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STAR ATTACHED ODD-EVEN GRACEFUL GRAPHS

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ABSTRACT

The Odd-Even graceful labeling of a graph G with q edges means that there is an injection f : V(G) to {1,3,5,…,2q+1} such that, when each edge uv is assigned the label $|f(u)-f(v)|$, the resulting edge labels are {2,4,6,…,2q }.A graph which admits an Odd-Even graceful labeling is called an Odd-Even graceful graph. In this paper, we prove that some graphs namely <$B_{m,n}$: w>, (P_n , S_m), ($H_n \odot mK_1$), <$K_{1,n}^{(1)}$,$K_{1,n}^{(2)}$, … , $K_{1,n}^{(m)}$> are Odd-Even graceful.

Key words and phrases: *Graceful labeling, Odd graceful labeling, Odd-Even graceful graph.*

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1. INTRODUCTION

By a graph, we mean a finite undirected graph without loops or multiple edges. A path on *n* vertices is denoted by P_n . G^+ is a graph obtained from the graph G by attaching a pendant vertex at each vertex of G. Graph labeling, where the vertices are assigned certain values subject to some conditions, have often motivated by practical problems. In the last five decades enormous work has been done on this subject [1]. The concept of graceful labeling was introduced by Rosa[3] in 1967.

A function *f* is a graceful labeling of a graph G with q edges if *f* is an injection from the vertices of G to the set {0,1,2,…,q} such that, when each edge uv is assigned the label $|f(u)-f(v)|$, the resulting edge labels are distinct. Golomb[2] subsequently called such labeling are graceful and this is now the popular term. A graph G with q edges to be odd-graceful if there is an injection *f* from V(G) to {0,1,2,…,2q−1} such that, when each edge *uv* is assigned the label $|f(u)-f(v)|$,the resulting edge labels are {1,3,…,2q−1}. Sridevi R, Navaneethakrishnan S defined the Odd-Even graceful labeling [4]. The Odd-Even graceful labeling of a graph G with q edges is an injection *f* : V(G) to {1,3,5,…,2q+1} such that, when each edge uv is assigned the label $|f(u)-f(v)|$, the resulting edge labels are {2,4,6,…,2q }. A graph which admits an Odd-Even graceful labeling is called an Odd-Even graceful graph. In this paper, we prove that some graphs namely <$B_{m,n}$: w>, (P_n , S_m) , ($H_n \odot mK_1$), <$K_{1,n}^{(1)}$,$K_{1,n}^{(2)}$,, … , $K_{1,n}^{(m)}$> are Odd-Even graceful.

2. MAIN RESULTS

Definition: 2.1 A (p, q)-graph G = (V,E) is said to be an Odd-Even graceful if there exists an injection f from V into { 1,3,5,…,2q+1} such that f*(E)={2,4,6,…,2q } where f*(uv) = |f(u) − f(v)| for any edge uvϵE.

Theorem: 2.2 <$B_{m,n}$: w> is an Odd-Even graceful graph.

Proof: Let G = <$B_{m,n}$: w>.

Let V(G) = {(u_{i},u_{1j}: 1≤ $i \le 3, 1 \le j \le m$) ; (u_{3k}:1 $\le k \le n$)} .

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Let
E(G) = {(u_1u_2)∪ (u_2u_3) ∪ (u_1u_{1j}: $1 \leq j \leq m$) ∪(u_3u_{3j} :$1 \leq j \leq n$)} and $|V(G)|$ = m+n+3 and $|E(G)|$ = m+n+2.

Define f: V(G) →{1,3,5,…,(2m+2n+5)} as

f(u_1) = 1, f(u_2) = 2n+5 , f(u_3) = 3

f(u_{1i}) = 2(m+n−i) + 7,1≤ $i \leq m$

f(u_{3i}) = 2i+3, 1≤ $i \leq n$.

The induced edge labels are given in the set E as follows:
E_1 = {$|f(u_1) - f(u_{1i})|$: 1≤ $i \leq m$}
= {2(m+n+3)−2i : 1≤ $i \leq m$}

E_2 = {$|f(u_3) - f(u_{3i})|$: 1≤ $i \leq n$}
= {2i : 1≤ $i \leq n$}

E_3 = {$|f(u_1) - f(u_2)|, |f(u_2) - f(u_3)|$ }
= {(2n+4), (2n+2)} .

Thus, $E = E_1 \cup E_2 \cup E_3$
= {2, 4, 6,…, (2m+2n+4)} .

It shows the edge labels are distinct.

Hence, <$B_{m,n}$: w> is an Odd-Even graceful graph.

For example, the Odd-Even graceful labeling of < $B_{8,9}$: w> is shown in figure 2.3.

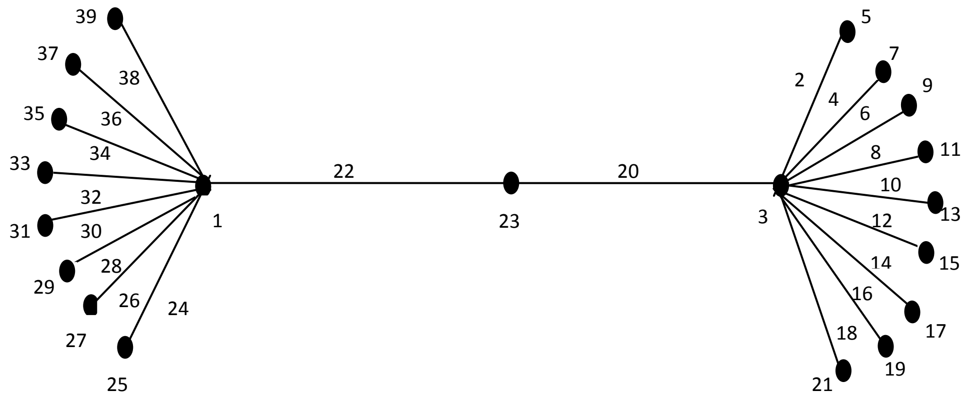

Figure 2.3: < $B_{8,9}$: w>

Theorem: 2.4 (P_n , S_m) is an Odd-Even graceful graph.

Proof: Let G = (P_n , S_m).

Let V(G) = {(v_i : $1 \leq i \leq n$), (u_i^j : $1 \leq i \leq n, 1 \leq j \leq m$)} .

Let
E(G) = {($v_i v_{i+1}$): $1 \leq i \leq n-1$)} ∪ {($v_i u_i^j$) : $1 \leq i \leq n, 1 \leq j \leq m$)} and $|V(G)|$ = n(m+1) and $|E(G)|$ = n(m+1)−1.

Case (i): When n is odd.

Define f: V(G) → {1,3,5,…,(2mn+2n−1)} as

f(v_1) = 1

f(v_{2i+1}) = (2m+3)+(2m+2)(i−1), $1 \leq i \leq \frac{n-1}{2}$

f(v_{2i}) = (2mn+2n−$2m-1) - (2m+2)(i-1), 1 \leq i \leq \frac{n-1}{2}$

f(u_j^{2i}) = 3+2(j−1) $+ (2m+2)(i-1);\ 1 \leq j \leq m, 1 \leq i \leq \frac{n-1}{2}$

f($u_j^{(2i-1)}$) = (2q+1)−2(j−1)−(2m+2)(i−1); $1 \leq j \leq m, 1 \leq i \leq \frac{n+1}{2}$.

The induced edge labeling are given in the set E as follows:

$E_1 = \{|f(v_{n+1-i}) - f(v_{n-i})| : 1 \leq i \leq n-1\}$
 = {(2m+2)i : $1 \leq i \leq n-1\}$

$E_2 = \{\left|f(v_{n+1-i}) - f(u_{m+1-j}^{n+1-i})\right| : 1 \leq i \leq n, 1 \leq j \leq m\}$
 = {(2m+2)(i−1) $+ 2j : 1 \leq i \leq n, 1 \leq j \leq m\}$.

Therefore, E= $E_1 \cup E_2$
 = {2, 4, 6,…, (2mn+4m−4)}.

It shows the edge labels are distinct.

Hence, (P_n , S_m) is an Odd-Even graceful graph.

For example, the Odd-Even graceful labeling of (P_5 , S_6) is shown in figure 2.5.

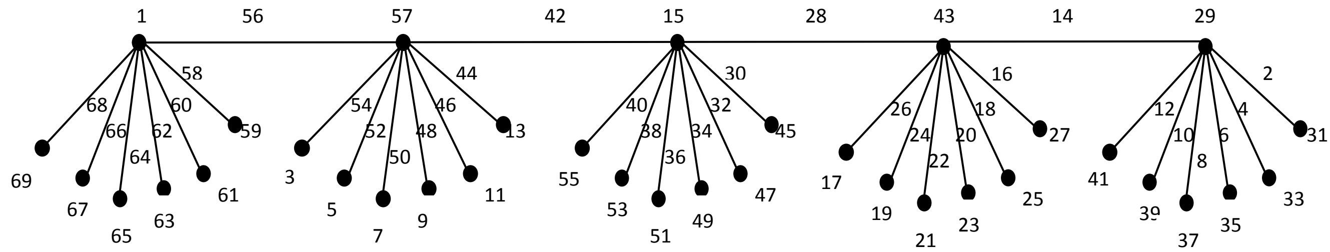

Figure 2.5. (P_5, S_6)

Case (i): When n is even.

Define f: V(G) →{1,3,5,…,(2mn+2n−1)} **as**

f(v_1) = 1

f(v_{2i+1}) = (2m+3)+(2m+2)(i−1),$1 \leq i \leq \frac{n}{2} - 1$

f(v_{2i}) = (2mn+2n−$2m-1) - (2m+2)(i-1), 1 \leq i \leq \frac{n}{2}$

f(u_j^{2i}) = 3+2(j−1) $+ (2m+2)(i-1);\ 1 \leq j \leq m, 1 \leq i \leq \frac{n}{2}$

f($u_j^{(2i-1)}$) = (2q+1)−2(j−1)−(2m+2)(i−1); $1 \leq j \leq m, 1 \leq i \leq \frac{n}{2}$.

The edge labels are same as in case (i).

It shows the edge labels are distinct.

Hence, (P_n , S_m) is an Odd-Even graceful graph.

For example, the Odd-Even graceful labeling of (P_6, S_5) is shown in figure 2.6.

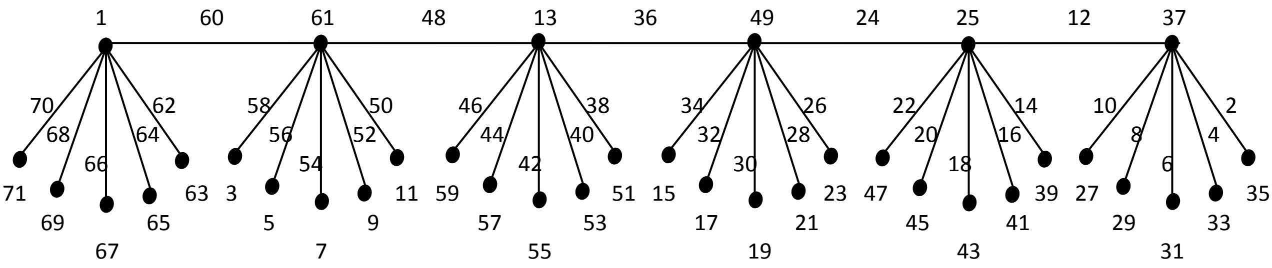

Figure 2.6: $(P_6\ , S_5)$

Theorem: 2.7 ($H_n \odot mK_1$) is an Odd-Even graceful graph

Proof: Let G = ($H_n \odot mK_1$) .

Let V(G) = $\{(u_i, v_i : 1 \leq i \leq n), (u_i^j, v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$.

Let
E (G) =$\{(u_iu_{i+1}) \cup (v_iv_{i+1}) : 1 \leq i \leq n-1)\} \cup \{(u_iu_i^j) \cup (v_iv_i^j) : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{(u_{(n+1)/2}\ v_{(n+1)/2})\}$
and $|V(G)|$ = 2n (m+1) and $|E(G)|$ = 2n (m+1)−1.

Case (i): When n is odd.

Define f: V(G) →{1,3,5,…,(4mn+4n−1)}as

$f(u_{2i-1}) = (2m+2)(i-1) + 1\ , 1 \leq i \leq \frac{n+1}{2}$

$f(u_{2i}) = (2q+1) - 2m - (2m+2)(i-1), 1 \leq i \leq \frac{n-1}{2}$

$f(v_{2i-1}) = 3n(m+1) - m - (2m+2)(i-1), 1 \leq i \leq \frac{n+1}{2}$

$f(v_{2i}) = n(m+1) + (m+2) + (2m+2)(i-1), 1 \leq i \leq \frac{n-1}{2}$.

For $1 \leq i \leq \frac{n+1}{2}$,

$f(v_{(2i-1)}^j) = n(m+1) - (m-2) + 2(j-1) + (2m+2)(i-1), 1 \leq j \leq m$

$f(u_{(2i-1)}^j) = (2q+1) - 2(j-1) - (2m+2)(i-1), 1 \leq j \leq m$

For $1 \leq i \leq \frac{n-1}{2}$,

$f(u_{2i}^j) = 3 + 2(j-1) + (2m+2)(i-1), 1 \leq j \leq m$

$f(v_{2i}^j) = 3n(m+1) - (m+2) - 2(j-1) - (2m+2)(i-1), 1 \leq j \leq m$.

The induced edge labeling are given in the set E as follows:

$E_1 = \{|f(u_{n+1-i}) - f(u_{n-i})| : 1 \leq i \leq n-1\}$
$\quad = \{2n(m+1) + (2m+2)i : 1 \leq i \leq n-1\}$

$E_2 = \{|f(u_{n+1-i}) - f(u_{n+1-i}^{m+1-j})| : 1 \leq i \leq n, 1 \leq j \leq m\}$
$\quad = \{2n(m+1) + (2m+2)(i-1) - 2j : 1 \leq i \leq n, 1 \leq j \leq m\}$

$E_3 = \{|f(u_{(n+1)/2}) - f(v_{(n+1)/2})|\}$
$\quad = \{2n(m+1)\}$

$E_4 = \{|f(v_{n+1-i}) - f(v_{n-i})| : 1 \le i \le n-1\}$
$= \{(2m+2)i : 1 \le i \le n-1\}$

$E_5 = \{|f(v_{n+1-i}) - f(v_{n+1-i}^{m+1-j})| : 1 \le i \le n, 1 \le j \le m\}$
$= \{(2m+2)(i-1) + 2j : 1 \le i \le n, 1 \le j \le m\}$.

Therefore, E = $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$
$= \{2, 4, 6, \ldots, (4mn+4n-2)\}$.

It shows the edge labels are distinct.

Hence, $(H_n \odot mK_1)$ is an Odd-Even graceful graph.

For example, the Odd-Even graceful labeling of $(H_5 \odot 4K_1)$ is shown in figure 2.8.

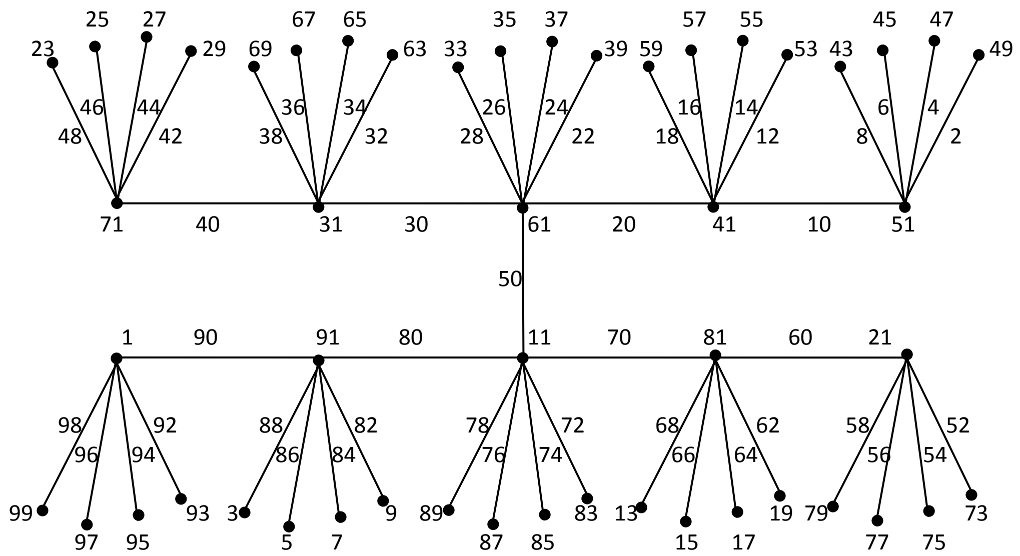

Figure 2.8: $(H_5 \odot 4K_1)$

Case (i): When n is even .

E(G)= $\{(u_iu_{i+1}) \cup (v_iv_{i+1}) : 1 \le i \le n-1)\} \cup \{(u_iu_i^j) \cup (v_iv_i^j) : 1 \le i \le n, 1 \le j \le m\} \cup (u_{(n+1)/2}\ v_{(n+1)/2}\}$

Define f: V(G) $\rightarrow \{1,3,5,\ldots,(4mn+4n-1)\}$ as

$f(u_{2i-1}) = (2m+2)(i-1) + 1, 1 \le i \le \frac{n}{2}$

$f(u_{2i}) = (2q+1) - 2m - (2m+2)(i-1), 1 \le i \le \frac{n}{2}$

$f(v_{2i-1}) = n(m+1)+1 + (2m+2)(i-1), 1 \le i \le \frac{n}{2}$

$f(v_{2i}) = 3n(m+1) - (2m+1) - (2m+2)(i-1), 1 \le i \le \frac{n}{2}$.

For $1 \le i \le \frac{n}{2}$,

$f(v_{(2i-1)}^j) = 3n(m+1) - 1 - 2(j-1) - (2m+2)(i-1), 1 \le j \le m$

$f(u_{(2i-1)}^j) = (2q+1) - 2(j-1) - (2m+2)(i-1) , 1 \le j \le m$

For $1 \le i \le \frac{n}{2}$,

$f(u_{2i}^j) = 3 + 2(j-1) + (2m+2)(i-1), 1 \le j \le m$

$f(v_{2i}^{j})$ = n(m+1)+3 +2(j−1)+(2m+2)(i−1), $1 \leq j \leq m$.

The induced edge labeling are given in the set E as follows:

$E_1 = \{|f(u_{n+1-i}) - f(u_{n-i})| : 1 \leq i \leq n-1\}$
$= \{2n(m+1)+(2m+2)i : 1 \leq i \leq n-1\}$

$E_2 = \{|f(u_{n+1-i}) - f(u_{n+1-i}^{m+1-j})| : 1 \leq i \leq n, 1 \leq j \leq m\}$
$= \{2n(m+1)+(2m+2)(i-1) - 2j : 1 \leq i \leq n, 1 \leq j \leq m\}$

$E_3 = \{|f(u_{(n+2)/2}) - f(u_{n/2})|\}$
$= \{2n(m+1)\}$

$E_4 = \{|f(v_{n+1-i}) - f(v_{n-i})| : 1 \leq i \leq n-1\}$
$= \{(2m+2)i : 1 \leq i \leq n-1\}$

$E_5 = \{|f(v_{n+1-i}) - f(v_{n+1-i}^{m+1-j})| : 1 \leq i \leq n, 1 \leq j \leq m\}$
$= \{(2m+2)(i-1) + 2j : 1 \leq i \leq n, 1 \leq j \leq m\}$.

Therefore, E = $E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$
= {2, 4, 6,…, (4mn+4n−2)}.

It shows the edge labels are distinct.

Hence, $(H_n \odot mK_1)$ is an Odd − Even graceful graph.

For example, the Odd-Even graceful labeling of $(H_6 \odot 3K_1)$ is shown in figure 2.9

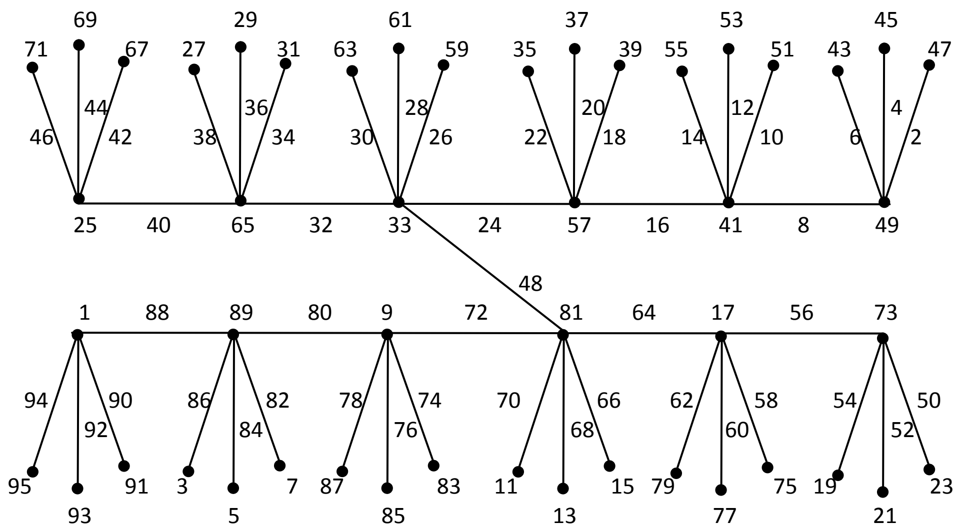

Figure.2.9: $(H_6 \odot 3K_1)$

Definition: 2.10[5] Consider two stars $K_{1,n}^{(1)}$ and $K_{1,n}^{(2)}$. Then G = < $K_{1,n}^{(1)}$, $K_{1,n}^{(2)}$ > is the graph obtained by joining apex (central) vertices of stars to a new vertex x.

Note that G has (2n+3) vertices and (2n+2) edges.

Definition: 2.11[5] Consider t copies of stars namely $K_{1,n}^{(1)}$,$K_{1,n}^{(2)}$, … , $K_{1,n}^{(t)}$. Then G = < $K_{1,n}^{(1)}$,$K_{1,n}^{(2)}$, … , $K_{1,n}^{(t)}$> is the graph obtained by joining apex vertices of each $K_{1,n}^{(m-1)}$ and $K_{1,n}^{(m)}$ to a new vertex $x_{(m-1)}$ where 2≤m≤t.

Note that G has t(n+2)-1 vertices and t(n+2)-2 edges.

Theorem: 2.12 $<K_{1,n}^{(1)}, K_{1,n}^{(2)}, \dots, K_{1,n}^{(m)}>$ is an Odd-Even graceful graph.

Proof: Let G = $<K_{1,n}^{(1)}, K_{1,n}^{(2)}, \dots, K_{1,n}^{(m)}>$ **.**

Let V(G) = $\{(u_i^{(j)}: 1 \le i \le n; 1 \le j \le m), (w_i: 1 \le i \le m-1), (v_i: 1 \le i \le m)\}$.

Let
E(G) = $\{(v_j u_i^j): 1 \le i \le n; 1 \le j \le m\} \cup \{(w_i v_i) \cup (w_i v_{i+1}): 1 \le i \le m-1\}$ and $|V(G)|$ = m(n+2) −1 and $|E(G)|$= m(n+2) −2 .

Define f: V(G) →{ 1,3,5,…,(2mn+4m−3)} as

$f(u_i^j) = (2m+1)+2(n+1)(j-1)+2(i-1),\ 1 \le i \le n; 1 \le j \le m$

$f(w_j) = (2m+1)+2(n+1)(j-1)+2(n-1)+2,\ 1 \le j \le m-1$

$f(v_{m-(j-1)}) = 2j-1, 1 \le j \le m$.

The induced edge labeling are given in the set E as follows:
$E_1 = \{|f(u_i^j) - f(v_j)| : 1 \le i \le n, 1 \le j \le m\}$
$\quad = \{2i+2(n+2)(j-1) : 1 \le i \le n; 1 \le j \le m\}$

$E_2 = \{|f(w_j) - f(v_j)| : 1 \le j \le m\} \cup \{|f(w_j) - f(v_{j+1})| : 1 \le j \le m\}$
$\quad = \{2(n+2)i-2 : 1 \le i \le n\} \cup \{2(n+2)i : 1 \le i \le n\}$.

Therefore, E = $E_1 \cup E_2$
$\quad\quad = \{2, 4, 6, \dots, (2mn+4m-4)\}$.

It shows the edge labels are distinct.

Hence, $<K_{1,n}^{(1)}, K_{1,n}^{(2)}, \dots, K_{1,n}^{(m)}>$ is an Odd-Even graceful graph.

For example, the Odd-Even graceful labeling of $<K_{1,8}^{(1)}, \dots, K_{1,8}^{(4)}>$ is shown in figure 2.13.

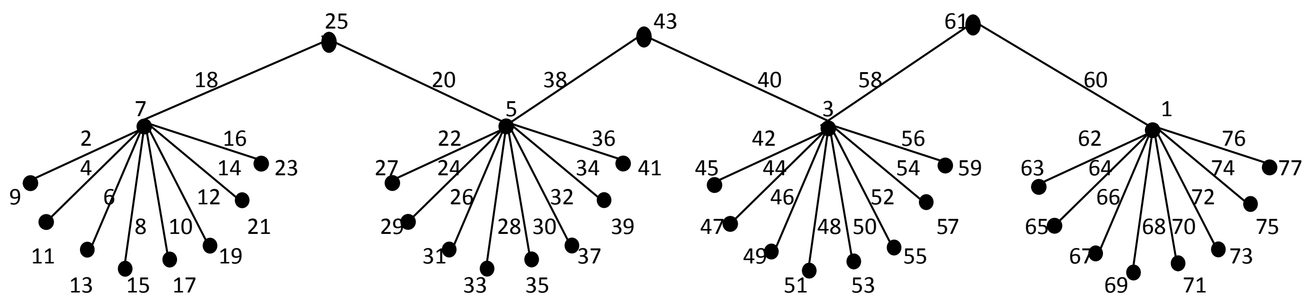

Figure 2.13: $<K_{1,8}^{(1)}, \dots, K_{1,8}^{(4)}>$

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