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PERIODIC SOLUTIONS FOR NON-LINEAR SYSTEMS OF DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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ABSTRACT

In this work, we investigate a periodic solution for non-linear systems of differential equations with boundary conditions by using the numerical analytic method, which was introduced by (Samoilenko, A. M.), These investigations lead us to improving and extending the above method. Also we expand the results obtained by Samoilenko to change the periodic system of non-linear differential equations with initial condition to periodic a system of non-linear differential equations with boundary conditions conditions.

Keyword and Phrases: *Numerical-analytic methods, existence of periodic solutions, nonlinear system, boundary conditions.*

I. INTRODUCTION

The study of periodic solutions for non-linear system of differential equations with boundary conditions and boundary integral conditions is a very important branch in the differential equation theory. Many results about the existence and approximation of periodic solutions for system of non–linear differential equations have been obtained by the numerical analytic methods that were proposed by Samoilenko [5] which had been later applied in many studies [2,3,6,7,8,9].

In this paper, we prove the existence and uniqueness of periodic solution for non-linear system of differential equations with the boundary conditions by using the method of Samoilenko [5].

Consider the following problem:

$$\left.\begin{array}{l}\dfrac{dx}{dt}=Ax+f(t,x,y,z)\\ A_1x(0)+A_2x(T)=e_1\\ \dfrac{dy}{dt}=By+g(t,x,y,z)\\ B_1y(0)+B_2y(T)=e_2\\ \dfrac{dz}{dt}=Cz+h(t,x,y,z)\\ C_1z(0)+C_2z(T)=e_3\end{array}\right\} \tag{P}$$

where $x\in D\subset R^n, y\in D_1\subset R^m$ and $z\in D_2\subset R^k$.The domains D, D_1 and D_2 are closed and bounded.

Let the vector functions $f(t,x,y,z)$, $g(t,x,y,z)$ and $h(t,x,y,z)$ are defined and continuous on the domain:

$$(t,x,y,z)\in R^1\times D\times D_1\times D_2 \tag{1}$$

and periodic in t of period T. also $A=(A_{ij})$, $A_1=(A_{1ij})$, $A_2=(A_{2ij})$,

$B=(B_{ij})$, $B_1=(B_{1ij})$, $B_2=(B_{2ij})$, $C=(C_{ij})$, $C_1=(C_{1ij})$, $C_2=(C_{2ij})$ are $n\times n$ non-negative matrices, also $e_1=(e_{11},e_{12},\dots)$, $e_2=(e_{21},e_{22},\dots)$, $e_3=(e_{31},e_{32},\dots)$ are positive constant vectors.

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Suppose that the functions $f(t,x,y,z)$, $g(t,x,y,z)$ and $h(t,x,y,z)$ satisfy the following inequalities:

$$\|f(t,x,y,z)\| \le M_1, \|g(t,x,y,z)\| \le M_2, \|h(t,x,y,z)\| \le M_3 \tag{2}$$

$$\|f(t,x_1,y_1,z_1)-f(t,x_2,y_2,z_2)\| \le K_1\|x_1-x_2\| + K_2\|y_1-y_2\| + K_3\|z_1-z_2\| \tag{3}$$

$$\|g(t,x_1,y_1,z_1)-g(t,x_2,y_2,z_2)\| \le L_1\|x_1-x_2\| + L_2\|y_1-y_2\| + L_3\|z_1-z_2\| \tag{4}$$

$$\| h(t,x_1,y_1,z_1)-h(t,x_2,y_2,z_2) \| \le P_1\|x_1-x_2\| + P_2\|y_1-y_2\| + P_3\|z_1-z_2\| \tag{5}$$

for all $t \in R^1$, $x, x_1, x_2 \in D$, $y, y_1, y_2 \in D_1$, $z, z_1, z_2 \in D_2$,

where $M_1, M_2, M_2, K_1, K_2, K_3, L_1, L_2, L_3, P_1, P_2, P_3$ are positive constants Provided that

$$\|e^{A(t-s)}\| \le \frac{\gamma_1}{\lambda_1}, \quad \|e^{B(t-s)}\| \le \frac{\gamma_2}{\lambda_2}, \quad \|e^{C(t-s)}\| \le \frac{\gamma_3}{\lambda_3} \tag{6}$$

where $\gamma_1, \gamma_2, \gamma_3, \lambda_1, \lambda_2, \lambda_3$ are positive conistants , and $\|.\| = \max_{t\in[0,T]}|.|$.

We define non-empty sets as follows:

$$\left.\begin{aligned} D_\alpha &= D - \frac{T}{2}\frac{\gamma_1}{\lambda_1}M_1 - \frac{\gamma_1}{\lambda_1}\|\alpha\|T \\ D_\beta &= D_1 - \frac{T}{2}\frac{\gamma_2}{\lambda_2}M_2 - \frac{\gamma_2}{\lambda_2}\|\beta\|T \\ D_\delta &= D_2 - \frac{T}{2}\frac{\gamma_3}{\lambda_3}M_3 - \frac{\gamma_3}{\lambda_3}\|\delta\|T \end{aligned}\right\} \tag{7}$$

Furthermore, we suppose that the largest eigen- value of the matrix

$$\Omega = \begin{pmatrix} K_1\frac{\gamma_1}{\lambda_1}\frac{T}{2} & K_2\frac{\gamma_1}{\lambda_1}\frac{T}{2} & K_3\frac{T}{2}\frac{\gamma_1}{\lambda_1} \\ L_1\frac{\gamma_2}{\lambda_2}\frac{T}{2} & L_2\frac{\gamma_2}{\lambda_2}\frac{T}{2} & L_3\frac{\gamma_2}{\lambda_2}\frac{T}{2} \\ P_1\frac{\gamma_3}{\lambda_3}\frac{T}{2} & P_2\frac{\gamma_3}{\lambda_3}\frac{T}{2} & P_3\frac{\gamma_3}{\lambda_3}\frac{T}{2} \end{pmatrix} \text{ Does not exceed unity}$$

$$\frac{\omega_1}{3} + \frac{\omega_4}{6} + \frac{2\omega_2}{3} + \frac{2{\omega_1}^2}{3} < 1, \tag{8}$$

where $\omega_1 = \frac{T}{2}\left(K_1\frac{\gamma_1}{\lambda_1} + L_2\frac{\gamma_2}{\lambda_2} + P_3\frac{\gamma_3}{\lambda_3}\right)$,

$$\omega_2 = \frac{T^2}{2}\left(\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}(K_2L_1 - K_1L_2) + \frac{\gamma_1}{\lambda_1}\frac{\gamma_3}{\lambda_3}(K_3P_1 - K_1P_3) + \frac{\gamma_2}{\lambda_2}\frac{\gamma_3}{\lambda_3}(L_3P_2 - L_2P_3)\right),$$

$$\omega_3 = \frac{T^3}{2}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}\frac{\gamma_3}{\lambda_3}\left(K_1(L_2P_3 - L_3P_2) + K_2(L_3P_1 - L_1P_3) + K_3(L_1P_2 - L_2P_1)\right),$$

$$\omega_4 = 8{\omega_1}^3 + 180\omega_3 + 36\omega_1\omega_2 + 12(81{\omega_3}^2 + 12\omega_3{\omega_1}^3 + 54\omega_1\omega_2\omega_3 - 3{\omega_1}^2{\omega_2}^2 - 12{\omega_2}^3)^{(1/2)})^{(1/3)}.$$

Define a sequence of functions:

$\{x_m(t,x_0,y_0,z_0), y_m(t,x_0,y_0,z_0), z_m(t,x_0,y_0,z_0)\}_{m=0}^{\infty}$ by:

$$\begin{aligned} x_{m+1}(t,x_0,y_0,z_0) = x_0e^{At} + \int_0^t e^{A(t-s)}\,[f(s,x_m(s,x_0,y_0,z_0),\ y_m(s,x_0,y_0,z_0), z_m(s,x_0,y_0,z_0)) - \alpha \\ - \frac{A}{(e^{AT}-E)}\int_0^T e^{A(T-s)}\, f(s,x_m(s,x_0,y_0,z_0), y_m(s,x_0,y_0,z_0),\ z_m(s,x_0,y_0,z_0))ds]ds \end{aligned} \tag{9}$$

with

$x_0(t,x_0,y_0,z_0) = {}_{x_0}e^{\mathrm{At}}$

where $\alpha = \dfrac{\mathrm{A}}{A_2(e^{\mathrm{AT}}-\mathrm{E})}[A_1x_0 + A_2x_0e^{\mathrm{AT}} - e_1]$; $det(e^{\mathrm{AT}}-\mathrm{E}) \neq 0$ and $det(A_2(e^{\mathrm{AT}}-\mathrm{E})) \neq 0$.

$$y_{m+1}(t,x_0,y_0,z_0) = y_oe^{Bt} + \int_0^t e^{B(t-s)}\,[g(s,x_m(s,x_0,y_0,z_0),,y_m(s,x_0,y_0,z_0)\,,z_m(s,x_0,y_0,z_0))$$
$$-\beta - \frac{\mathrm{B}}{(e^{\mathrm{BT}}-\mathrm{E})}\int_0^T e^{B(T-s)}\,g(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0)\,,z_m(s,x_0,y_0,z_0))ds]\,ds \qquad (10)$$

With

$y_0(t,x_0,y_0,z_0) = y_oe^{Bt}$

where $\beta = \dfrac{\mathrm{B}}{B_2(e^{\mathrm{BT}}-\mathrm{E})}[B_1y_0 + B_2y_oe^{BT} - e_2]$; $\det(e^{\mathrm{BT}}-\mathrm{E}) \neq 0$

and $\det\big(B_2(e^{\mathrm{BT}}-\mathrm{E})\big) \neq 0$.

and

$$z_{m+1}(t,x_0,y_0,z_0) = y_oe^{Ct} + \int_0^t e^{C(t-s)}\,[h\,(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0)\,,z_m(s,x_0,y_0,z_0))$$
$$-\delta - \frac{\mathrm{C}}{(e^{\mathrm{CT}}-\mathrm{E})}\int_0^T e^{C(T-s)}\,h(\,s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0),z_m(s,x_0,y_0,z_0))ds]ds \qquad (11)$$

with

$z_0(t,x_0,y_0,z_0) = z_oe^{Ct}$

where $\delta = \frac{\mathrm{C}}{C_2(e^{\mathrm{CT}}-\mathrm{E})}[C_1y_0 + C_2y_oe^{CT} - e_3]$; $det(e^{CT}-E) \neq 0$ and $det\Big(C_2(e^{\mathrm{CT}}-\mathrm{E})\Big) \neq 0$. m = 0, 1, 2, … ,

By using lemma3.1 [5], we can state and proof the following lemma:

Lemma: 1 Suppose that the functions $f(t,x,y\,,z), g(t,x,y\,,z)$ and $f(t,x,y\,,z)$ be vectors which are defined in the interval $[0,T]$, then the following inequality holds:

$$\begin{pmatrix} \|F_1(t,x_0,y_0,z_0)\| \\ \|F_2(t,x_0,y_0,z_0)\| \\ \|F_3(t,x_0,y_0,z_0)\| \end{pmatrix} \leq \begin{pmatrix} \alpha_1(t)\dfrac{\gamma_1}{\lambda_1}M_1 \\ \alpha_2(t)\dfrac{\gamma_2}{\lambda_2}M_2 \\ \alpha_3(t)\dfrac{\gamma_3}{\lambda_3}M_3 \end{pmatrix}, \qquad (12)$$

for $0 \leq t \leq T$, $\alpha_1(t) \leq \frac{T}{2}$, $\alpha_2(t) \leq \frac{T}{2}$, $\alpha_3(t) \leq \frac{T}{2}$,

where

$$F_1(t,x_0,y_0,z_0) = \int_0^t e^{A(t-s)}\,[f(s,x_0,y_0,z_0) - \frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f(s,x_0,y_0,z_0)ds]ds$$

$$F_2(t,x_0,y_0,z_0) = \int_0^t e^{B(t-s)}\,[g(s,x_0,y_0,z_0) - \frac{\mathrm{B}}{(e^{\mathrm{BT}}-\mathrm{E})}\int_0^T e^{B(T-s)}\,g(s,x_0,y_0,z_0)ds]ds$$

$$F_3(t,x_0,y_0,z_0) = \int_0^t e^{C(t-s)}\,[h(s,x_0,y_0,z_0) - \delta - \frac{\mathrm{C}}{(e^{\mathrm{CT}}-\mathrm{E})}\int_0^T e^{C(T-s)}\,h(s,x_0,y_0,z_0)ds\,]ds$$

and

$$\alpha_1(t) = \frac{t\big(2e^{\|A\|(T-t)} - e^{\|A\|T} - \|E\|\big) + T\big(e^{\|A\|T} - e^{\|A\|(T-t)}\big)}{(e^{\|A\|T} - \|E\|)}$$

$$\alpha_2(t) = \frac{t\big(2e^{\|B\|(T-t)} - e^{\|B\|T} - \|E\|\big) + T\big(e^{\|B\|T} - e^{\|B\|(T-t)}\big)}{(e^{\|B\|T} - \|E\|)}$$

$$\alpha_3(t) = \frac{t\big(2e^{\|C\|(T-t)} - e^{\|C\|T} - \|E\|\big) + T\big(e^{\|C\|T} - e^{\|C\|(T-t)}\big)}{(e^{\|C\|T} - \|E\|)}$$

Proof:

$$\|F_1(t,x_0,y_0,z_0)\| \le \left(\|E\| - \frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\int_0^t \left\|e^{A(t-s)}\right\| \|f(s,x_0,y_0,z_0)\| ds$$
$$+ \left(\frac{e^{\|A\|T} - e^{\|A\|(T-t)}}{e^{\|A\|T} - \|E\|}\right)\int_t^T \left\|e^{A(t-s)}\right\| \|f(s,x_0,y_0,z_0)\| \, ds$$
$$\le \alpha_1(t)\frac{\gamma_1}{\lambda_1} M_1 \tag{13}$$

and similarly

$$\|F_2(t,x_0,y_0,z_0)\| \le \alpha_2(t)\frac{\gamma_2}{\lambda_2} M_2 \tag{14}$$

and

$$\|F_3(t,x_0,y_0,z_0)\| \le \alpha_3(t)\frac{\gamma_3}{\lambda_3} M_3 \tag{15}$$

From (13), (14) and (15) we conclude that the inequality (12) holds . ∎

II. APPROXIMATION OF PERIODIC SOLUTION FOR (P)

The investigation of approximate solution of (P) will be introduced by the following theorem:

Theorem: 1 Let the vector functions $f(t,x,y,z)$, $g(t,x,y,z)$ and $h(t,x,y,z)$ are defined and continuous on the domain (1) and periodic in t of period T. Suppose that these functions satisfy the inequalities (2), (3), (4), (5) and the conditions (6), (7) and (8), then there exist a sequences of functions (9), (10) and (11), converges uniformly on the domain:

$$(t,x_0,y_0,z_0) \in [0,T] \times D_\alpha \times D_\beta \times D_\delta \tag{16}$$

to the limit functions $\begin{pmatrix} x^0(t,x_0,y_0,z_0) \\ y^0(t,x_0,y_0,z_0) \\ z^0(t,x_0,y_0,z_0) \end{pmatrix}$ wich is continuous in the domain (1) periodic in t of period T and satisfies the following vector form:

$$\begin{pmatrix} x(t,x_0,y_0,z_0) \\ y(t,x_0,y_0,z_0) \\ z(t,x_0,y_0,z_0) \end{pmatrix} = \begin{pmatrix} x_0 e^{At} + \int_0^t e^{A(t-s)}\left[f(s,x,y,z) - \alpha - \frac{A}{(e^{AT}-E)}\int_0^T e^{A(T-s)} f(s,x,y,z)ds\right]ds \\ y_o e^{Bt} + \int_0^t e^{B(t-s)}\left[g(s,x,y,z) - \beta - \frac{B}{(e^{BT}-E)}\int_0^T e^{B(T-s)} g(s,x,y,z)ds\right]ds \\ z_o e^{Ct} + \int_0^t e^{C(t-s)}\left[h(s,x,y,z) - \delta - \frac{C}{(e^{CT}-E)}\int_0^T e^{C(T-s)} h(s,x,y,z)ds\right]ds \end{pmatrix}, \tag{17}$$

and it is a unique solution of (BVP) which satisfies the following inequality:

$$\begin{pmatrix} \|x^0(t,x_0,y_0,z_0) - x_m(t,x_0,y_0,z_0)\| \\ \|y^0(t,x_0,y_0,z_0) - y_m(t,x_0,y_0,z_0)\| \\ \|z^0(t,x_0,y_0,z_0) - z_m(t,x_0,y_0,z_0)\| \end{pmatrix} \le \Omega^m (E\text{-}\Omega)^{-1}\Psi_1$$

where $\Psi_1 = \begin{pmatrix} \frac{T}{2}\frac{\gamma_1}{\lambda_1}M_1 + \frac{\gamma_1}{\lambda_1}\|\alpha\|T \\ \frac{T}{2}\frac{\gamma_2}{\lambda_2}M_2 + \frac{\gamma_2}{\lambda_2}\|\beta\|T \\ \frac{T}{2}\frac{\gamma_3}{\lambda_3}M_3 + \frac{\gamma_3}{\lambda_3}\|\delta\|T \end{pmatrix}$

provided that

$$\begin{pmatrix} \|x(t,x_0,y_0,z_0) - x_0\| \\ \|y(t,x_0,y_0,z_0) - y_0\| \\ \|z(t,x_0,y_0,z_0) - z_0\| \end{pmatrix} \le \begin{pmatrix} \frac{T}{2}\frac{\gamma_1}{\lambda_1}M_1 + \frac{\gamma_1}{\lambda_1}\|\alpha\|T \\ \frac{T}{2}\frac{\gamma_2}{\lambda_2}M_2 + \frac{\gamma_2}{\lambda_2}\|\beta\|T \\ \frac{T}{2}\frac{\gamma_3}{\lambda_3}M_3 + \frac{\gamma_3}{\lambda_3}\|\delta\|T \end{pmatrix} \tag{18}$$

for all $t \in [0,T]$ and $x_0 \in D_\alpha$, $y_0 \in D_\beta$ and $z_0 \in D_\delta$

Proof: Setting m=0 in (9), (10) and (11) and by using Lemma 1, we have

$$\|x_1(t,x_0,y_0,z_0)-x_0\| = \left\| x_0 e^{\mathrm{At}} + \int_0^t e^{A(t-s)}\,[f(s,x_0,y_0,z_0)-\alpha - \frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f(s,x_0,y_0,z_0)ds]ds - x_0 e^{\mathrm{At}} \right\|$$

$$\le \left\| \int_0^t e^{A(t-s)}\,[f(s,x_0,y_0,z_0) - \frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f(s,x_0,y_0,z_0)ds]ds \right\| + \left\| \int_0^t e^{A(t-s)}\,\alpha\, ds \right\|$$

$$\le \alpha_1(t)\frac{\gamma_1}{\lambda_1}M_1 + \frac{\gamma_1}{\lambda_1}\|\alpha\|T$$

Hence $x_1(t,x_0,y_0,z_0) \in D_\alpha$ for all t ∈ [0,T]

Then by mathematical induction we can prove that

$$\|x_m(t,x_0,y_0,z_0)-x_0\| \le \alpha_1(t)\frac{\gamma_1}{\lambda_1}M_1 + \frac{\gamma_1}{\lambda_1}\|\alpha\|T \tag{19}$$

Which given $x_m(t,x_0,y_0,z_0) \in D_\alpha$ for all t ∈ [0,T] .

Similarly

$$\|y_1(t,x_0,y_0,z_0)-y_0\| \le \alpha_2(t)\frac{\gamma_2}{\lambda_2}M_2 + \frac{\gamma_2}{\lambda_2}\|\beta\|T$$

Hence $y_1(t,x_0,y_0,z_0) \in D_\beta$ for all t ∈ [0,T]

and

$$\|y_m(t,x_0,y_0,z_0)-y_0\| \le \alpha_2(t)\frac{\gamma_1}{\lambda_1}M_2 + \frac{\gamma_1}{\lambda_1}\|\beta\|T \tag{20}$$

then $y_m(t,x_0,y_0,z_0) \in D_\beta$ for all t ∈ [0,T] .

and

$$\|z_1(t,x_0,y_0,z_0)-z_0\| \le \alpha_3(t)\frac{\gamma_3}{\lambda_3}M_3 + \frac{\gamma_3}{\lambda_3}\|\delta\|T$$

Hence
$z_1(t,x_0,y_0,z_0) \in D_\delta$ for all t ∈ [0,T] ,

Also

$$\|z_m(t,x_0,y_0,z_0)-z_0\| \le \alpha_3(t)\frac{\gamma_3}{\lambda_3}M_3 + \frac{\gamma_3}{\lambda_3}\|\delta\|T \tag{21}$$

Then $z_m(t,x_0,y_0,z_0) \in D_\delta$ for all t ∈ [0,T] .

Next, we shall prove that the sequence of functions (9), 10) and (11) converges uniformly on the domain (16).

When m= 0 and by using Lemma 1 and the inequalities (3), (4) and (5) ,we have

$$\|x_2(t,x_0,y_0,z_0)-x_1(t,x_0,y_0,z_0)\| \le \left(\|E\| - \frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|\mathrm{A}\|\mathrm{T}}-\|E\|}\right)\int_0^t \left\|e^{A(t-s)}\right\| \|f(s,x_1(s,x_0,y_0,z_0), ,y_1(s,x_0,y_0,z_0),z_1(s,x_0,y_0,z_0)) - f(s,x_0,y_0,z_0)\|ds$$

$$+\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|A\|T}-\|E\|}\right)\int_t^T \left\|e^{A(t-s)}\right\| \|f(s,x_1(s,x_0,y_0,z_0),y_1(s,x_0,y_0,z_0)\,,$$
$$,z_1(s,x_0,y_0,z_0)\big)-f(s,x_0,y_0,z_0)\|\,ds$$

$$\leq \alpha_1(t)\frac{\gamma_1}{\lambda_1}(K_1\|x_1(t,x_0,y_0,z_0)-x_0\|+K_2\|y_1(t,x_0,y_0,z_0)-y_0\|$$
$$+K_3\|z_1(t,x_0,y_0,z_0)-z_0\|)$$

Then by mathematical induction we can prove that

$$\|x_{m+1}(t,x_0,y_0,z_0)-x_m(t,x_0,y_0,z_0)\| \leq \alpha_1(t)\frac{\gamma_1}{\lambda_1}(K_1\|x_m(t,x_0,y_0,z_0)-x_{m-1}(t,x_0,y_0,z_0)\|$$
$$+K_2\|y_m(t,x_0,y_0,z_0)-y_{m-1}(t,x_0,y_0,z_0)\|$$
$$+K_3\|z_m(t,x_0,y_0,z_0)-z_{m-1}(t,x_0,y_0,z_0)\|) \tag{22}$$

And similarly

$$\|y_2(t,x_0,y_0,z_0)-y_1(t,x_0,y_0,z_0)\| \leq \alpha_2(t)\frac{\gamma_2}{\lambda_2}(L_1\|x_1(t,x_0,y_0,z_0)-x_0\|+L_2\|y_1(t,x_0,y_0,z_0)-y_0\|$$
$$+L_3\|z_1(t,x_0,y_0,z_0)-z_0\|)$$

and

$$\|y_{m+1}(t,x_0,y_0,z_0)-y_m(t,x_0,y_0,z_0)\| \leq \alpha_2(t)\frac{\gamma_2}{\lambda_2}(L_1\|x_m(t,x_0,y_0,z_0)-x_{m-1}(t,x_0,y_0,z_0)\|$$
$$+L_2\|y_m(t,x_0,y_0,z_0)-y_{m-1}(t,x_0,y_0,z_0)\|$$
$$+L_3\|z_m(t,x_0,y_0,z_0)-z_{m-1}(t,x_0,y_0,z_0)\|) \tag{23}$$

Also

$$\|z_2(t,x_0,y_0,z_0)-z_1(t,x_0,y_0,z_0)\| \leq \alpha_3(t)\frac{\gamma_3}{\lambda_3}(P_1\|x_1(t,x_0,y_0,z_0)-x_0\|+P_2\|y_1(t,x_0,y_0,z_0)-y_0\|$$
$$+P_3\|z_1(t,x_0,y_0,z_0)-z_0\|)$$

Also by mathematical induction we can prove that

$$\|z_{m+1}(t,x_0,y_0,z_0)-z_m(t,x_0,y_0,z_0)\| \leq \alpha_3(t)\frac{\gamma_3}{\lambda_3}(P_1\|x_m(t,x_0,y_0,z_0)-x_{m-1}(t,x_0,y_0,z_0)\|$$
$$+P_2\|y_m(t,x_0,y_0,z_0)-y_{m-1}(t,x_0,y_0,z_0)\|$$
$$+P_3\|z_m(t,x_0,y_0,z_0)-z_{m-1}(t,x_0,y_0,z_0)\|)\,. \tag{24}$$

Rewrite (22), (23) and (24) by vector form i. e.

$$\Psi_{m+1}(t) \leq \Omega(t)\Psi_m(t) \tag{25}$$

$$\Psi_{m+1}=\begin{pmatrix}\|x_{m+1}(t,x_0,y_0,z_0)-x_m(t,x_0,y_0,z_0)\|\\ \|y_{m+1}(t,x_0,y_0,z_0)-y_m(t,x_0,y_0,z_0)\|\\ \|z_{m+1}(t,x_0,y_0,z_0)-z_m(t,x_0,y_0,z_0)\|\end{pmatrix}$$

$$\Psi_m=\begin{pmatrix}\|x_m(t,x_0,y_0,z_0)-x_{m-1}(t,x_0,y_0,z_0)\|\\ \|y_m(t,x_0,y_0,z_0)-y_{m-1}(t,x_0,y_0,z_0)\|\\ \|z_m(t,x_0,y_0,z_0)-z_{m-1}(t,x_0,y_0,z_0)\|\end{pmatrix}$$

and

$$\Omega(t)=\begin{pmatrix}K_1\frac{\gamma_1}{\lambda_1}\alpha_1(t) & K_2\frac{\gamma_1}{\lambda_1}\alpha_1(t) & K_3\frac{\gamma_1}{\lambda_1}\alpha_1(t)\\ L_1\frac{\gamma_2}{\lambda_2}\alpha_2(t) & L_2\frac{\gamma_2}{\lambda_2}\alpha_2(t) & L_3\frac{\gamma_2}{\lambda_2}\alpha_2(t)\\ P_1\frac{\gamma_3}{\lambda_3}\alpha_3(t) & P_2\frac{\gamma_3}{\lambda_3}\alpha_3(t) & P_3\frac{\gamma_3}{\lambda_3}\alpha_3(t)\end{pmatrix}$$

Now we take the maximum value for the both sides of the inequalities (25) we get

$$\Psi_{m+1} \leq \Omega\Psi_m \tag{26}$$

where $\Omega = \max_{t\in[0,T]} \Omega(t)$

$$\Omega = \begin{pmatrix} K_1\frac{\gamma_1}{\lambda_1}\frac{T}{2} & K_2\frac{\gamma_1}{\lambda_1}\frac{T}{2} & K_3\frac{\gamma_1}{\lambda_1}\frac{T}{2} \\ L_1\frac{\gamma_2}{\lambda_2}\frac{T}{2} & L_2\frac{\gamma_2}{\lambda_2}\frac{T}{2} & L_3\frac{\gamma_2}{\lambda_2}\frac{T}{2} \\ P_1\frac{\gamma_3}{\lambda_3}\frac{T}{2} & P_2\frac{\gamma_3}{\lambda_3}\frac{T}{2} & P_3\frac{\gamma_3}{\lambda_3}\frac{T}{2} \end{pmatrix},$$

and by repetition of (26) we find that $\Psi_{m+1} \le \Omega^m \Psi_1$ and also we get

$$\sum_{i=1}^{m} \Psi_i \le \sum_{i=1}^{m} \Omega^{i-1}\Psi_1 \quad , \tag{27}$$

by using (8) then the sequence (27) is uniformly convergent that is

$$\lim_{m\to\infty} \sum_{i=1}^{m} \Omega^{i-1}\Psi_1 = \sum_{i=1}^{\infty} \Omega^{i-1}\Psi_1 = (E-\Omega)^{-1}\Psi_1 \tag{28}$$

Let

$$\lim_{m\to\infty} \begin{pmatrix} x_m(t,x_0,y_0,z_0) \\ y_m(t,x_0,y_0,z_0) \\ z_m(t,x_0,y_0,z_0) \end{pmatrix} = \begin{pmatrix} x^0(t,x_0,y_0,z_0) \\ y^0(t,x_0,y_0,z_0) \\ z^0(t,x_0,y_0,z_0) \end{pmatrix} \tag{29}$$

Since the sequence of function (9), (10) and (11) is defined and continuous in the domain (16) then the limiting function $\begin{pmatrix} x^0(t,x_0,y_0,z_0) \\ y^0(t,x_0,y_0,z_0) \\ z^0(t,x_0,y_0,z_0) \end{pmatrix}$ is also defined and continuous in the same domain.

Moreover, by Lemma 1, the relation (29) and proceeding (9), (10) and (11) to the limit $\begin{pmatrix} x^0(t,x_0,y_0,z_0) \\ y^0(t,x_0,y_0,z_0) \\ z^0(t,x_0,y_0,z_0) \end{pmatrix}$ when m→∞ the equality (28) is satisfied for all $m \ge 0$, and this show that the limiting function $\begin{pmatrix} x^0(t,x_0,y_0,z_0) \\ y^0(t,x_0,y_0,z_0) \\ z^0(t,x_0,y_0,z_0) \end{pmatrix}$ is the solution of (P).

Finally, we have to show that $\begin{pmatrix} x(t,x_0,y_0,z_0) \\ y(t,x_0,y_0,z_0) \\ z(t,x_0,y_0,z_0) \end{pmatrix}$ is a unique solution of (BVP).

Let $\begin{pmatrix} \overline{x}(t,x_0,y_0,z_0) \\ \overline{y}(t,x_0,y_0,z_0) \\ \overline{z}(t,x_0,y_0,z_0) \end{pmatrix}$ be another solution of (P), i. e.

$$\overline{x}(t,x_0,y_0,z_0) = x_0 e^{At} + \int_0^t e^{A(t-s)} [f(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0), \overline{z}(s,x_0,y_0,z_0)) - \alpha$$
$$- \tfrac{A}{(e^{AT}-E)} \int_0^T e^{A(T-s)} f(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),\overline{z}(s,x_0,y_0,z_0))ds]ds$$

$$\overline{y}(t,x_0,y_0,z_0) = y_o e^{Bt} + \int_0^t e^{B(t-s)} [g(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0), \overline{z}(s,x_0,y_0,z_0)) - \beta$$
$$- \frac{B}{(e^{BT}-E)} \int_0^T e^{B(T-s)} g(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),\overline{z}(s,x_0,y_0,z_0)ds]ds$$

$$\overline{z}(t,x_0,y_0,z_0) = z_o e^{Ct} + \int_0^t e^{C(t-s)} [h(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),\overline{z}(s,x_0,y_0,z_0)) - \beta$$
$$- \frac{C}{(e^{CT}-E)} \int_0^T e^{C(T-s)} h(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),\overline{z}(s,x_0,y_0,z_0))ds]ds$$

$$\|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\| \le \Big\|\int_0^t e^{A(T-s)}\,[f(s,x(s,x_0,y_0,z_0),y(s,x_0,y_0,z_0),z(s,x_0,y_0,z_0)) \\ -f\big(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),\overline{z}(s,x_0,y_0,z_0)\big) \\ -\frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f\big(s,x(s,x_0,y_0,z_0),y(s,x_0,y_0,z_0),z(s,x_0,y_0,z_0)\big) \\ -f\big(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),\overline{z}(s,x_0,y_0,z_0)\big)ds\big]ds\Big\|$$

$$\le \left(\|E\|-\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|\mathrm{A}\|\mathrm{T}}-\|E\|}\right)\int_0^t \big\|e^{A(t-s)}\big\|\,\|\,f(s,x(s,x_0,y_0,z_0), \\ ,y_1(s,x_0,y_0,z_0),z(s,x_0,y_0,z_0))-f(s,\overline{x}(s,x_0,y_0,z_0), \\ ,\overline{y}(s,x_0,y_0,z_0),\overline{z}(s,x_0,y_0,z_0))\|ds+ \\ +\left(\frac{e^{\|A\|T}-e^{\|A\|(T-t)}}{e^{\|\mathrm{A}\|\mathrm{T}}-\|E\|}\right)\int_t^T \big\|e^{A(t-s)}\big\|\|\,f(s,x(s,x_0,y_0,z_0),y(s,x_0,y_0,z_0), \\ ,z(s,x_0,y_0,z_0)-f(s,\overline{x}(s,x_0,y_0,z_0),\overline{y}(s,x_0,y_0,z_0),\overline{z}(s,x_0,y_0,z_0)\|ds$$

$$\therefore \|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\| \le \alpha_1(t)\frac{\gamma_1}{\lambda_1}(K_1\|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\| \\ +K_2\|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\|+ \\ +K_3\|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\|) \tag{30}$$

Now similarly

$$\|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\| \le \alpha_2(t)\frac{\gamma_2}{\lambda_2}(L_1\|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\| \\ +L_2\|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\|+ \\ +L_3\|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\|) \tag{31}$$

Also

$$\|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\| \le \alpha_3(t)\frac{\gamma_3}{\lambda_3}(P_1\|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\| \\ +P_2\|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\| \\ +P_3\|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\|) \tag{32}$$

Then we can rewrite the inequalities (30), (31) and (32) by the vector form:

$$\begin{pmatrix}\|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\|\\ \|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\|\\ \|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\|\end{pmatrix} \le \Omega\begin{pmatrix}\|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\|\\ \|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\|\\ \|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\|\end{pmatrix} \tag{33}$$

Then by the condition (8),

$$\begin{pmatrix}\|x(t,x_0,y_0,z_0)-\overline{x}(t,x_0,y_0,z_0)\|\\ \|y(t,x_0,y_0,z_0)-\overline{y}(t,x_0,y_0,z_0)\|\\ \|z(t,x_0,y_0,z_0)-\overline{z}(t,x_0,y_0,z_0)\|\end{pmatrix} \to \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\therefore \begin{pmatrix}x(t,x_0,y_0,z_0)\\ y(t,x_0,y_0,z_0)\\ z(t,x_0,y_0,z_0)\end{pmatrix} = \begin{pmatrix}\overline{x}(t,x_0,y_0,z_0)\\ \overline{y}(t,x_0,y_0,z_0)\\ \overline{z}(t,x_0,y_0,z_0)\end{pmatrix}$$

which proves the solution is a unique and this complete the proof.

III. EXISTENCE OF PERIODIC SOLUTION FOR (P)

The problem of existence solution of (P) is uniquely connected with existence of zero of the functions $\Delta_1(x_0,y_0,z_0)$, $\Delta_2(x_0,y_0,z_0)$ and $\Delta_3(x_0,y_0,z_0)$ which has the form:

$$\Delta_1(x_0,y_0,z_0)=\alpha+\frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0),z^0(s,x_0,y_0,z_0)\big)ds \tag{34}$$

$\Delta_1\colon D_\alpha\times D_\beta\times D_\delta\to R^n$

$$\Delta_2(x_0,y_0,z_0)=\beta+\frac{\mathrm{B}}{(e^{\mathrm{BT}}-\mathrm{E})}\int_0^T e^{B(T-s)}\,g(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0),z^0(s,x_0,y_0,z_0)\big)ds \tag{35}$$

$\Delta_2: D_\alpha \times D_\beta \times D_\delta \to R^n$

$$\Delta_3(x_0,y_0,z_0) = \delta + \frac{\mathrm{C}}{(e^{\mathrm{CT}}-\mathrm{E})}\int_0^T e^{C(T-s)}\, h(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0),z^0(s,x_0,y_0,z_0))ds \tag{36}$$

$\Delta_3: D_\alpha \times D_\beta \times D_\delta \to R^n$

The functions $\Delta_1(x_0,y_0,z_0)$, $\Delta_2(x_0,y_0,z_0)$ and $\Delta_3(x_0,y_0,z_0)$ are approximately determined from the following sequences:

$$\Delta_{1m}(x_0,y_0,z_0) = \alpha + \frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\, f(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0),z_m(s,x_0,y_0,z_0))ds \tag{37}$$

$\Delta_{1m}: D_\alpha \times D_\beta \times D_\delta \to R^n$

$$\Delta_{2m}(x_0,y_0,z_0) = \beta + \frac{\mathrm{B}}{(e^{\mathrm{BT}}-\mathrm{E})}\int_0^T e^{B(T-s)}\, g(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0),z_m(s,x_0,y_0,z_0))ds \tag{38}$$

$\Delta_{2m}: D_\alpha \times D_\beta \times D_\delta \to R^n$

$$\Delta_{3m}(x_0,y_0,z_0) = \delta + \frac{\mathrm{C}}{(e^{\mathrm{CT}}-\mathrm{E})}\int_0^T e^{C(T-s)}\, h(s,x_m(s,x_0,y_0,z_0),y_m(s,x_0,y_0,z_0),z_m(s,x_0,y_0,z_0))ds \tag{39}$$

$\Delta_{3m}: D_\alpha \times D_\beta \times D_\delta \to R^n$

Theorem: 2 Let all assumptions and conditions of Theorem 2.1 be satisfied then the following inequality:

$$\begin{pmatrix} \|\Delta_1(x_0,y_0,z_0)-\Delta_{1m}(x_0,y_0,z_0)\| \\ \|\Delta_2(x_0,y_0,z_0)-\Delta_{2m}(x_0,y_0,z_0)\| \\ \|\Delta_3(x_0,y_0,z_0)-\Delta_{3m}(x_0,y_0,z_0)\| \end{pmatrix} \le \begin{pmatrix} N_1\frac{\gamma_1}{\lambda_1}K_1 & N_1\frac{\gamma_1}{\lambda_1}K_2 & N_1\frac{\gamma_1}{\lambda_1}K_3 \\ N_2\frac{\gamma_2}{\lambda_2}L_1 & N_2\frac{\gamma_2}{\lambda_2}L_2 & N_2\frac{\gamma_2}{\lambda_2}L_3 \\ N_3\frac{\gamma_3}{\lambda_3}P_1 & N_3\frac{\gamma_3}{\lambda_3}P_2 & N_3\frac{\gamma_3}{\lambda_3}P_3 \end{pmatrix} \Omega^m(E-\Omega)^{-1}\Psi_1, \tag{40}$$

holds

where $N_1 = \dfrac{\|\mathrm{A}\|\mathrm{T}}{(e^{\|\mathrm{A}\|\mathrm{T}}-\|E\|)}, N_2 = \dfrac{\|\mathrm{B}\|\mathrm{T}}{(e^{\|\mathrm{B}\|\mathrm{T}}-\|E\|)}, N_3 = \dfrac{\|\mathrm{C}\|\mathrm{T}}{(e^{\|\mathrm{C}\|\mathrm{T}}-\|E\|)}$

Proof: From the equations (34) and (37) we have

$$\begin{aligned}\|\Delta_1(x_0,y_0,z_0)-\Delta_{1m}(x_0,y_0,z_0)\| = {} & \|\alpha + \frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\, f(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0), \\ & z^0(s,x_0,y_0,z_0))ds - \alpha - \frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\, f(\, s,x_m(s,x_0,y_0,z_0), \\ & ,y_m(s,x_0,y_0,z_0),z_m(s,x_0,y_0,z_0))ds\| \\ & \le N_1\frac{\gamma_1}{\lambda_1}K_1\|x^0(t,x_0,y_0,z_0)-x_m(t,x_0,y_0,z_0)\| \\ & +N_1\frac{\gamma_1}{\lambda_1}K_2\|y^0(t,x_0,y_0,z_0)-y_m(t,x_0,y_0,z_0)\| \\ & +N_1\frac{\gamma_1}{\lambda_1}K_3\|z^0(t,x_0,y_0,z_0)-z_m(t,x_0,y_0,z_0)\|\end{aligned}$$

$$\|\Delta_1(x_0,y_0,z_0)-\Delta_{1m}(x_0,y_0,z_0)\| \le \langle (N_1\frac{\gamma_1}{\lambda_1}K_1 \;\; N_1\frac{\gamma_1}{\lambda_1}K_2 \;\; N_1\frac{\gamma_1}{\lambda_1}K_3)\,,\, \Omega^m(E-\Omega)^{-1}\Psi_1\rangle = q_m \tag{41}$$

Similarly from the equation (35) and (38) we have

$$\|\Delta_2(x_0,y_0,z_0)-\Delta_{2m}(x_0,y_0,z_0)\| \le \langle (N_2\frac{\gamma_2}{\lambda_2}P_1 \;\; N_2\frac{\gamma_2}{\lambda_2}P_2 \;\; N_2\frac{\gamma_2}{\lambda_2}P_3)\,,\Omega^m(E-\Omega)^{-1}\Psi_1\rangle \quad = v_m \tag{42}$$

and

$$\|\Delta_3(x_0,y_0,z_0)-\Delta_{3m}(x_0,y_0,z_0)\| \le \langle \left(N_3\frac{\gamma_3}{\lambda_3}P_1 \quad N_3\frac{\gamma_3}{\lambda_3}P_2 \quad N_3\frac{\gamma_3}{\lambda_3}P_3 \right), \Omega^m (E-\Omega)^{-1}\Psi_1 \rangle \;=\; w_m \tag{43}$$

We rewrite (41), (42) and (43) by the vector form, Then we get (40).

Now, we prove the following theorem taking into account that the inequality (41), (42) and (43) will be satisfied for all $m \ge 0$.

Theorem: 3 Let the (P) be defined in the intervals $[a,b]$, $[c,d]$ and $[\,i,\mathrm{j}]$ on R^1 and periodic in t of period T, suppose that for $m \ge 0$ the sequences of functions $\Delta_{1m}(x_0,y_0,z_0)$, $\Delta_{2m}(x_0,y_0,z_0)$ and $\Delta_{3m}(x_0,y_0,z_0)$ which are defined in (37), (38) and (39) satisfy the inequalities:

$$\left.\begin{array}{l} \min\limits_{\substack{x_0\in I_1\\ y_0\in I_2\\ z_0\in I_3}} \Delta_{1m}(x_0,y_0,z_0) \le -q_m \\ \max\limits_{\substack{x_0\in I_1\\ y_0\in I_2\\ z_0\in I_3}} \Delta_{1m}(x_0,y_0,z_0) \ge q_m \end{array}\right\} \tag{44}$$

$$\left.\begin{array}{l} \min\limits_{\substack{x_0\in I_1\\ y_0\in I_2\\ z_0\in I_3}} \Delta_{2m}(x_0,y_0,z_0) \le -v_m \\ \max\limits_{\substack{x_0\in I_1\\ y_0\in I_2\\ z_0\in I_3}} \Delta_{2m}(x_0,y_0,z_0) \ge v_m \end{array}\right\} \tag{45}$$

$$\left.\begin{array}{l} \min\limits_{\substack{x_0\in I_1\\ y_0\in I_2\\ z_0\in I_3}} \Delta_{3m}(x_0,y_0,z_0) \le -w_m \\ \max\limits_{\substack{x_0\in I_1\\ y_0\in I_2\\ z_0\in I_3}} \Delta_{3m}(x_0,y_0,z_0) \ge w_m \end{array}\right\} \tag{46}$$

Then the (P) has periodic solution $x=x(t,x_0,y_0,z_0)$, $y=y(t,x_0,y_0,z_0)$ and $z=z(t,x_0,y_0,z_0)$ suchthat:

$x_0 \in I_1 = \;[a+\frac{T}{2}\frac{\gamma_1}{\lambda_1}M_1+\frac{\gamma_1}{\lambda_1}\|\alpha\|T,\;\; b-\frac{T}{2}\frac{\gamma_1}{\lambda_1}M_1-\frac{\gamma_1}{\lambda_1}\|\alpha\|T]$

$y_0 \in I_2 = [\,c+\frac{T}{2}\frac{\gamma_2}{\lambda_2}M_2+\frac{\gamma_2}{\lambda_2}\|\beta\|T,\;\; d-\frac{T}{2}\frac{\gamma_2}{\lambda_2}M_2-\frac{\gamma_2}{\lambda_2}\|\beta\|T]$ and

$z_0 \in I_3 = [\,i+\frac{T}{2}\frac{\gamma_3}{\lambda_3}M_3+\frac{\gamma_3}{\lambda_3}\|\delta\|T,\;\; j-\frac{T}{2}\frac{\gamma_3}{\lambda_3}M_3-\frac{\gamma_3}{\lambda_3}\|\delta\|T]$

Proof: Let x_1,x_2 be any points in the interval I_1, y_1,y_2 be any points in the interval I_2, and z_1,z_2 be any points in the interval I_3.

$$\left.\begin{array}{l} \Delta_{1m}(x_1,y_1,z_1) = \min\limits_{\substack{x_0\in I_1\\ y_0\in I_2\\ z_0\in I_3}} \Delta_{1m}(x_0,y_0,z_0) \\ \Delta_{1m}(x_2,y_2,z_2) = \max\limits_{\substack{x_0\in I_1\\ y_0\in I_2\\ z_0\in I_3}} \Delta_{1m}(x_0,y_0,z_0) \end{array}\right\} \tag{47}$$

$$\left.\begin{aligned} \Delta_{2m}(x_1,y_1,z_1) &= \min_{\substack{x_0 \in I_1 \\ y_0 \in I_2 \\ z_0 \in I_3}} \Delta_{2m}(x_0,y_0,z_0) \\ \Delta_{2m}(x_2,y_2,z_2) &= \max_{\substack{x_0 \in I_1 \\ y_0 \in I_2 \\ z_0 \in I_3}} \Delta_{2m}(x_0,y_0,z_0) \end{aligned}\right\} \tag{48}$$

$$\left.\begin{aligned} \Delta_{3m}(x_1,y_1,z_1) &= \min_{\substack{x_0 \in I_1 \\ y_0 \in I_2 \\ z_0 \in I_3}} \Delta_{3m}(x_0,y_0,z_0) \\ \Delta_{3m}(x_2,y_2,z_2) &= \max_{\substack{x_0 \in I_1 \\ y_0 \in I_2 \\ z_0 \in I_3}} \Delta_{3m}(x_0,y_0,z_0) \end{aligned}\right\} \tag{49}$$

By using the inequalities(41), (42), (43), (44), (45) and (46) we have

$$\left.\begin{aligned} \Delta_1(x_1,y_1,z_1) &= \Delta_{1m}(x_1,y_1,z_1) + \big(\Delta_1(x_1,y_1,z_1) - \Delta_{1m}(x_1,y_1,z_1)\big) < 0 \\ \Delta_1(x_2,y_2,z_2) &= \Delta_{1m}(x_2,y_2,z_2) + (\Delta_1(x_2,y_2,z_2) - \Delta_{1m}(x_2,y_2,z_2)) > 0 \end{aligned}\right\} \tag{50}$$

$$\left.\begin{aligned} \Delta_2(x_1,y_1,z_1) &= \Delta_{2m}(x_1,y_1,z_1) + \big(\Delta_2(x_1,y_1,z_1) - \Delta_{2m}(x_1,y_1,z_1)\big) < 0 \\ \Delta_2(x_2,y_2,z_2) &= \Delta_{2m}(x_2,y_2,z_2) + (\Delta_2(x_2,y_2,z_2) - \Delta_{2m}(x_2,y_2,z_2)) > 0 \end{aligned}\right\} \tag{51}$$

$$\left.\begin{aligned} \Delta_3(x_1,y_1,z_1) &= \Delta_{3m}(x_1,y_1,z_1) + \big(\Delta_3(x_1,y_1,z_1) - \Delta_{3m}(x_1,y_1,z_1)\big) < 0 \\ \Delta_3(x_2,y_2,z_2) &= \Delta_{3m}(x_2,y_2,z_2) + (\Delta_3(x_2,y_2,z_2) - \Delta_{3m}(x_2,y_2,z_2)) > 0 \end{aligned}\right\} \tag{52}$$

And from the continuity of the functions $\Delta_1(x_1,y_1,z_1)$, $\Delta_2(x_2,y_2,z_2)$ and $\Delta_3(x_2,y_2,z_2)$ and the inequalities (50), (51) and (52) then there exist an isolated points $(x^0,y^0,z^0) = (x_0,y_0,z_0)$ and $x^0 \in [x_1, x_2]$, $y^0 \in [y_1, y_2]$ and $z^0 \in [z_1, z_2]$ where $\Delta_1(x_0,y_0,z_0) = \Delta_2(x_0,y_0,z_0) = \Delta_3(x_0,y_0,z_0) = 0$ this means that (P) has a periodic solution $x = x(t,x_0,y_0,z_0)$, $y = y(t,x_0,y_0,z_0)$ and $z = z(t,x_0,y_0,z_0)$.

V. STABILITY OF SOLUTION FOR (P) [4]

In this section, we study the stability theorems of a periodic solution for (P).

Theorem: 4 Let the vector functions $\Delta_1(x_0,y_0,z_0), \Delta_2(x_0,y_0,z_0)$ and $\Delta_3(x_0,y_0,z_0)$, be defined by the following forms:

$$\Delta_1(x_0,y_0,z_0) = \alpha + \frac{\mathrm{A}}{(e^{\mathrm{AT}} - \mathrm{E})}\int_0^T e^{A(T-s)} f(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0),z^0(s,x_0,y_0,z_0))ds \tag{53}$$

$\Delta_1 \colon D_\alpha \times D_\beta \times D_\delta \to R^n$

$$\Delta_2(x_0,y_0,z_0) = \beta + \frac{\mathrm{B}}{(e^{\mathrm{BT}} - \mathrm{E})}\int_0^T e^{B(T-s)} g(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0),z^0(s,x_0,y_0,z_0)ds \tag{54}$$

$\Delta_2 \colon D_\alpha \times D_\beta \times D_\delta \to R^n$

$$\Delta_3(x_0,y_0,z_0) = \delta + \frac{\mathrm{C}}{(e^{\mathrm{CT}} - \mathrm{E})}\int_0^T e^{C(T-s)} h(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0),z^0(s,x_0,y_0,z_0)ds \tag{55}$$

$\Delta_3 \colon D_\alpha \times D_\beta \times D_\delta \to R^n$

where $x^0(t,x_0,y_0,z_0), y^0(t,x_0,y_0,z_0)$ and $z^0(t,x_0,y_0,z_0)$ are the limits of the sequences (9), (10) and (11) respectively, then the following inequality holds:

$$\begin{pmatrix} \|\Delta_1(x_0,y_0,z_0)\| \\ \|\Delta_2(x_0,y_0,z_0)\| \\ \|\Delta_3(x_0,y_0,z_0)\| \end{pmatrix} \le \begin{pmatrix} N_1\frac{\gamma_1}{\lambda_1}M_1+\|\alpha\| \\ N_2\frac{\gamma_2}{\lambda_2}M_2+\|\beta\| \\ N_3\frac{\gamma_3}{\lambda_3}M_3+\|\delta\| \end{pmatrix} \tag{56}$$

Proof: From the properties of the functions $x^0(t,x_0,y_0,z_0), y^0(t,x_0,y_0,z_0)$ and $z^0(t,x_0,y_0,z_0)$ are fixative in Theorem2, then the functions $\Delta_1(x_0,y_0,z_0)$, $\Delta_2(x_0,y_0,z_0)$ and $\Delta_3(x_0,y_0,z_0)$ are continuous and bounded in the domain (1).

By using (53), we get

$$\begin{aligned}\|\Delta_1(x_0,y_0,z_0)\| &= \|\alpha+\frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}f(s,x^0(s,x_0,y_0,z_0),y^0(s,x_0,y_0,z_0),z^0(s,x_0,y_0,z_0)ds\| \\ &\le \frac{\|\mathrm{A}\|\mathrm{T}}{(e^{\|\mathrm{A}\|\mathrm{T}}-\|\mathrm{E}\|)}\frac{\gamma_1}{\lambda_1}M_1+\|\alpha\|\|\Delta_1(x_0,y_0,z_0)\| \\ &\le N_1\frac{\gamma_1}{\lambda_1}M_1+\|\alpha\| \end{aligned} \tag{57}$$

By using (54) we get

$$\|\Delta_2(x_0,y_0,z_0)\| \le N_2\frac{\gamma_2}{\lambda_2}M_2+\|\beta\| \tag{58}$$

And by using (55) we get

$$\|\Delta_3(x_0,y_0,z_0)\| \le N_3\frac{\gamma_3}{\lambda_3}M_3+\|\delta\| \tag{59}$$

then we rewrite (57), (58) and (59) by the vector form, we get (56)

Theorem: 5 Let all assumptions and conditions of Theorems 3, 4 be given, then for all $x_0,x_0^1,x_0^2 \in D_\alpha, y_0, y_0^1,y_0^2 \in D_\beta$ and $z_0,z_0^1,z_0^2 \in D_\delta$ the following inequality holds:

$$\begin{pmatrix} \|\Delta_1(x_0^1,y_0^1,z_0^1)-\Delta_1(x_0^2,y_0^2,z_0^2)\| \\ \|\Delta_2(x_0^1,y_0^1,z_0^1)-\Delta_2(x_0^2,y_0^2,z_0^2)\| \\ \|\Delta_3(x_0^1,y_0^1,z_0^1)-\Delta_3(x_0^2,y_0^2,z_0^2)\| \end{pmatrix} \le \begin{pmatrix} \mathrm{N}_1\frac{\gamma_1}{\lambda_1}E_1 & \mathrm{N}_1\frac{\gamma_1}{\lambda_1}E_2 & \mathrm{N}_1\frac{\gamma_1}{\lambda_1}E_3 \\ \mathrm{N}_2\frac{\gamma_2}{\lambda_2}E_4 & \mathrm{N}_2\frac{\gamma_2}{\lambda_2}E_5 & \mathrm{N}_2\frac{\gamma_2}{\lambda_2}E_6 \\ N_3\frac{\gamma_3}{\lambda_3}E_7 & N_3\frac{\gamma_3}{\lambda_3}E_8 & N_3\frac{\gamma_3}{\lambda_3}E_9 \end{pmatrix}\begin{pmatrix} \|x_0^1-x_0^2\| \\ \|y_0^1-y_0^2\| \\ \|z_0^1-z_0^2\| \end{pmatrix}, \tag{60}$$

where

$\mathrm{E}_1=\mathrm{K}_1(\mathrm{V}_9+\mathrm{V}_{13}+\mathrm{V}_{17})+\mathrm{H}_1,\quad \mathrm{E}_2=\mathrm{K}_1(\mathrm{V}_{10}+\mathrm{V}_{12}+\mathrm{V}_{16})$,
$\mathrm{E}_3=\mathrm{K}_1(\mathrm{V}_{11}+\mathrm{V}_{14}+\mathrm{V}_{15}),\quad \mathrm{E}_4=\mathrm{K}_2(\mathrm{V}_9+\mathrm{V}_{13}+\mathrm{V}_{17})$,
$\mathrm{E}_5=\mathrm{K}_2(\mathrm{V}_{10}+\mathrm{V}_{12}+\mathrm{V}_{16})+\mathrm{H}_2,\quad \mathrm{E}_6=\mathrm{K}_2(\mathrm{V}_{11}+\mathrm{V}_{14}+\mathrm{V}_{15})$,
$\mathrm{E}_7=\mathrm{K}_3(\mathrm{V}_9+\mathrm{V}_{13}+\mathrm{V}_{17}),\quad \mathrm{E}_8=\mathrm{K}_3(\mathrm{V}_{10}+\mathrm{V}_{12}+\mathrm{V}_{16})$,
$\mathrm{E}_9=\mathrm{K}_3(\mathrm{V}_{11}+\mathrm{V}_{14}+\mathrm{V}_{15})+\mathrm{H}_3$,

$$H_1=\frac{1}{\|A_2\|\mathrm{T}}\left[\|A_1\|\frac{\lambda_1}{\gamma_1}+\|A_2\|\right],\quad H_2=\frac{1}{\|B_2\|\mathrm{T}}\left[\|B_1\|\frac{\lambda_2}{\gamma_2}+\|B_2\|\right],$$

$$H_3=\frac{1}{\|C_2\|T}\left[\|C_1\|\frac{\lambda_3}{\gamma_3}+\|C_2\|\right],$$

$$V_1=W_4W_1Q_1, V_2=W_4W_2Q_2, V_3=\frac{T}{2}\frac{\gamma_2}{\lambda_2}L_1W_1W_2W_4Q_1,$$

$$\mathrm{V}_4=\mathrm{K}_2\mathrm{W}_1\mathrm{W}_2\mathrm{W}_4\mathrm{Q}_2,\quad \mathrm{V}_5=\frac{\mathrm{T}}{2}\frac{\gamma_3}{\lambda_3}\mathrm{P}_1\mathrm{W}_5\mathrm{W}_3\mathrm{W}_1\mathrm{Q}_1\mathrm{W}_5\mathrm{W}_3\mathrm{Q}_3,\quad \mathrm{V}_6=\mathrm{W}_5\mathrm{W}_3\mathrm{Q}_3,$$

$$\mathrm{V}_7=(1-\mathrm{G}_1\mathrm{G}_4)^{-1}, \mathrm{V}_8=\frac{\mathrm{T}}{2}\frac{\gamma_3}{\lambda_3}\mathrm{P}_2\mathrm{W}_6\mathrm{W}_3\mathrm{W}_2\mathrm{Q}_2,\quad \mathrm{V}_9=\mathrm{W}_7\mathrm{V}_1,$$

$\mathrm{V}_{10}=\mathrm{W}_7(\mathrm{V}_4+\mathrm{G}_1\mathrm{V}_8),\ \mathrm{V}_{11}=\mathrm{W}_7\mathrm{G}_1\mathrm{V}_7,\ \mathrm{V}_{12}=\mathrm{W}_8\mathrm{V}_2$,
$\mathrm{V}_{13}=\mathrm{W}_8(\mathrm{V}_3+\mathrm{G}_2\mathrm{V}_5),\ \mathrm{V}_{14}=\mathrm{W}_8\mathrm{G}_2\mathrm{V}_6,\ \mathrm{V}_{15}=\mathrm{W}_7\mathrm{V}_7$
$\mathrm{V}_{16}=\mathrm{W}_7(\mathrm{V}_8+\mathrm{G}_4\mathrm{V}_4), \mathrm{V}_{17}=\mathrm{W}_7\mathrm{G}_4\mathrm{V}_1$,

$$W_1=\left(1-\frac{T}{2}\frac{\gamma_1}{\lambda_1}K_1\right)^{-1},$$

$$W_2=\left(1-\frac{T}{2}\frac{\gamma_2}{\lambda_2}K_2\right)^{-1},\qquad W_3=\left(1-\frac{T}{2}\frac{\gamma_3}{\lambda_3}K_3\right)^{-1},$$

$$W_4 = \left(1 - \frac{T^2}{4}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}L_1K_2W_1W_2\right)^{-1}, \quad W_5 = \left(1 - \frac{T^2}{4}\frac{\gamma_3}{\lambda_3}\frac{\gamma_1}{\lambda_1}P_1K_3W_3W_1\right)^{-1},$$

$$W_6 = \left(1 - \frac{T^2}{4}\frac{\gamma_2}{\lambda_2}\frac{\gamma_3}{\lambda_3}P_2L_3W_3W_2\right)^{-1}, \qquad W_7 = (1 - G_4G_1)^{-1},$$

$$W_8 = (1 - G_2G_3)^{-1},$$

$$Q_1 = \frac{\|A\|\frac{\gamma_1}{\lambda_1}T}{\|A_2\|(e^{\|A\|T} - \|E\|)}\left[\|A_1\| + \|A_2\|\frac{\gamma_1}{\lambda_1}\right] + \frac{\gamma_1}{\lambda_1},$$

$$Q_2 = \frac{\|B\|\frac{\gamma_2}{\lambda_2}T}{\|B_2\|(e^{\|B\|T} - \|E\|)}\left[\|B_1\| + \|B_2\|\frac{\gamma_2}{\lambda_2}\right] + \frac{\gamma_2}{\lambda_2},$$

$$Q_3 = \frac{\|C\|\frac{\gamma_3}{\lambda_3}T}{\|C_2\|(e^{\|C\|T} - \|E\|)}\left[\|C_1\| + \|C_2\|\frac{\gamma_3}{\lambda_3}\right] + \frac{\gamma_3}{\lambda_3},$$

$$G_1 = \frac{T}{2}\frac{\gamma_1}{\lambda_1}K_3W_1W_4 + \frac{T^2}{4}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}L_3K_2W_1W_2W_4,$$

$$G_2 = \frac{T}{2}\frac{\gamma_2}{\lambda_2}L_3W_2W_4 + \frac{T^2}{4}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}L_1K_3W_1W_2W_4,$$

$$G_3 = \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_2W_5W_3 + \frac{T^2}{4}\frac{\gamma_3}{\lambda_3}\frac{\gamma_1}{\lambda_1}P_1K_2W_5W_3W_1W_1,$$

$$G_4 = \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_1W_6W_3 + \frac{T^2}{4}\frac{\gamma_2}{\lambda_2}\frac{\gamma_3}{\lambda_3}P_2L_1W_6W_3W_2,$$

Proof: By using (53) we get

$$\begin{aligned}\|\Delta_2(x_0^1,y_0^1,z_0^1) - \Delta_2(x_0^2,y_0^2,z_0^2)\| \le{}& \frac{\|A\|}{(e^{\|A\|T} - \|E\|)}\int_0^T \|e^{A(T-s)}\|\,\|f(s,x^0(s,x_0^1,y_0^1,z_0^1),y^0(s,x_0^1,y_0^1,z_0^1),\\ &,z^0(s,x_0^1,y_0^1,z_0^1)) - f(s,x^0(s,x_0^2,y_0^2,z_0^2),y^0(s,x_0^2,y_0^2,z_0^2),\\ &,z^0(s,x_0^2,y_0^2,z_0^2))\|ds +\\ &+\frac{\|A\|}{\|A_2\|(e^{\|A\|T} - \|E\|)}[\|A\|_1 + \|A_2\|\|e^{AT}\|]\|x_o^1 - x_o^2\|\\ \le{}& \frac{\|A\|T}{(e^{\|A\|T} - \|E\|)}\frac{\gamma_1}{\lambda_1}(K_1\|x^0(t,x_0^1,y_0^1,z_0^1) - x^0(t,x_0^2,y_0^2,z_0^2)\|\\ &+K_2\|y^0(t,x_0^1,y_0^1,z_0^1) - y^0(t,x_0^2,y_0^2,z_0^2)\|\\ &+K_3\|z^0(t,x_0^1,y_0^1,z_0^1) - z^0(t,x_0^2,y_0^2,z_0^2)\|\\ &+\frac{1}{\|A_2\|T}\left[\|A_1\|\frac{\lambda_1}{\gamma_1} + \|A_2\|\right]\|x_0^1 - x_0^2\|)\end{aligned}$$

$$\begin{aligned}\therefore \|\Delta_1(x_0^1,y_0^1,z_0^1) - \Delta_1(x_0^2,y_0^2,z_0^2)\| \le{}& N_1\frac{\gamma_1}{\lambda_1}(K_1\|x^0(t,x_0^1,y_0^1,z_0^1) - x^0(t,x_0^2,y_0^2,z_0^2)\|\\ &+K_2\|y^0(t,x_0^1,y_0^1,z_0^1) - y^0(t,x_0^2,y_0^2,z_0^2)\|\\ &+K_3\|z^0(t,x_0^1,y_0^1,z_0^1) - z^0(t,x_0^2,y_0^2,z_0^2)\| + H_1\|x_0^1 - x_0^2\|)\end{aligned} \tag{61}$$

Similarly

$$\begin{aligned}\|\Delta_2(x_0^1,y_0^1,z_0^1) - \Delta_2(x_0^2,y_0^2,z_0^2)\| \le{}& N_2\frac{\gamma_2}{\lambda_2}[L_1\|x^0(t,x_0^1,y_0^1,z_0^1) - x^0(t,x_0^2,y_0^2,z_0^2)\|\\ &+L_2\|y^0(t,x_0^1,y_0^1,z_0^1) - y^0(t,x_0^2,y_0^2,z_0^2)\|\\ &+L_3\|z^0(t,x_0^1,y_0^1,z_0^1) - z^0(t,x_0^2,y_0^2,z_0^2)\| + H_2\|y_0^1 - y_0^2\|]\end{aligned} \tag{62}$$

Also

$$\begin{aligned}\|\Delta_3(x_0^1,y_0^1,z_0^1) - \Delta_3(x_0^2,y_0^2,z_0^2)\| \le{}& N_3\frac{\gamma_3}{\lambda_3}[P_1\|x^0(t,x_0^1,y_0^1,z_0^1) - x^0(t,x_0^2,y_0^2,z_0^2)\|\\ &+P_2\|y^0(t,x_0^1,y_0^1,z_0^1) - y^0(t,x_0^2,y_0^2,z_0^2)\|\\ &+P_3\|z^0(t,x_0^1,y_0^1,z_0^1) - z^0(t,x_0^2,y_0^2,z_0^2)\| + H_3\|z_0^1 - z_0^2\|]\end{aligned} \tag{63}$$

where

$$x(t,x_0^k,y_0^k,z_0^k)=x_0^k e^{\mathrm{At}}+\int_0^t e^{A(t-s)}\,[f(s,x(s,x_0^k,y_0^k,z_0^k),y(s,x_0^k,y_0^k,z_0^k),$$
$$z(s,x_0^k,y_0^k,z_0^k))-\frac{\mathrm{A}}{A_2(e^{\mathrm{AT}}-\mathrm{E})}[A_1x_0^k+A_2x_0^ke^{\mathrm{AT}}-e_1]$$
$$-\frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f(s,x(s,x_0^k,y_0^k,z_0^k),y(s,x_0^k,y_0^k,z_0^k),\ z(s,x_0^k,y_0^k,z_0^k))ds]ds. \tag{64}$$

$$y(t,x_0^k,y_0^k,z_0^k)=y_0^k e^{\mathrm{Bt}}+\int_0^t e^{B(t-s)}\,[g(s,x(s,x_0^k,y_0^k,z_0^k),y(s,x_0^k,y_0^k,z_0^k),z(s,x_0^k,y_0^k,z_0^k))$$
$$-\frac{\mathrm{B}}{B_2(e^{\mathrm{BT}}-\mathrm{E})}[B_1y_0^k+B_2y_0^ke^{\mathrm{BT}}-e_2]$$
$$-\frac{\mathrm{B}}{(e^{\mathrm{BT}}-\mathrm{E})}\int_0^T e^{B(T-s)}\,g(s,x(s,x_0^k,y_0^k,z_0^k),y(s,x_0^k,y_0^k,z_0^k),z(s,x_0^k,y_0^k,z_0^k))ds]ds \tag{65}$$

$$z(t,x_o^k,y_o^k,z_o^k)=z_0e^{Ct}+\int_0^t e^{C(t-s)}\,[h(s,x(s,x_0^k,y_0^k,z_0^k),y(s,x_0^k,y_0^k,z_0^k),$$
$$,y(s,x_0^k,y_0^k,z_0^k))-\delta-\frac{\mathrm{C}}{C_2(e^{\mathrm{CT}}-\mathrm{E})}[C_1z_0^k+C_2z_0^ke^{\mathrm{CT}}-e_3]$$
$$-\frac{\mathrm{C}}{(e^{\mathrm{CT}}-\mathrm{E})}\int_0^T e^{C(T-s)}\,h(s,x(s,x_0^k,y_0^k,z_0^k),y(s,x_0^k,y_0^k,z_0^k),y(s,x_0^k,y_0^k,z_0^k))ds\,]ds \tag{66}$$

Since $\begin{pmatrix}x^0(t,x_0,y_0,z_0)\\ y^0(t,x_0,y_0,z_0)\\ z^0(t,x_0,y_0,z_0)\end{pmatrix}$ satisfy the equation (16), from(64), (65) and(66), we get

$$\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\|\le\|x_0^1-x_0^2\|\|e^{\mathrm{At}}\|+\left\|\int_0^t e^{A(t-s)}[\,f(s,x^0(s,x_0^1,y_0^1,z_0^1),y^0(s,x_0^1,y_0^1,z_0^1),\right.$$
$$z^0(s,x_0^1,y_0^1,z_0^1)-f(s,x^0(s,x_0^2,y_0^2,z_0^2),y^0(s,x_0^2,y_0^2,z_0^2),z^0(s,x_0^2,y_0^2,z_0^2))$$
$$-\frac{\mathrm{A}}{(e^{\mathrm{AT}}-\mathrm{E})}\int_0^T e^{A(T-s)}\,f(s,x^0(s,x_0^1,y_0^1,z_0^1),y^0(s,x_0^1,y_0^1,z_0^1),$$
$$,z^0(s,x_0^1,y_0^1,z_0^1))-f(s,x^0(s,x_0^2,y_0^2,z_0^2),y^0(s,x_0^2,y_0^2,z_0^2),$$
$$,z^0(s,x_0^2,y_0^2,z_0^2))ds]ds\|$$
$$+\left\|e^{A(t-s)}\right\|\frac{\|\mathrm{A}\|\mathrm{T}}{\|A_2\|(e^{\|\mathrm{A}\|\mathrm{T}}-\|\mathrm{E}\|)}[\|A_1\|+\|A_2\|\|e^{\mathrm{AT}}\|\|x_0^1-x_0^2\|]$$

$$\le\left(\frac{\|\mathrm{A}\|\frac{\gamma_1}{\lambda_1}T}{\|A_2\|(e^{\|\mathrm{A}\|\mathrm{T}}-\|\mathrm{E}\|)}\left[\|A_1\|+\|A_2\|\frac{\gamma_1}{\lambda_1}\right]+\frac{\gamma_1}{\lambda_1}\right)\|x_0^1-x_0^2\|$$
$$+\alpha_1(t)\tfrac{\gamma_1}{\lambda_1}(\,K_1\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\|$$
$$+K_2\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\|$$
$$+K_3\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\|)$$

$$\le\ \left(1-\frac{T}{2}\frac{\gamma_1}{\lambda_1}\,K_1\right)^{-1}(Q_1\|x_0^1-x_0^2\|$$
$$+\frac{T}{2}\frac{\gamma_1}{\lambda_1}\,K_2\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\|$$
$$+\frac{T}{2}\frac{\gamma_1}{\lambda_1}\,K_3\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\|$$

$$\therefore\ \|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\|\le\ W_1\,Q_1\|x_0^1-x_0^2\|+\frac{T}{2}\frac{\gamma_1}{\lambda_1}\,K_2W_1\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\|$$
$$+\frac{T}{2}\frac{\gamma_1}{\lambda_1}\,K_3W_1\|\,z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\| \tag{67}$$

Similarly

$$\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\| \le W_2Q_2\|y_0^1-y_0^2\|+\frac{T}{2}\frac{\gamma_2}{\lambda_2}L_1W_2\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\| +\frac{T}{2}\frac{\gamma_2}{\lambda_2}L_3W_2\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\| \quad (68)$$

Also

$$\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\| \le W_3Q_3\|z_0^1-z_0^2\|+\frac{T}{2}\frac{\gamma_3}{\lambda_3}P_1W_3\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\| +\frac{T}{2}\frac{\gamma_3}{\lambda_3}P_2W_3\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\| \quad (69)$$

From(67) and(68) we get

$$\begin{aligned}\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\| &\le \left(1-\frac{T^2}{4}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}L_1K_2W_1W_2\right)^{-1}(W_1\,Q_1\|x_0^1-x_0^2\| \\ &+\frac{T}{2}\frac{\gamma_1}{\lambda_1}K_2W_1\,W_2Q_2\|y_0^1-y_0^2\| \\ &+\left(\frac{T}{2}\frac{\gamma_1}{\lambda_1}K_3W_1+\frac{T^2}{4}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}L_3K_2W_1W_2\right)\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\|) \\ &\le W_4W_1\,Q_1\|x_0^1-x_0^2\|+\frac{T}{2}\frac{\gamma_1}{\lambda_1}K_2W_1\,W_2W_4Q_2\|y_0^1-y_0^2\|+(\frac{T}{2}\frac{\gamma_1}{\lambda_1}K_3W_1W_4 \\ &+\frac{T^2}{4}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}L_3K_2W_1W_2W_4)\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\|\end{aligned}$$

$$\therefore \|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\| \le V_1\|x_0^1-x_0^2\|+V_4\|y_0^1-y_0^2\| +G_1\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\| \quad (70)$$

and

$$\begin{aligned}\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\| &\le \left(1-\frac{T^2}{4}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}L_1K_2W_1W_2\right)^{-1}(W_2Q_2\|y_0^1-y_0^2\| \\ &+\frac{T}{2}\frac{\gamma_2}{\lambda_2}L_1W_2W_1Q_1\|x_0^1-x_0^2\|+(\frac{T}{2}\frac{\gamma_2}{\lambda_2}L_3W_2 \\ &+\frac{T^2}{4}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}L_1K_3W_1W_2)\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\|) \\ &\le W_4\,W_2Q_2\|y_0^1-y_0^2\|+\frac{T}{2}\frac{\gamma_2}{\lambda_2}L_1W_1W_2W_4\,Q_1\|x_0^1-x_0^2\|+(\frac{T}{2}\frac{\gamma_2}{\lambda_2}L_3W_2W_4 \\ &+\frac{T^2}{4}\frac{\gamma_1}{\lambda_1}\frac{\gamma_2}{\lambda_2}L_1K_3W_1W_2W_4)\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\|\end{aligned}$$

$$\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\| \le V_2\|y_0^1-y_0^2\|+V_3\|x_0^1-x_0^2\| +G_2\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\| \quad (71)$$

From (69) and (67) we get

$$\begin{aligned}\|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\| &\le \left(1-\frac{T^2}{4}\frac{\gamma_3}{\lambda_3}\frac{\gamma_1}{\lambda_1}P_1K_3W_3W_1\right)^{-1}(W_3Q_3\|z_0^1-z_0^2\| \\ &+\frac{T}{2}\frac{\gamma_3}{\lambda_3}P_1W_3\,W_1\,Q_1\|x_0^1-x_0^2\|+(\frac{T}{2}\frac{\gamma_3}{\lambda_3}P_2W_3+ \\ &+\frac{T^2}{4}\frac{\gamma_3}{\lambda_3}\frac{\gamma_1}{\lambda_1}P_1K_2W_3W_1)\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\|) \\ &\le W_5W_3Q_3\|z_0^1-z_0^2\|+\frac{T}{2}\frac{\gamma_3}{\lambda_3}P_1W_5W_3\,W_1\,Q_1\|x_0^1-x_0^2\|+\left(\frac{T}{2}\frac{\gamma_3}{\lambda_3}P_2W_5W_3+\right. \\ &\left.+\frac{T^2}{4}\frac{\gamma_3}{\lambda_3}\frac{\gamma_1}{\lambda_1}P_1K_2W_5W_3W_1\right)\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\|\end{aligned}$$

$$\therefore \|z^0(t,x_0^1,y_0^1,z_0^1)-z^0(t,x_0^2,y_0^2,z_0^2)\| \le V_6\|z_0^1-z_0^2\|+V_5\|x_0^1-x_0^2\| +G_3\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\| \quad (72)$$

Also from (69) and (68) we get

$$\| z^0(t,x_0^1,y_0^1,z_0^1) - z^0(t,x_0^2,y_0^2,z_0^2)\| \le \left(1-\frac{T^2}{4}\frac{\gamma_2}{\lambda_2}\frac{\gamma_3}{\lambda_3}P_2L_3W_3W_2\right)^{-1}(W_3Q_3\|z_0^1-z_0^2\| + \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_2W_3\,W_2Q_2\|y_0^1-y_0^2\| + \left(\frac{T}{2}\frac{\gamma_3}{\lambda_3}P_1W_3+\frac{T^2}{4}\frac{\gamma_2}{\lambda_2}\frac{\gamma_3}{\lambda_3}P_2L_1W_3W_2\right)\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\|)$$

$$\le W_6W_3Q_3\|z_0^1-z_0^2\| + \frac{T}{2}\frac{\gamma_3}{\lambda_3}P_2\,W_6W_3\,W_2Q_2\|y_0^1-y_0^2\| + \left(\frac{T}{2}\frac{\gamma_3}{\lambda_3}P_1W_6W_3 + \frac{T^2}{4}\frac{\gamma_2}{\lambda_2}\frac{\gamma_3}{\lambda_3}P_2L_1\,W_6W_3W_2\right)\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\|$$

$$\therefore \| z^0(t,x_0^1,y_0^1,z_0^1) - z^0(t,x_0^2,y_0^2,z_0^2)\| \le V_7\|z_0^1-z_0^2\| + V_8\|y_0^1-y_0^2\| + G_4\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\| \tag{73}$$

Now we substitute (73) in (70) we get

$$\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\| \le V_1\|x_0^1-x_0^2\| + V_4\|y_0^1-y_0^2\| + G_1V_7\|z_0^1-z_0^2\| + G_1V_8\|y_0^1-y_0^2\| + G_1G_4\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\|$$

$$\le (1-G_1\,G_4)^{-1}(V_1\|x_0^1-x_0^2\| + (V_4+G_1V_8)\|y_0^1-y_0^2\| + G_1V_7\|z_0^1-z_0^2\|)$$

$$\le W_7V_1\|x_0^1-x_0^2\| + W_7(V_4+G_1V_8)\|y_0^1-y_0^2\| + W_7G_1V_7\|z_0^1-z_0^2\|$$

$$\|x^0(t,x_0^1,y_0^1,z_0^1)-x^0(t,x_0^2,y_0^2,z_0^2)\| \le V_9\|x_0^1-x_0^2\| + V_{10}\|y_0^1-y_0^2\| + V_{11}\|z_0^1-z_0^2\| \tag{74}$$

Now we substitute (72) in (71) we get

$$\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\| \le V_2\|y_0^1-y_0^2\| + V_3\|x_0^1-x_0^2\| + G_2V_6\|z_0^1-z_0^2\| + G_2V_5\|x_0^1-x_0^2\| + G_2C_3\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\|$$

$$\le (1-G_2G_3)^{-1}(V_2\|y_0^1-y_0^2\| + (V_3+G_2V_5)\|x_0^1-x_0^2\| + G_2V_6\|z_0^1-z_0^2\|)$$

$$\le W_8V_2\|y_0^1-y_0^2\| + W_8(V_3+G_2V_5)\|x_0^1-x_0^2\| + W_8G_2V_6\|z_0^1-z_0^2\|$$

$$\|y^0(t,x_0^1,y_0^1,z_0^1)-y^0(t,x_0^2,y_0^2,z_0^2)\| \le V_{12}\|y_0^1-y_0^2\| + V_{13}\|x_0^1-x_0^2\| + V_{14}\|z_0^1-z_0^2\| \tag{75}$$

Substitute (70) in (73)

$$\| z^0(t,x_0^1,y_0^1,z_0^1) - z^0(t,x_0^2,y_0^2,z_0^2)\| \le V_7\|z_0^1-z_0^2\| + V_8\|y_0^1-y_0^2\| + G_4V_1\|x_0^1-x_0^2\| + G_4V_4\|y_0^1-y_0^2\|G_4G_1\| z^0(t,x_0^1,y_0^1,z_0^1) - z^0(t,x_0^2,y_0^2,z_0^2)\|$$

$$\le (1-G_4G_1)^{-1}(\,V_7\|z_0^1-z_0^2\| + (V_8+G_4V_4)\|y_0^1-y_0^2\| + G_4V_1\|x_0^1-x_0^2\|)$$

$$\le W_7\,V_7\|z_0^1-z_0^2\| + W_7(V_8+G_4V_4)\|y_0^1-y_0^2\| + W_7\,G_4V_1\|x_0^1-x_0^2\|$$

$$\therefore \| z^0(t,x_0^1,y_0^1,z_0^1) - z^0(t,x_0^2,y_0^2,z_0^2)\| \le V_{15}\|z_0^1-z_0^2\| + V_{16}\|y_0^1-y_0^2\| + V_{17}\|x_0^1-x_0^2\| \tag{76}$$

Now Substitute (74), (75) and (76) in(61)

$$\|\Delta_1(x_0^1,y_0^1,z_0^1)-\Delta_1(x_0^2,y_0^2,z_0^2)\| \le \mathrm{N}_1\frac{\gamma_1}{\lambda_1}\,[(K_1(V_9+V_{13}+V_{17})+H_1)\|x_0^1-x_0^2\| + K_2(V_{10}+V_{12}+V_{16})\|y_0^1-y_0^2\| + K_3(V_{11}+V_{14}+V_{15})\|z_0^1-z_0^2\|]$$

$$\therefore \|\Delta_1(x_0^1,y_0^1,z_0^1)-\Delta_1(x_0^2,y_0^2,z_0^2)\| \le \mathrm{N}_1\frac{\gamma_1}{\lambda_1}[E_1\|x_0^1-x_0^2\| + E_2\|y_0^1-y_0^2\| + E_3\|z_0^1-z_0^2\|] \tag{77}$$

Now Substitute (74), (75) and (76) in(62)

$$\|\Delta_2(x_0^1, y_0^1, z_0^1) - \Delta_2(x_0^2, y_0^2, z_0^2)\| \le \mathrm{N}_2 \frac{\gamma_2}{\lambda_2}[L_1(V_9 + V_{13} + V_{17})\|x_0^1 - x_0^2\| + (L_2(V_{10} + V_{12} + V_{16}) + H_2)\|y_0^1 - y_0^2\| + L_3(V_{11} + V_{14} + V_{15})\|z_0^1 - z_0^2\|]$$

$$\therefore \|\Delta_2(x_0^1, y_0^1, z_0^1) - \Delta_2(x_0^2, y_0^2, z_0^2)\| \le \mathrm{N}_2 \frac{\gamma_2}{\lambda_2}[E_4\|x_0^1 - x_0^2\| + E_5\|y_0^1 - y_0^2\| + E_6\|z_0^1 - z_0^2\|] \quad (78)$$

Now Substitute (74), (75) and (76) in(63)

$$\|\Delta_3(x_0^1, y_0^1, z_0^1) - \Delta_3(x_0^2, y_0^2, z_0^2)\| \le N_3 \frac{\gamma_3}{\lambda_3}[P_1(V_9 + V_{13} + V_{17})\|x_0^1 - x_0^2\| + P_2(V_{10} + V_{12} + V_{16})\|y_0^1 - y_0^2\| + (P_3(V_{11} + V_{14} + V_{15}) + \mathrm{H}_3)\|z_0^1 - z_0^2\|]$$

$$\therefore \|\Delta_3(x_0^1, y_0^1, z_0^1) - \Delta_3(x_0^2, y_0^2, z_0^2)\| \le N_3 \frac{\gamma_3}{\lambda_3}[E_7\|x_0^1 - x_0^2\| + E_8\|y_0^1 - y_0^2\| + E_9\|z_0^1 - z_0^2\|] \quad (79)$$

Then we rewrite (77), (78) and (79) by the vector form we get (60). ▯

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