

(α, β) DERIVATIONS AND COMMUTATIVITY IN σ -PRIME RING

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ABSTRACT

Let R be a σ -prime ring with characteristic $\neq 2$ and d be a nonzero (α, β) derivation of R commuting with σ . It is prove that a non-zero (α, β) - derivation d associated with be a non-zero σ -ideal I of R which commutes σ and i) if $[d(x), x] = 0 \forall x \in I$ then R is commutative ii) If $d^2(I) = 0$ then $d = 0$. Also we prove that R must be commutative under some suitable conditions.

Keywords: σ -prime ring, σ -ideals, (α, β) derivations.

1. INTRODUCTION

Through out the present paper all rings will be associative. A ring R equipped with an involution σ is said to be σ -prime if $aRb = aR \sigma(b) = 0$ implies that $a = 0$ or $b = 0$. Recall that a ring R is prime if $aRb = 0$ implies that $a = 0$ or $b = 0$. Obviously, every prime ring with involution σ is σ -prime but the converse is in general not true. An ideal I of R is a σ -ideal if I is invariant under σ i.e, $\sigma(I) = I$. Set of all symmetric and skew-symmetric elements of R is defined as $Sa_\sigma(R) = \{x \in R / \sigma(x) = \pm x\}$. we shall use the basic commutator identities: $[xy, z] = x[y, z] + [x, z]y$ and $[x, yz] = y[x, z] + x[y, z]$. An additive mapping d from R to itself is a derivation if $d(xy) = d(x)y + xd(y)$. holds for all pairs $x, y \in R$. An additive mapping d from R to itself said to be (α, β) derivation if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ hold for all $x, y \in R$. A mapping $F: R \rightarrow R$ is said to be centralizing on a subset S of R if $[F(s), s] \in Z(R)$ for all $s \in S$, where $Z(R)$ denotes the center of R ; In the special case where $[F(s), s] = 0$ for all s in S , the mapping F is said to be commuting on S .

The history of commuting and centralizing mapping goes back to 1955 when Divinsky [1] prove that simple artinian ring is commutative if it has commuting non-trivial automorphism. Two years later Posner [2] have proved that the existence of non-zero centralizing derivation on a prime ring forces Ring to be commutative (Posner's second theorem). Mayne [3] prove the analogous result for centralizing automorphisms. P. H. Lee and T.K. Lee [4] have shown that if a prime ring of characteristic different from two has a non-zero derivation d satisfying $[d(R), d(R)]$ then R is commutative. Joso Vukman [5] have shown that a prime ring of characteristic not two possessing a non-zero derivation D from R to itself such that $x \rightarrow [D(x), x]$ is commuting on R then R is commutative. M. Bresar [6] describe the structure of arbitrary additive mapping which is centralizing on a prime ring. Later L. oukhtite and S. Salhi proved some conditions under which derivations of σ -prime ring are commutative (Refer [7], [8]). These results are motivation for our results.

We prove the following results

3. MAIN RESULTS

Lemma: 1 Let I be a non-zero σ -ideal of σ -prime ring R and d be a (α, β) derivation on R which commutes with σ , If $[d(x), x] = 0. \forall x \in I$ then R is commutative.

Proof: By Hypothesis $[d(x), x] = 0$ (1)

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Linearizing equation (1) we get

$$[d(x+y), x+y] = 0.$$

$$[d(x)+d(y), x+y] = 0.$$

$$[d(x),y]+ [d(y), x] = 0. \tag{2}$$

Replace y by yx and $d \neq 0$ be a (α, β) derivation.

$$[d(x), yx] + [d(yx), x] = 0.$$

$$[d(x), y]x + y[d(x), x] + [d(y)\alpha(x) + \beta(y)d(x), x] = 0.$$

$$[d(x), y]x + [d(y) \alpha(x), x] + [\beta(y)d(x), x] = 0.$$

$$[d(x), y]x + d(y)[\alpha(x), x] + [d(y), x] \alpha(x) + \beta(y) [d(x), x] + [\beta(y), x]d(x). \tag{3}$$

Replace $\alpha(x)$ by x in (3) and using (2) we obtain

$$[\beta(y), x]d(x) = 0. \tag{4}$$

For any $r \in R$, Replacing y by ry in above equation

$$[\beta(ry), x]d(x) = 0.$$

$$\Rightarrow [\beta(r)\beta y, x]d(x) = 0.$$

$$\Rightarrow \beta(r) [\beta y, x]d(x) + [\beta(r), x]\beta(y)d(x) = 0 \tag{4}$$

$$\Rightarrow [\beta(r), x] \beta(y)d(x) = 0 \quad \forall r \in R, x, y \in I. \text{ (By (4))}$$

$$\Rightarrow 0 = \beta^{-1}([\beta(r), x]) I \beta^{-1}(d(x)) = 0. \quad \forall x \in I, r \in R.$$

Since d commutes with σ and I is σ -ideal of R we have

$$0 = \beta^{-1}([\beta(r), x]) I \beta^{-1}(d(x)) = \beta^{-1}([\beta(r), x]) I \beta^{-1}(\sigma(d(x))).$$

By Lemma 1 of [7] we obtain $\beta^{-1}[\beta(r), x] = 0$ or $\beta^{-1}(d(x)) = 0$.

Case -1: if $\beta^{-1}[\beta(r), x] = 0 \quad \forall x \in I, r \in R$.

Then $[\beta(r), x] = 0$ (since β is automorphism).

Since β is automorphism, this implies that I central and hence R is commutative.

Lemma: 2 Let d be a (α, β) - derivation of σ -prime ring R satisfies $d\sigma = \pm \sigma d$ and let I be a non-zero σ -ideal of R . If $d^2(I) = 0$ then $d = 0$.

Proof: for any $x \in I$ $d^2(x) = 0$.

Replacing x by xy we obtain

$$d^2(xy) = 0 \text{ that is } d(d(xy)) = 0.$$

$$\Rightarrow d(d(x)\alpha(y) + \beta(x)d(y)) = 0.$$

$$\Rightarrow d^2(x) \alpha^2(y) + \beta(d(x)) d(\alpha(y)) + d(\beta(x)) \alpha(d(y)) + \beta^2(x) d^2(y) = 0.$$

The fact that $d^2(I) = 0$ we get

$$\beta(d(x)) d(\alpha(y)) + d(\beta(x)) \alpha(d(y)) = 0. \tag{5}$$

If we assume that $\beta d = d\beta$, $d\alpha = \alpha d$. Then (5) is reduced to

$$2 \beta (d(x)) d(\alpha(y)) = 0 \text{ imply that } \beta(d(x)) d(\alpha(y)) = 0 \text{ (since char } R \neq 2) \tag{6}$$

Replacing x by xz in (6) where $z \in I$ then we have

$$\begin{aligned} \beta (d(xz)) \alpha(d(y)) &= 0. \text{ (d commutes with } \alpha) \\ &= \beta (d(x) \alpha(z) + \beta(x)d(z)) \alpha(d(y)) \\ &= \beta (d(x)) \beta (\alpha(z)) \alpha(d(y)) + \beta^2 (x) \beta(d(z)) \alpha(d(y)) \end{aligned}$$

Replacing z by $d(z)$ we get

$$\Rightarrow \beta (d(x)) \beta (\alpha (d(z))) \alpha(d(y)) + \beta^2 (x) \beta(d^2 (z)) \alpha(d(y)) = 0.$$

$$\Rightarrow \beta (d(x)) \beta (\alpha(d(z))) \alpha(d(y)) = 0. \text{ (since } d^2 (z) = 0)$$

$$\Rightarrow \beta^{-1} (\beta (d(x)) \beta (\alpha(d(z))) \alpha(d(y))) = 0.$$

$$\Rightarrow d(x) \alpha(d(z)) \beta^{-1}(\alpha(d(y))) = 0. \forall x \in U.$$

$$\Rightarrow d(U) \alpha(d(z)) \beta^{-1}(\alpha(d(y))) = 0$$

Now we use the following lemma without proof

Let d be a non-zero derivation of 2-torsion free σ -prime ring R which commutes with σ and $U \not\subset Z(R)$ be a σ -Lie ideal of R . If $t \in R$ verifies $td(U) = 0$. Or $d(u)t = 0$ then $t = 0$. and hence

$$\alpha(d(z)) \beta^{-1}(\alpha(d(y))) = 0.$$

$$\Rightarrow \beta (\alpha(d(z))) R\alpha(d(y)) = 0.$$

since d commutes with σ and I is σ - ideal we have $\beta(\alpha(d(z))) R \alpha(d(y)) = \beta(\alpha(d(z))) R \alpha(\sigma (d(y))) = 0$.

By the definition of σ -prime ring we have either $\beta(\alpha(d(z))) = 0$. Or $\alpha(d(y)) = 0$.

If $\alpha(d(y)) = 0$ then $d(y) = 0 \forall y \in I$ (since α is automorphism)

Replace y by yr we have then

$$d(yr) = 0.$$

$$\Rightarrow d(y) \alpha(r) + \beta(y)d(r) = 0.$$

$$\Rightarrow \beta (y)d(r) = 0. \text{ (d(y) = 0)}$$

$$\Rightarrow \beta^{-1} (\beta (y) d(r)) = 0.$$

$$\Rightarrow IRd(r) = 0. \forall r \in R.$$

$$\Rightarrow d(r) = 0. \forall r \in R. \text{ This implies } d = 0.$$

Lemma: 3 Let d_1 and d_2 be (α, β) -derivations of R such that $d_1\sigma = \pm \sigma d_1$ and $d_2\sigma = \pm \sigma d_2$. and d_1 commutes with β . If $I \neq 0$ is a σ - ideal of R such that $d_2(I) \subset I$ and $d_1 d_2(I) = 0$ then $d_1 = 0$ or $d_2 = 0$.

Proof: Let $u, v \in I$ then

$$0 = d_1 d_2(uv) = d_1(d_2(u) \alpha(v) + \beta (u) d_2(v))$$

Again applying (α, β) -derivation d_1 then

$$= d_1(d_2(u)) \alpha^2(v) + \beta(d_2(u)) d_1(\alpha(v)) + d_1(\beta(u)) \alpha(d_2(v)) + \beta^2(u)d_1d_2(v).$$

Using the hypothesis the above equation can be written as

$$\beta (d_2(u)) d_1(\alpha(v)) + d_1(\beta(u)) \alpha(d_2(v)) = 0.$$

$$\text{Replacing } \alpha(v) \text{ by } d_2(v) \text{ in the above equation we get } d_1(\beta(u))\alpha(d_2(v)) = 0 \tag{7}$$

Replacing v by vw where $w \in I$ in equation (7) we get

$$\Rightarrow d_1(\beta(u)) \alpha(d_2(vw)) = 0 .$$

$$\Rightarrow d_1(\beta(u)) \alpha (d_2(v) \alpha(w) + \beta(v)d_2(w)) = 0.$$

$$\Rightarrow d_1(\beta(u))\alpha(d_2(v))\alpha^2(w) + d_1(\beta(u))\alpha(\beta(v)d_2(w)) = 0. \tag{8}$$

$$\Rightarrow d_1(\beta(u)) \alpha(\beta(v)) \alpha(d_2(w)) = 0. \text{ (By (7))}$$

$$\Rightarrow \alpha^{-1}(d_1(\beta(u))) R d_2(w) = 0. \forall u, w \in I \tag{9}$$

Now consider $\sigma(d_2(w)) = d_2(\sigma(w)) = d_2(w)$ (since I is σ ideal.)

Since R is σ -prime then from equation (9) we obtain

$$\Rightarrow \alpha^{-1} (d_1(\beta(u)))R d_2(w) = 0 = \alpha^{-1}(d_1(\beta(u)))R \sigma (d_2(w))$$

By the definition of σ -prime we obtain either $\alpha^{-1}(d_1(\beta(u))) = 0$ or $d_2(w) = 0$

Case - 1: If $\alpha^{-1}(d_1(\beta(u))) = 0$ then

$$(d_1(\beta(u))) = 0 \text{ (since } \alpha^{-1} \text{ is an automorphism)}$$

$$\Rightarrow \beta (d_1(u)) = 0 \forall u \in I. (d_1 \text{ commutes with } \beta)$$

$$\Rightarrow d_1(u) = 0 \forall u \in I.$$

For any $r \in R$ we have $d_1(ur) = 0$ then $d_1(u)\alpha(r) + \beta(u)d_1(r) = 0$

Since $d_1(u) = 0 \forall u \in I$ the above equation reduces to $\beta(u)d_1(r) = 0$.

$$\beta^{-1}(\beta(u) d_1(r)) = 0 = I \beta^{-1} (d_1(r)) = 0.$$

Since I is non-zero the last relation yields $d_1(r) = 0 \forall r \in R$. Hence $d_1 = 0$.

Similarly for any $r \in R$ $d_2(wr) = 0$ then $\beta(w)d_2(r) = 0$. (Using definition of (α, β) derivation).

$$\beta^{-1}(\beta(w) d_2(r)) = 0 = I d_2(r) = 0.$$

Since $I \neq 0$ the last relation yields $d_2(r) = 0 \forall r \in R$ ie $d_2 = 0$ on R .

Theorem: Let R be a σ -prime ring with characteristic $\neq 2$. I be a non-zero σ -ideal of R and d be a non-zero (α, β) derivation of R which commutes with σ . If $[d(x), x] \in Z(R) \forall x \in I$ then R is commutative.

Proof: Let $[d(x), x] \in Z(R) \forall x \in I$

$$\text{By linearizing above equation we get } [d(x), y] + [d(y), x] \in Z(R) \forall x, y \in I. \tag{10}$$

Replacing y by x^2 in (10) we have $[d(x), x^2] + [d(x) \alpha(x) + \beta(x)d(x), x] \in Z(R) \forall x \in I$

So $[d(x), x] \alpha(x) \in Z(R)$. For any $r \in R$ we have that

$$\alpha(r)\alpha(x) [d(x), x] = \alpha(x) [d(x), x] \alpha(r) = \alpha(x) \alpha(r) [d(x), x].$$

$$[\alpha(x), \alpha(r)][d(x), x] = 0 \tag{11}$$

Since $[d(x), x] \in Z(R)$, we get from (11)

$$[\alpha(x), \alpha(r)] R[d(x), x] = 0. \text{ Also, } I \text{ is } \sigma\text{-ideal so } \sigma([d(x), x]) = [d(x), x].$$

By σ -primeness of R we get either $[\alpha(x), \alpha(r)] = 0$ or $[d(x), x] = 0$

Case - 1: If $[d(x), x] = 0$ then by lemma 3 of [7] R is commutative.

Case - 2: If $[\alpha(x), \alpha(r)] = 0$ then $\alpha(x) \alpha(r) - \alpha(r) \alpha(x) = 0$.

$$\alpha(xr) - \alpha(rx) = \alpha[x, r] = 0 \quad \forall x \in I, r \in R.$$

Since $\alpha \neq 0$ is automorphism we get $[x, r] = 0 \quad \forall x \in I, r \in R$.

Therefore $I \subseteq Z(R)$. Replacing r by rs , where $r, s \in R$ in the above equation we get

$$[x, rs] = 0. \text{ Expanding this equation we get } rsx = rxs = srx$$

We conclude that $[r, s]x = 0$ then $[r, s] I = 0. \quad \forall r, s \in R$.

$$[r, s] = 0 \quad \forall r, s \in R. (I \neq 0)$$

Hence R is commutative.

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