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(α, β) DERIVATIONS AND COMMUTATIVITY IN σ-PRIME RING

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ABSTRACT

Let R be a σ-prime ring with characterstic ≠ 2 and d be a nonzero (α, β) derivation of R commuting with σ. It is prove that a non-zero (α, β)- derivation d associated with be a non-zero σ–ideal I of R which commutes σ and i) if [d(x),x] = 0 ∀ x∈I then R is commutative ii) If $d^2(I)$ =0 then d= 0. Also we prove that R must be commutative under some suitable conditions.

Keywords: *σ-prime ring, σ-ideals, (α, β) derivations.*

1. INTRODUCTION

Through out the present paper all rings will be associative. A ring R equipped with an involution σ is said to be σ-prime if aRb=aR σ(b)=0 implies that a=0 or b=0. Recall that a ring R is prime if aRb=0 implies that a=0 or b=0. Obviously, every prime ring with involution σ is σ-prime but the converse is in general not true. An ideal I of R is a σ-ideal if I is invariant under σ i.e, σ(I) = I. Set of all symmetric and skew-symmetric elements of R is defined as $Sa_\sigma(R)$ = {x∈R/ σ(x) = ± x). we shall use the basic commutator identities: [xy, z] = x[y, z] + [x, z]y and [x, yz] = y[x, z] + x[y, z].An additive mapping d from R to itself is a derivation if d(xy)=d(x)y+xd(y). holds for all pairs x, y∈R. An additive mapping d from R to itself said to be (α, β) derivation if d(xy)= d(x) α(y)+β(x)d(y) hold for all x, y∈R. A mapping F:R→R is said to be centralizing on a subset S of R if [F(s),s]∈Z(R) for all s∈S, where Z(R) denotes the center of R;. In the special case where [F(s), s] = 0 for all s in S, the mapping F is said to be commuting on S.

The history of commuting and centralizing mapping goes back to 1955 when Divinsky[1] prove that simple artinian ring is commutative if it has commuting non-trivial automorphism. Two years later Posner [2] have proved that the existence of non-zero centralizing derivation on a prime ring forces Ring to be commutative (Posner's second theorem).Mayne [3] prove the analogous result for centralizing automorphisms. P. H. Lee and T.K. Lee [4] have shown that if a prime ring of characterstic different from two has a noz-zero derivation d satisfying [d(R),d(R)] then R is commutative. Joso Vukman [5] have shown that a prime ring of characterstic not two possessing a non-zero derivation D from R to itself such that x→[D(x),x] is commuting on R then R is commutative. M. Bresar [6] describe the structure of arbitrary additive mapping which is centralizing on a prime ring. Later L. oukhtite and S. Salhi proved some conditions under which derivations of σ-prime ring are commutative(Refer[7], [8]).These results are motivation for our results.

We prove the following results

3. MAIN RESULTS

Lemma: 1 Let I be a non-zero σ-ideal of σ-prime ring R and 0≠d be a (α, β) derivation on R which commutes with σ, If [d(x),x] = 0. ∀x∈I then R is commutative.

Proof: By Hypothesis [d(x), x] = 0 (1)

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Linearizing equation (1) we get

$[d(x+y), x+y] = 0.$

$[d(x)+d(y), x+y] = 0.$

$[d(x),y)+ [d(y), x] = 0.$ (2)

Replace y by yx and $d\neq 0$ be a (α, β) derivation.

$[d(x), yx] + [d(yx), x] = 0.$

$[d(x), y]x + y[d(x), x] + [d(y)\alpha(x) + \beta(y)d(x), x] = 0.$

$[d(x), y]x + [d(y)\ \alpha(x), x] + [\beta(y)d(x), x] = 0.$

$[d(x), y]x + d(y)[\alpha(x), x] + [d(y), x]\ \alpha(x) + \beta(y)\ [d(x), x] + [\beta(y), x]d(x).$ (3)

Replace α(x) by x in (3) and using (2) we obtain

$[\beta(y), x]d(x) = 0.$ (4)

For any $r\in R$, Replacing y by ry in above equation

$[\beta(ry), x]d(x) = 0.$

$\Rightarrow [\beta(r)\beta y), x]d(x) = 0.$

$\Rightarrow \beta(r)\ [\beta y), x]d(x) +[\beta(r), x]\beta(y)d(x) =0$ (4)

$\Rightarrow [\beta(r), x]\ \beta(y)d(x) =0\ \forall r\in R,\ x, y\in I.$ (By (4))

$\Rightarrow 0 = \beta^{-1}([\beta(r), x])\ I\ \beta^{-1}(d(x) = 0. \forall x\in I,\ r\in R.$

Since d commutes with σ and I is σ-ideal of R we have

$0 =\beta^{-1}\ ([\beta(r), x])\ I\beta^{-1}(d(x) = \beta^{-1}([\beta(r), x])\ I\beta^{-1}\ (\sigma(d(x)).$

By Lemma 1 of [7] we obtain $\beta^{-1}\ [\beta(r), x] =0$ or $\beta^{-1}\ (d(x)) = 0.$

Case -1: if $\beta^{-1}\ [\beta(r), x] =0\ \ \forall\ \ x\in I,\ r\in R.$

Then $[\beta(r), x] = 0$ (since β is automorphism).

Since β is automorphism, this implies that I central and hence R is commutative.

Lemma: 2 Let d be a (α, β) - derivation of σ-prime ring R satisfies $d\sigma = \pm\ \sigma d$ and let I be a non-zero σ-ideal of R. If $d^2(I) = 0$ then d = 0.

Proof: for any $x\in I\ d^2(x) = 0.$

Replacing x by xy we obtain

$d^2(xy) = 0$ that is $d(d(xy) = 0.$

$\Rightarrow d(\ d(x)\alpha(y) + \beta(x)d(y)\ ) = 0.$

$\Rightarrow d^2(x)\ \alpha^2(y) + \beta(d(x))\ d(\alpha(y)) + d(\beta(x))\ \alpha(d(y)) + \beta^2(x)\ d^2(y) = 0.$

The fact that $d^2(I) = 0$ we get

$\beta(d(x))\ d(\alpha(y)) + d(\beta(x))\ \alpha(d(y)) = 0.$ (5)

If we assume that $\beta d = d\beta$, $d\alpha = \alpha d$. Then (5) is reduced to

$2\ \beta(d(x))\ d(\alpha(y)) = 0$ implie that $\beta(d(x))\ d(\alpha(y)) = 0$ (since char $R \neq 2$) (6)

Replacing x by xz in (6) where $z \in I$ then we have

$\beta(d(xz))\ \alpha(d(y)) = 0$. (d commutes with α)

$= \beta(d(x)\ \alpha(z) + \beta(x) d(z))\ \alpha(d(y))$

$= \beta(d(x))\ \beta(\alpha(z))\ \alpha(d(y)) + \beta^2(x)\ \beta(d(z))\ \alpha(d(y)$

Replacing z by d(z) we get

$\Rightarrow \beta(d(x))\ \beta(\alpha(d(z)))\ \alpha(d(y)) + \beta^2(x)\ \beta(d^2(z))\ \alpha(d(y) = 0.$

$\Rightarrow \beta(d(x))\ \beta(\alpha(d(z)))\ \alpha(d(y)) = 0.$ (since $d^2(z) = 0$)

$\Rightarrow \beta^{-1}(\beta(d(x)\ \beta(\alpha(d(z)))\ \alpha(d(y)) = 0.$

$\Rightarrow d(x)\ \alpha(d(z))\ \beta^{-1}(\alpha(d(y))) = 0.\ \forall\ x \in U.$

$\Rightarrow d(U)\ \alpha(d(z)\ \beta^{-1}(\alpha(d(y))) = 0$

Now we use the following lemma without proof

Let d be a non-zero derivation of 2-torsion free σ-prime ring R which commutes with σ and $U \not\subset Z(R)$ be a σ-Lie ideal of R. If $t \in R$ verifies $td(U) = 0$. Or $d(u)t = 0$ then $t = 0$. and hence

$\alpha(d(z))\ \beta^{-1}(\alpha(d(y))) = 0.$

$\Rightarrow \beta(\alpha(d(z)))\ R\alpha(d(y)) = 0.$

since d commutes with σ and I is σ- ideal we have $\beta(\alpha(d(z)))\ R\ \alpha(d(y)) = \beta(\alpha(d(z))\ R\ \alpha(\sigma(d(y))) = 0.$

By the definition of σ-prime ring we have either $\beta(\alpha(d(z)) = 0$. Or $\alpha(d(y) = 0$.

If $\alpha(d(y) = 0$ then $d(y) = 0\ \forall y \in I$ (since α is automorphism)

Replace y by yr we have then

$d(yr) = 0.$

$\Rightarrow d(y)\ \alpha(r) + \beta(y) d(r) = 0.$

$\Rightarrow \beta(y)) d(r) = 0.$ $(d(y) = 0)$

$\Rightarrow \beta^{-1}(\beta(y)\ d(r) = 0.$

$\Rightarrow IRd(r) = 0.\ \forall r \in R.$

$\Rightarrow d(r) = 0.\ \forall r \in R$. This implies $d = 0$.

Lemma: 3 Let d_1 and d_2 be (α, β) -derivations of R such that $d_1\sigma = \pm \sigma d_1$ and $d_2\sigma = \pm \sigma d_2$.and d_1 commutes with β. If $I \neq 0$ is a σ- ideal of R such that $d_2(I) \subset I$ and $d_1 d_2(I) = 0$ then $d_1 = 0$ or $d_2 = 0$.

Proof: Let $u, v \in I$ then

$0 = d_1 d_2(uv) = d_1(d_2(u)\ \alpha(v) + \beta(u)\ d_2(v))$

Again applying (α, β)-derivation d_1 then

$= d_1(d_2(u))\,\alpha^2(v) + \beta(d_2(u))\,d_1(\alpha(v)) + d_1(\beta(u))\,\alpha(d_2(v)) + \beta^2(u)d_1d_2(v).$

Using the hypothesis the above equation can be written as

$\beta(d_2(u))\,d_1(\alpha(v)) + d_1(\beta(u))\,\alpha(d_2(v)) = 0.$

Replacing $\alpha(v)$ by $d_2(v)$ in the above equation we get $d_1(\beta(u))\alpha(d_2(v)) = 0$ (7)

Replacing v by vw where $w\in I$ in equation (7) we get

$\Rightarrow d_1(\beta(u))\,\alpha(d_2(vw)) = 0$.

$\Rightarrow d_1(\beta(u))\,\alpha\,(d_2(v)\,\alpha(w) + \beta(v)d_2(w)) = 0.$

$\Rightarrow d_1(\beta(u))\alpha(d_2(v))\alpha^2(w) + d_1(\beta(u)\alpha(\beta(v)d_2(w)) = 0.$ (8)

$\Rightarrow d_1(\beta(u))\,\alpha(\beta(v))\,\alpha(d_2(w)) = 0.$ (By (7))

$\Rightarrow \alpha^{-1}(d_1(\beta(u)))\,R\,d_2(w) = 0.\ \forall u, w\in I$ (9)

Now consider $\sigma(d_2(w) = d_2(\sigma(w)) = d_2(w)$ (since I is σ ideal.)

Since R is σ-prime then from equation (9) we obtain

$\Rightarrow \alpha^{-1}(d_1(\beta(u)))R\,d_2(w) = 0 = \alpha^{-1}(d_1(\beta(u)))R\,\sigma(d_2(w))$

By the definition of σ-prime we obtain either $\alpha^{-1}(d_1(\beta(u))) = 0$ or $d_2(w) = 0$

Case - 1: If $\alpha^{-1}(d_1(\beta(u))) = 0$ then

$(d_1(\beta(u)) = 0$ (since α^{-1} is an automorphism)

$\Rightarrow \beta(d_1(u) = 0\ \forall u\in I.$($d_1$ commutes with β)

$\Rightarrow d_1(u) = 0\ \forall u\in I.$

For any $r\in R$ we have $d_1(ur) = 0$ then $d_1(u)\alpha(r)+\beta(u)d_1(r) = 0$

Since $d_1(u) = 0\ \forall u\in I$ the above equation reduces to $\beta(u)d_1(r) = 0.$

$\beta^{-1}(\beta(u)\,d_1(r)) = 0 = I\ \beta\text{-}1\ (d_1(r)) = 0.$

Since I is non-zero the last relation yields $d_1(r) = 0\ \forall r\in R$. Hence $d_1 = 0$.

Similarly for any $r\in R$ $d_2(wr) = 0$ then $\beta(w)d_2(r) = 0$. (Using definition of (α, β) derivation).

$\beta^{-1}(\beta(w)\,d_2(r)) = 0 = I\,d_2(r) = 0.$

Since $I \neq 0$ the last relation yields $d_2(r) = 0\ \forall r\in R$ ie $d_2 = 0$ on R.

Theorem: Let R be a σ-prime ring with characterstic≠ 2. I be a non -zero σ-ideal of R and d be a non-zero (α, β) derivation of R which commutes with σ. If $[d(x), x] \in Z(R)\ \forall x\in I$ then R is commutative.

Proof: Let $[d(x), x] \in Z(R)\ \forall x\in I$

By linearizing above equation we get $[d(x), y] + [d(y), x] \in Z(R)\ \forall x, y\in I.$ (10)

Replacing y by x^2 in (10) we have $[d(x), x^2] + [d(x)\,\alpha(x) + \beta(x)d(x), x]\in Z(R)\ \forall x\in I$

So $[d(x), x]\, \alpha(x) \in Z(R)$. For any $r \in R$ we have that

$\alpha(r)\alpha(x)\, [d(x), x] = \alpha(x)\, [d(x), x]\;\; \alpha(r) = \alpha(x)\, \alpha(r)\, [d(x), x]$.

$[\alpha(x), \alpha(r)][d(x),x]=0$ (11)

Since $[d(x), x] \in Z(R)$, we get from (11)

$[\alpha(x), \alpha(r)]\;\; R[d(x), x]\;\; = 0$. Also, I is σ- ideal so $\sigma([d(x),x]) = [d(x),x]$.

By σ-primeness of R we get either $[\alpha(x), \alpha(r)] = 0$ or $[d(x), x] = 0$

Case - 1: If $[d(x), x] = 0$ then by lemma 3 of [7] R is commutative.

Case - 2: If $[\alpha(x), \alpha(r)] = 0$ then $\alpha(x)\, \alpha(r)- \alpha(r)\, \alpha(x) = 0$.

$\alpha(xr)- \alpha(rx) = \alpha[x, r] = 0\ \forall\ x\in I,\ r\in R$.

Since $\alpha \neq 0$ is automorphism we get $[x, r] = 0\ \forall x\in I,\ r\in R$.

Therefore $I \subseteq Z(R)$. Replacing r by rs, where $r, s\in R$ in the above equation we get

$[x, rs] = 0$. Expanding this equation we get $rsx= rxs = srx$

We conclude that $[r, s]x = 0$ then $[r, s]\, I = 0$. $\forall\ r, s\in R$.

$[r, s]\;\; = 0\ \forall\ r, s\in R$. $(I\neq 0)$

Hence R is commutative.

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