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ON A NEW NONLINEAR RETARDED INEQUALITIES APPLICABLE TO CERTAIN RETARDED PARTIAL INTEGRODIFFERENTIAL EQUATIONS

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ABSTRACT

In this article a two dimensional nonlinear retarded integral inequalities which can be used as ready and powerful tool in the analysis of class of integrodiffrerntial equations is presented. Application of nonlinear retarded inequality is also presented.

Keywords and Phrases: *Retarded, integral inequality, Boundedness, explicit bound, integrodifferential.*

1. INTRODUCTION

The celebrated Gronwall-Bellman [4,10] inequality states that if u and f are nonnegative continuous functions on an interval $[a,b]$ satisfying

$$u(t) \le c + \int_a^t f(s)u(s)ds \tag{1}$$

for some constant $c \ge 0$, then

$$u(t) \le c\ \exp\left(\int_a^t f(s)ds\right),\ t\ \in [a,b] \tag{2}$$

Inequality (1), provides an explicit bound on the unknown function and hence furnishes a handy tool in the study of qualitative and quantitative properties of solution of differential and integral equations, it has become one of the very few classic and most influential results in the theory and applications of inequalities. Due to its fundamental importance many generalizations and analogous results of (1), have been established.Such inequalities are in general known as Gronwall-Bellman type inequalities in the literature[see 1-3,5-8,11-15,].

The aim of the paper is to extend results which proved by Kim [9] to obtain a new generalizations some formal famous inequalities, which can be used as handy tools to study qualitative as well as quantitative properties of solutions of some nonlinear partial integrodifferential equations.

2 MAIN RESULTS

Let R denotes the set of real numbers, $R_+ = [0,\infty)$. Also, $J_1 = [x_0, X)$ and $J_2 = [y_0, Y)$ be the given subset of R,$\Delta = J_1 \times J_2$.$D_1 z(x,y), D_2 z(x,y)$ be partial derivative of z with respective x and y respectively.

Theorem: 2.1 Let $u, a, c \in C(\Delta, R_+), a$ and c be nondecreasing in each variables $f_i, g_i, h_i \in C(\Delta, R_+), i = 1,\cdots,n$ and let $\alpha_i \in C^1(J_1, J_1)$ be nondecreasing with $\alpha_i(t) \le t, i = 1,\cdots,n$ and $\beta_i \in C^1(J_2, J_2)$ be nondecreasing with $\beta_i(t) \le t, i = 1,\cdots,n$. Suppose $p > q > 0$ are constants $\varphi \in C(R_+, R_+)$ is an increasing function with $\varphi(\infty) = \infty$ and $\psi(u)$ is a nondecreasing continuous function for $u \in R$ with $\psi(u) > 0$ for $u > 0$. If

$$u^p(x,y) \le a(x,y) + c(x,y) \sum_{i=1}^{n} \int_{\alpha_i(x_0)}^{\alpha_i(x)} [f_i(s,t) u^q(s,t)\ (\psi(u(s,t)) + \int_{\alpha_i(x_0)}^{s} \int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta) d\eta d\sigma) + g_i(s,t) u^q(s,t)] dt ds \tag{3}$$

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for all $(x,y) \in \Delta$, then

$$u(x,y) \le [G_1^{-1}(G_1(k_1(x_0,y)) + c(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)(1+\int_{\alpha_i(x_0)}^{s}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)d\eta d\sigma)dtds)]^{\frac{1}{p-q}} \tag{4}$$

for all $x \in [x_0,x_1]\times[y_0,y_1]$, where

$$k_1(x_0,y) = [a(x,y)]^{\frac{p-q}{p}} + \frac{p-q}{p}c(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} g_i(s,t)dtds$$

$$G_1(r) = \int_{r_0}^{r}\frac{ds}{\psi(s^{\frac{1}{p-q}})}, \quad r \ge r_0 > 0.$$

G_1^{-1} denotes the inverse function of G_1 and $(x_1,y_1) \in \Delta$ is chosen so that

$$G_1(k_1(x_0,y)) + c(x,y)\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(s,t)(1+\int_{\alpha_i(x_0)}^{s}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)d\eta d\sigma)]dtds \in Dom(G_1^{-1})$$

Proof: Let us first assume that $a(x,y) > 0$. Fixing any number $\bar{x}$ and $\bar{y}$ with $x_0 \le x \le \bar{x}$ and $y_0 \le y \le \bar{y}$, define a positive function $z(x,y)$ as right hand side of (3). Then

$$z(x,y) = a(\bar{x},\bar{y}) + c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(s,t)u^q(s,t)\,(\psi(u(s,t)) + \int_{\alpha_i(x_0)}^{s}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)d\eta d\sigma) + g_i(s,t)u^q(s,t)]dtds$$

Then $z(x,y) > 0$, $z(x_0,y) = z(x,y_0) = a(\bar{x},\bar{y})$ and (3) can be restated

$$u(x,y) \le [z(x,y)]^{\frac{1}{p}} \tag{5}$$

clearly, $z(x,y)$ is continuous nondecreasing function for all $x \in J_1$, $y \in J_2$ and

$$D_1 z(x,y) = c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(\alpha_i(x),t)u^q(\alpha_i(x),t)(\psi(u(\alpha_i(x),t)) + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)\psi(\sigma,\eta)d\eta d\sigma) + g_i(\alpha_i(x),t)u^q(\alpha_i(x),t)]\alpha_i'(x)dt$$

Using (5), we deduce

$$D_1 z(x,y) \le c(\bar{x},\bar{y})[z(x,y)]^{\frac{q}{p}}\sum_{i=1}^{n}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(\alpha_i(x),t)(\psi(u(\alpha_i(x),t)) + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)\psi(u(\sigma,\eta))d\eta d\sigma) + g_i(\alpha_i(x),t)]\alpha_i'(x)dt$$

Using the monotonicity of $z(t)$, we deduce

$$\frac{D_1 z(x,y)}{[z(x,y)]^{\frac{q}{p}}} \le c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(\alpha_i(x),t)(\psi(u(\alpha_i(x),t)) + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)\psi(u(\sigma,\eta))d\eta d\sigma) + g_i(\alpha_i(x),t)]\alpha_i'(x)dt \tag{6}$$

Keeping y fixed and integrating inequality (6) from x_0 to x and making change of variables, we get

$$z^{\frac{p-q}{p}}(x,y) \le (a(\bar{x},\bar{y}))^{\frac{p-q}{p}} + \frac{p-q}{p}c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(s,t)\,(\psi(z^{\frac{1}{p}}(s,t)) + \int_{\alpha_i(x_0)}^{s}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)\psi(z^{\frac{1}{p}}(\sigma,\eta))d\eta d\sigma) + g_i(s,t)]dtds \tag{7}$$

Now, define a function $k_1(x,y)$ by

$$k_1(x,y) = (a(\bar{x},\bar{y}))^{\frac{p-q}{p}} + \frac{p-q}{p}c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(\bar{x})}\int_{\beta_i(y_0)}^{\beta_i(\bar{y})} g_i(s,t)dtds + \frac{p-q}{p}c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(\bar{x})}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)(\psi(z^{\frac{1}{p}}(s,t)) + \int_{\alpha_i(x_0)}^{s}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)\psi(z^{\frac{1}{p}}(\sigma,\eta))d\eta d\sigma)dtds$$

Then, (7) can be restated as

$$[z(x,y)]^{\frac{1}{p}} \le [k_1(x,y)]^{\frac{1}{p-q}} \tag{8}$$

We know, $u(x,y) \le [z(x,y)]^{\frac{1}{p}} \le [k_1(x,y)]^{\frac{1}{p-q}}$ and since ψ is nondecreasing,

$\psi[u(\sigma,\eta)] \le \psi[z^{\frac{1}{p}}(\sigma,\eta)] \le \psi[k_1^{\frac{1}{p-q}}(s,t)]$ for $\sigma \in [\alpha_i(t_0),\alpha_i(t)]$,and $\eta \in [\beta_i(t_0),\beta_i(t)]$

$$\begin{aligned}k_1(x,y) \le {} & (a(\bar{x},\bar{y}))^{\frac{p-q}{p}} + \frac{p-q}{p}c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(\bar{x})}\int_{\beta_i(y_0)}^{\beta_i(\bar{y})} g_i(s,t)dtds \\ & + \tfrac{p-q}{p}c(\bar{x},\bar{y})\textstyle\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)} f_i(s,t)(\psi(k_1^{\frac{1}{p-q}}(s,t)) \\ & + \textstyle\int_{\alpha_i(x_0)}^{s}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)\psi(k_1^{\frac{1}{p-q}}(\sigma,\eta))d\eta d\sigma)dtds\end{aligned}$$

Here, we observe that $k(x,y)$ is a continuous nondecreasing function for all $x \in J_1, y \in J_2$ and

$$\begin{aligned}D_1k_1(x,y) \le {} & \frac{p-q}{p}c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(\alpha_i(x),t)(\psi\left(k_1^{\frac{1}{p-q}}(s,t)\right) \\ & + \int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)\psi(k_1^{\frac{1}{p-q}}(\sigma,\eta))d\eta d\sigma)]\alpha'_i(x)dt\end{aligned}$$

$$D_1k_1(x,y) \le \tfrac{p-q}{p}c(\bar{x},\bar{y})\psi(k_1^{\frac{1}{p-q}}(x,y))\textstyle\sum_{i=1}^{n}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(\alpha_i(x),t)(1+\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)d\eta d\sigma)]\alpha'_i(x)dt \tag{9}$$

Using monotonicity of k_1 and ψ_1, we have

$$\frac{D_1k_1(x,y)}{\psi(k_1^{\frac{1}{p-q}}(x,y))} \le \frac{p-q}{p}c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(\alpha_i(x),t)(1+\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)d\eta d\sigma)]\alpha'_i(x)dt$$

From the definition of G_1 and Keeping y fixed in (9) and integrating from x_0 to x and making change of variables, we have

$$G_1(k_1(x,y)) \le G_1(k_1(x_0,y)) + \tfrac{p-q}{p}c(\bar{x},\bar{y})\textstyle\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(s,t)\ (1+\int_{\alpha_i(x_0)}^{s}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)d\eta d\sigma)]dtds \tag{10}$$

Using (8),(10) in (5), we have

$$u(x,y) \le \{G_1^{-1}[G_1(k_1(x_0,y)) + \frac{p-q}{p}c(\bar{x},\bar{y})\sum_{i=1}^{n}\int_{\alpha_i(x_0)}^{\alpha_i(x)}\int_{\beta_i(y_0)}^{\beta_i(y)}[f_i(s,t)(1+\int_{\alpha_i(x_0)}^{s}\int_{\beta_i(y_0)}^{t} h_i(\sigma,\eta)d\eta d\sigma)]dtds]\}^{\frac{1}{p-q}}$$

Taking $x=\bar{x}, y=\bar{y}$ in the above inequality since $\bar{x}$ and $\bar{y}$ are arbitrary, we get required inequality.

Remark: (1) If we put $h_i(\sigma,\eta)=0$ in above Theorem , we get corollary 2.2 in [9]

3 APPLICATION TO PARTIALINTEGRODIFFERENTIAL EQUATIONS

In this section we show that our one of the result is useful in proving global existence of the solution of certain non-linear partial integrodifferential equation with time delay. Consider the nonlinear integrodifferential equation involving several retarded arguments,

$$\begin{aligned}D_2(z^{p-1}(x,y)D_1z(x,y)) = {} & F(x,y,z(x-l_1(x),y-m_1(y)),\cdots,z(x-l_n(x),y-m_n(y)), \\ & \textstyle\int_{x_0}^{x}\int_{y_0}^{y} H_i(x,y,s,t,z(s-l_1(s),t-m_1(t)),\cdots,\cdots z(s-l_n(s),t-m_n(t)))dtds)\end{aligned} \tag{11}$$

with the initial boundary conditions

$$z^p(x,y_0)=e_1(x),\quad z^p(x_0,y)=e_2(y),\quad e_1(x_0)=e_2(y_0)=0$$

where $p>1$ is constant, $F \in C(\Delta\times R^n\times R,R), H_i \in C(\Delta\times\Delta\times R^n,R), e_1 \in C^1(J_1,R) e_2 \in C^1(J_2,R$ and $l_i \in C^1(J_1,J_1), m_i \in C^1(J_2,J_2)$ such that
$0<(l_i)'(x) \le 1, 0<(m_i)'(y) \le 1, l_i(x_0)=x_0, m_i(y_0)=y_0, i=1,\cdots,n.$

The following theorem deals with a boundedness on the solution of the problem (11)

Theorem: 3.1 Assume that $F: \Delta \times R^n \times R \to R$ is a continuous function for which there exists continuous nonnegative functions $f_i(x,y), g_i(x,y); i = 1,\cdots n$ such that

$$|F(x,y,u_1,u_2,\cdots u_n,v)| \le \sum_{i=1}^{n} |u_i|^q \{f_i(x,y)(\psi(|u_i| + |v|)) + g_i(x,y)|u_i|^q\} \tag{12}$$

and the function $H_i: \Delta \times \Delta \times R^n \to R$ is a continuous function for which there exist continuous nonnegative function $h_i(x,y), i = 1,\cdots,n$ such that

$$|H_i(x,y,s,t,u_1,u_2 \cdots u_n)| \le \sum_{i=1}^{n} h_i(x,y)\psi(|u_i|) \tag{13}$$

and

$$|e_1(x) + e_2(y)| \le a(x,y) \tag{14}$$

where $a(x,y) \in C(\Delta, R_+)$ is nondecreasing in each variables, $p > q > 0$ are constants and $\psi(u)$ is a nondecreasing continuous function for $u \in R$ with $\psi(u) > 0$ for $u > 0$. Let

$$M_i = \max_{x \in J_1} \frac{1}{1 - l'_i(x)}, \quad N_i = \max_{y \in J_2} \frac{1}{1 - m'_i(y)}, \quad i = 1,2,\cdots,n \tag{15}$$

if $z(x,y)$is any solution of problem (11) with the initial boundary condition, then

$$|z(x,y)| = \{G_1^{-1}[G_1(k_1(x_0,y)) + (p-q)\sum_{i=1}^{n} \int_{x_0 - l_i(x_0)}^{x - l_i(x)} \int_{y_0 - m_i(y_0)}^{y - m_i(y)} \overline{f_i}(\sigma,\tau)$$
$$(1 + \int_{x_0 - l_i(x_0)}^{s} \int_{y_0 - m_i(y_0)}^{t} \overline{h_i}(\phi,\gamma) d\gamma d\phi) d\tau d\sigma]\}^{\frac{1}{p-q}}$$

for all $(x,y) \in [x_0,x_1] \times [y_0,y_1]$, where G is in Theorem 2.1 and

$$k(x_0,y) = [a(x,y)]^{\frac{p-q}{p}} + (p-q)\sum_{i=1}^{n} \int_{x_0 - l_i(x_0)}^{x - l_i(x)} \int_{y_0 - m_i(y_0)}^{y - m_i(y)} \overline{g_i}(\sigma,\tau) d\tau d\sigma.$$

$$\overline{f_i} = f_i(\sigma + l_i(s), \tau + m_i(t)) M_i N_i$$

$$\overline{h_i} = g_i(\phi + l_i(\xi), \gamma + m_i(\eta)) M_i N_i$$

$$\overline{g_i} = h_i(\sigma + l_i(s), \tau + m_i(t)) M_i N_i$$

Proof: It is easy to see that the solution $z(x,y)$ of problem (11) with the initial boundary condition satisfies the equivalent integral equation

$$z^p(x,y) = e_1(x) + e_2(y) + p\int_{x_0}^{x}\int_{y_0}^{y} F(s,t,z(s - l_1(s), t - m_1(t)),\cdots, z(s - l_n(s), t - m_n(t)),$$
$$\int_{x_0}^{s}\int_{y_0}^{t} H_i(s,t,\xi,\eta, z(\xi - l_1(\xi), \eta - m_1(\eta)),\cdots\cdots, z(\xi - l_n(\xi), \eta - m_n(\eta))) d\eta d\xi) dt ds \tag{16}$$

using (12),(13),(14),(15) and making change of variable we have

$$|z^p(x,y)| \le a(x,y) + p\sum_{i=1}^{n} \int_{x_0 - l_i(x_0)}^{x - l_i(x)} \int_{x_0 - m_i(y_0)}^{y - m_i(y)} [|z(\sigma,\tau)|^q \overline{f_i}(\sigma,\tau)$$
$$\times (\psi(|z(\sigma,\tau)|) + \int_{x_0 - l_i(x_0)}^{\sigma} \int_{x_0 - m_i(y_0)}^{\tau} \overline{h_i}(\phi,\nu)\psi(|z(\phi,\nu)|) d\nu d\phi) + \overline{g_i}(\sigma,\tau)|z(\sigma,\tau)|^q] d\tau d\sigma \tag{17}$$

$$\overline{f_i} = f_i(\sigma + l_i(s), \tau + m_i(t)) M_i N_i,$$

$$\overline{h_i} = h_i(\phi + l_i(\xi), \nu + m_i(\eta)) M_i N_i,$$

$$\overline{g_i} = g_i(\sigma + l_i(s), \tau + m_i(t)) M_i N_i,$$

$$\sigma, s, \phi, \xi \in J_1, \tau, t, \nu, \eta \in J_2.$$

Now, applying Theorem 2.1 to above inequality, we get

$$|z(x,y)| = \{G_1^{-1}[G_1(k_1(x_0,y))$$
$$+(p-q)\sum_{i=1}^{n} \int_{x_0 - l_i(x_0)}^{x - l_i(x)} \int_{y_0 - m_i(y_0)}^{y - m_i(y)} \overline{f_i}(\sigma,\tau)\,(1 + \int_{x_0 - l_i(x_0)}^{s} \int_{y_0 - m_i(y_0)}^{t} \overline{h_i}(\phi,\gamma) d\gamma d\phi) d\tau d\sigma]\}^{\frac{1}{p-q}}$$

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