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S_{α}– OPEN SETS IN TOPOLOGICAL SPACES

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ABSTRACT

In this paper, we investigate a new class of semi open sets called S_{α}-open sets in topological spaces and its properties are studied.

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1. INTRODUCTION AND PRELIMINARIES

Throughout this paper, a space means a topological space on which no separation axioms are assumed unless otherwise explicitly stated. In 1963 Levine [9] initiated semi open sets and gave their properties. Mathematicians gave in several papers interesting and different new types of sets. In 1965, O. Njastad [11] introduced α- closed sets. We recall the following definitions and characterizations. The closure (resp., interior) of a subset A of X is denoted by c l A (resp., int A),A subset A of X is said to be semi open [9] (resp, pre open [10], α- open [11], regular open [13]) set if A⊂ cl int A (resp., A⊂ int clA, A⊂ int cl int A, A=int cl (A) The complement of semi open (resp., pre open, α- open, regular open) set is said to be semi closed (resp., pre closed, α- closed, regular closed) The intersection of all semi closed (resp., pre closed, α- closed, regular closed) sets of X containing A is called semi closure (resp., pre closure, α–closure, regular closure) and denoted by scl A (resp., pcl A, αcl A, rcl A). The union of all semi open (resp., pre open, α- open) sets of X contained in A is called the semi interior (resp., pre interior, α-interior) and denoted by s int A (resp., p int A α int A). The family of all semi open (resp., pre open, α- open, regular open, semi closed, pre closed, α- closed, regular closed) subsets of a topological space X is denoted by SO (X) (resp., PO (X), αO(X) ,RO (X), SC(X), PC(X), αC(X),RC(X)).

Definition: 1.1 A topological space (X, τ) is said to be
1. Extremally disconnected of cl V∈τ, for every V∈τ.
2. Locally indiscrete if every open subset of X is closed.
3. Hyperconnected if every nonempty open subset of X is dense.

Lemma: 1.2
1. If X is a locally is indiscrete space, then each semi open subset of X is closed and hence each semi closed subset of X is open [2].
2. A topological space X is hyperconnected if and only if R0(X) = {∅, X} [6]

Theorem 1.3 .Let (X, τ)be a topological space. Then SO(X, τ) =SO(X, αO(X))[3].

Theorem: 1.4[9] Let (X, τ) be a topological space.
1. Let A⊂X. Then A∈SO (X, τ) if and only if cl A = cl int A.
2. If {Aγ: γ∈Γ} is a collection of semi open sets in a topological space (X, τ), then ∪{Aγ: γ∈Γ} is semi open.

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Theorem: 1. 5 If Y is a semi open subspace of a space X, then a subset A of Y, is a semi open set in X if and only if A is semi open set in Y [12].

Theorem: 1.6 [4] Let (X, τ) be a topological space.

If $A \in \tau$, and $B \in SO(X)$, then $A \cap B \in SO(x)$.

Theorem: 1.7 Let X and Y be spaces. If $A \subset X$ and $B \subset Y$ then $s\ int_{xxy}$ (AXB) =s int_x (A)X s int_y (B)[1].

Definition: 1.8 The subset A of a space X is said to be S_{p-} open [13] if for each $x \in A$, there exists a pre closed set F such that $x \in F \subset A$.

Theorem: 1.9 [4] Let A be any subset of a space X. Then $A \in SC(X)$ if and only if int cl $A \subset A$.

Theorem: 1.10 [12] A subset A of a space X is dense in X if and only if A is semi dense in X.

Theorem: 1.11 [7] A space X is extremely disconnected if and only if RO(x) =RC(X).

2. Sα-open sets

In this section, we introduce and study the concept of Sα- open sets in topological spaces and study some of its roperties.

Definition: 2.1 A semi open set A of a topological space X is said to be Sα-open if for each $x \in A$, there exists a α-closed set F such that $x \in F \subset A$. A subset B of a topological space X is $S_{\alpha-}$closed, if X-B is $S_{\alpha-}$open.

The family of $S_{\alpha-}$open subsets of X is denoted by $S_\alpha O(X)$.

Theorem: 2.2 A subset A of a topological space X is $S_{\alpha-}$open if and only if A is semi open and it is a union of α-closed sets.

Proof: Let A be $S_{\alpha-}$open. Then A is semi open $x \in A$ implies, there exists α-closed set F_x Such that $x \in F_x \subset A$ Hence $\bigcup_{x \in A} F_x \subset A$. But $x \in A$, $x \in F_x$ implies $A \subset \bigcup_{x \in A} F_x$. This completes one half of the proof.

Let A be semi open and $A = \bigcup_{i \in I} F_i$, where each F_i is α-closed. Let $x \in A$. Then $x \in$ some $F_i \subset A$. Hence A is S_α-open.

The following result shows that any union of S_α-open sets in S_α-open.

Theorem: 2.3 Let $\{A_\alpha : \alpha \in \Delta\}$ be a family of $S_{\alpha-}$open sets in a topological space X. Then $\bigcup_{\alpha \in \Delta} A_\alpha$ is an S_α -open set.

Proof: The union of an arbitrary semi open sets is semi open by theorem 1.4. Suppose that $x \in \bigcup_{\alpha \in \Delta} A_\alpha$. This implies that there exists $\alpha_0 \in \Delta$ such that $x \in A_{\alpha 0}$ and as $A_{\alpha 0}$ is an S_α –open set, there exists a α-closed set F in X such that $x \in F \subset A_{\alpha 0} \subset \bigcup_{\alpha \in \Delta} A_\alpha$. Therefore $\bigcup_{\alpha \in \Delta} A_\alpha$ is a S_α-open set.

From theorem 2.3, it is clear that any intersection of S_α –closed sets of a topological space X is S_α –closed. The following example shows that the intersection of two S_α–open sets need not be S_α–open.

Example: 2.4 Let X= {a, b, c}

τ = {∅, {a}, {b}, {a, b}, X}

S_α–open sets = {∅, {a, c}, {b, c}, X}{a, c} ∩{b, c}={c} is not an S_α–open set

Theorem 2.5: A subset G of the topological space X is S_α -open if and only if for each $x\in G$, there exists an S_α–open set H such that $x\in H\subset G$.

Proof: Let G be a S_α–open set in X. Then for each $x\in G$, we have G is an S_α–open set such that $x\in G\subset G$.

Conversely, let for each $x\in G$, there exists an S_α–open set H such that $x\in H\subset G$. Than G is a union of S_α–open sets, hence by theorem 2.3, G is S_α–open.

Theorem: 2.6
1. Regular closed set is S_α–open set.
2. Regular open set is S_α–closed set.

Proof:
1. 1. Let A be regular closed in a topological space X. A=cl int A. A is semi open. A is α- closed. $x\in A$ implies $x\in A\subset A$. Hence A is S_α–open.
2. Obvious.

Theorem 2.7: If a space X is a T_1–space, then $S_\alpha(X) = SO(X)$.

Proof: $S_\alpha(X)$□$\subset SO(X)$. Let $A\in SO(X)$. Let $x\in A$. AS X is a T_1–space, $\{x\}$ is closed. Every closed set in X is α-closed. Hence $x\in\{x\}\subset A\in S\alpha O(X)$. This completes the proof.

Theorem: 2.8 If the family of all semi open subsets of a topological space is a topology on X, then the family of SαO (X) is also a topology on X.

Proof: Obvious.

Theorem: 2.9 If a space X is hyperconnected, then the only S_α–open sets of X are ∅ and X.

Proof: Let $A\subset X$ such that A is S_α–open in X. If A=X, there is nothing to prove. If $A\neq X$ we have to prove A=∅. As A is S_α–open, for each $x\in A$, there exists a α–closed set F such that $x\in F\subset A$. So $X\text{-}A\subset X\text{-}F$. X-A is semi closed. Therefore int cl(X-A) $\subset$ (X-A). Since X is hyper connected, then by definition 1.1 and theorem 1.10 scl(int cl (X-A))=X$\subset$X-A. Hence X-A=X. So A=∅.

Theorem: 2.10 If a topological space X is locally indiscrete, then every semi open set is S_α–open.

Proof: Let A be semi open in X.

Then $A\subset$ cl int A. As X is locally indiscrete, int A is closed. Hence int A=cl int A. So, cl int A= int $A\subset A$. So A is regular closed. By theorem 2.6(1)-A is S_α–open.

Theorem: 2.11 If a topological space (X,τ) is T_1 or locally indiscrete, then $\tau\subset S_\alpha 0$ (X).

Proof: Let (X, τ) be T_1.As every open set is semi open, $\tau\subset S0(x) = S_\alpha 0$ (X).

Let (X, τ) be locally indiscrete then $\tau\subset SO(x)\subset S\alpha O$ (X).

Theorem: 2.12 If B in clopen subset of a space X and A is S_α–open in X, then $A\cap B\in S_\alpha 0(X)$.

Proof: Let A be S_α–open .So A is semi open. B is open and closed in x. Then by theorem 1.6, $A\cap B$ is semi open in X. Let $x\in A\cap B$. Then $x\in A$ and $x\in B$. Since A is S_α–open, there exists a α–closed set F such that $x\in F\subset A$. B is closed and hence α–closed. $F\cap B$ is α–closed. $x\in F\cap B\subset A\cap B$. So $A\cap B$ is S_α–open.

Theorem: 2.13 Let X be a locally indiscrete and $A\subset X$, $B\subset X$. It $A\in S_\alpha 0(X)$ and B is open, and then $A\cap B$ is S_α–open in X.

Proof: Follows from theorem 2.12.

Theorem: 2.14 Let X be extremally disconnected and $A\subset X$, $B\subset X$. If $A\in S_\alpha 0$ (X) and $B\in R0(X)$ then $A\cap B$ is S_α–open in X.

Proof: Let $A\in S_\alpha 0$ (X) and $B\in R0(X)$. Hence A is semi open. By Theorem 1.6, $A\cap B\in S0$ (X).

Let $x \in A \cap B$. This implies $x \in A$ and $x \in B$. As A is S_α –open, there exists a α –closed set F such that $x \in F \subset A$. X is extremally disconnected. By Theorem 1.11. B is a regular closed set. This implies $F \cap B$ is α-closed. $x \in F \cap B \subset A \cap B$. So $A \cap B$ is S_α – open.

3. S_α- Operations

Definition: 3.1 A subset N of a topological space X is called S_α - neighborhood of a subset A of X, if there exists an S_α-open set U such that $A \subset U \subset N$. When $A=\{x\}$, we say N is S_α- neighborhood of x.

Definition: 3.2 A point $x \in X$ is said to be an S_α- interior point of A, if there exists an S_α-open set U containing x such that $x \in U \subset A$. The set of all $S\alpha$-interior points of A is said to be S_α- interior of A and it is denoted by S_α- int A.

Theorem: 3.3 Let A be any subset of a topological space X. If x is a S_α-interior point of A, then there exists a semi closed set F of X containing x such that $F \subset A$.

Proof: Let $x \in S_\alpha$- int A. Then there exists a S_α-open set U containing x such that $U \subset A$. Since U is in S_α-open set, there exists a α-closed set F such that $x \in F \subset U \subset A$.

Theorem: 3.4 For any subset A of a topological space X, the following statements are true
1. The S_α-interior of A is the union of all S_α-open sets contained in A.
2. S_α- int A is the largest S_α-open set contained in A.
3. A is S_α- open set if and only of $A=S_\alpha$ int A.

Proof: obvious.

From 3, are see S_α int S_α int A= S_α int A.

Theorem: 3.5 If A and B are any subsets of a topological space X. Then,
1. S_α int $\emptyset=\emptyset$ and S_α int X=X
2. S_α int $A \subset A$
3. if $A \subset B$, then S_α int $A \subset S_\alpha$ int B
4. S_α int $A \cup S\alpha$ int $B \subset S\alpha$int $(A \cup B)$
5. S_α int $(A \cap B)$□$\subset S_\alpha$ int $A \cap S_\alpha$ int B
6. S_α int (A-B)□$\subset S_\alpha$ int A-S_α int B

Proof: 1-5, obvious.
6. Let $x \in S_\alpha$ int (A-B). There exists an S_α-open set U such that $x \in U \subset A-B$. That is $U \subset A$. $U \cap B=\emptyset$ and $x \notin B$. Hence $x \in S_\alpha$ int A, $x \notin S_\alpha$ int B. Hence $x \in S_\alpha$ int A-S_α int B. This completes the proof.

Definition: 3.6 Intersection of S_α-closed sets containing F is called the S_α-closure of F and is denoted by S_α cl F.

Theorem: 3.7 Let A be a subset of the space X. $x \in X$ is in S_α-closed of A if and only if $A \cap U \neq \emptyset$, for every S_α-open set U containing x.

Proof: To prove the theorem, let us prove the contra positive.
$x \notin$scl A $\Leftrightarrow$There exists an S_α-open set U containing x that does not intersect A. Let $x \notin S_\alpha$ cl A. X-S_α cl A is an S_α-open set containing x that does not intersect A. Let U be an S_α-open set set containing x that does not intersect A. X-U is a S_α –closed set containing A. S_α cl $A \subset$ (X-U)
$x \notin X-U \Rightarrow x \notin S_\alpha$cl A.

Theorem: 3.8 Let A be any subset of a space X .$A \cap F \neq \emptyset$ for every α closed set F of X containing x, then the point x is in the S_α- closure of A.

Proof: Let U be any S_α- open set containing x. So, there exists a α-closed set F such that $x \in F \subset U$. $A \cap F \neq \emptyset$ implies $A \cap U \neq \emptyset$ for every S_α -open set U containing x. Hence $x \in S_\alpha$ cl A, by theorem 3.7

Theorem: 3.9 For any subset F of a topological space X, the following statements are true.
1. S_α cl F is the intersection of all. S_{α}- closed sets in X containing F.
2. S_α cl F is the smallest . S_α –closed set containing F.
3. F is S_α closed if and only if F= S_α cl F.

Proof: Obvious.

Theorem: 3.10 If F and E are any subsets of a topological space X, then
1. S_α cl ∅=∅ and. S_α cl X=X
2. For any subset F of X, F⊂ S_α cl F.
3. If F⊂E, then S_α cl F⊂ S_α cl E.
4. S_α cl F∪ S_α cl E ⊂ S_α cl (F∪E).
5. S_α cl (F∩E)⊂ S_α cl F ∩ S_α cl E.

Proof: Obvious.

Theorem: 3.11 For any subset A of a topological space X, the following statements are true.
1. X- S_α cl A= S_α int(X-A).
2. X- S_α int A= S_α cl A.
3. S_α int A= X- S_α cl A.

Proof:
1. X- S_α cl A is a S_α-open set contained in (X-A). Hence X- S_α cl A ⊂ S_α int (X-A).

If X- S_α cl A ≠ S_α int(X-A), then X- S_α int (X-A) is a S_α closed set properly contained in S_α cl A, a contradiction. Hence X- S_α cl A= S_α int(X-A). 2&3 follow from 1.

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