarrow u_1' = u_2' \quad (2.10)$$

This implies that

$$\phi = u_1 - u_2 = \theta, \theta \text{ independent of } t \quad (2.11)$$

From (2.7) and (2.11) we obtain

$$\int_0^T (A\theta, \theta) dt = 0 \quad \forall \theta \in V \quad (2.12)$$

We deduce from (1.2)

$$\theta \in H_0^1(\Omega) + W^{2,q}(\Omega) \cap W_0^{1,q}(\Omega) \quad (2.13)$$

By (2.12) and (2.13) and using theorem of Green we have  $(A\theta, \theta) = 0 \Rightarrow \theta = 0$ .

where the uniqueness of solution.

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#### REFERENCES:

- [1] R.A. Adams, Sobolev Spaces, Academic Press, (1976).
- [2] H. Brezis, Analyse fonctionnelle, théorie et applications. Masson (1983).
- [3] H. Brezis, Equations et inéquations non linéaires dans les espaces vectoriels en dualité, Ann. Ins. Fourier; 18, (1968), 115-175.
- [4] G.Duvaut, J.L. Lions, Les inéquations en mécanique et en physique. Dunod. Paris. (1972).
- [5] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires. Dunod. (1969).
- [6] Luc Tartar, Topics in non linear analysis. Université de Paris-Sud, Publications Mathematiques d'Orsay, novembre (1978).
- [7] M. Meflah, A Nonlinear Elasticity Problem by Elliptic Regularization Technics, Int. J. Contemp. Math. Sciences, Vol 6, 2011, no. 25, 1221-1229.
- [8] M. Meflah, B Merouani, A Nonlinear Elasticity Problem Governed by Lamé System, Applied Mathematical Sciences, Vol.4, 2010, no. 36, 1785-1796.
- [9] B. Merouani, M. Meflah, A. Boulaouad, The Generalized and perturbed Lamé system, Applied Mathematical Sciences Vol.2, 2008, no. 49, 2425-2430.
- [10] F. Messelmi, B. Merouani, M. Meflah, Nonlinear thermoelasticity problem, Analele Universitatii Oradea, Fasc. Matematica, Tom XV (2008), 207-217.
- [11] G. Prodi, Soluzioni periodiche dell'equazione delle onde com termine dissipativo nonlineare. Rend. Sem. Mat. Padova, 35 (1965).
- [12] V. Patron, P. Perline, Méthode de la théorie mathématique de l'élasticité, Editions Mir, Moscou, 1981.
- [13] Taylor M.E, Partial differential non linear equations III Springer. (1996).
- [14] K. Yosida, Functional Analysis, Springer-Verlag Berlin Heidelberg New York 1968.

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