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STRONG TOTAL NEAR EQUITABLE DOMINATION IN GRAPHS

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ABSTRACT

Let $G=(V,E)$ be a graph, let $D \subseteq V$ and u be any vertex in D. Then the out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V-D)|$. A subset $D \subseteq V(G)$ is called a near equitable dominating set of G if for every $v \in V-D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. A near equitable dominating set is called a strong total near equitable dominating set (stned-set) if for every vertex $v \in D$ there exists $u \in D$ such that u is adjacent to v and $|od_D(u) - od_D(v)| \leq 1$. The minimum cardinality of stned-set of G is called the strong total near equitable domination number of G and is denoted by $\gamma_{stne}(G)$. In this paper, we initiate a study of this parameter.

Mathematics Subject Classification: *05C.*

Keywords: *Equitable Domination Number, Near Equitable Domination Number, Total Near Equitable Domination Number, Strong Total Near Equitable Domination Number.*

1. INTRODUCTION

By a graph $G=(V,E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m, respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [4].

Let $G=(V,E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

A subset S of V is called a dominating set if $N[S] = V$. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ $(\Gamma(G))$. An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [7]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [8]. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [7]. E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi [5] introduced the concept of total domination in graphs. A dominating set D of a graph G is a total dominating set if every vertex of V is adjacent to some vertex of D. The cardinality of a smallest total dominating set in a graph G is called the total domination number of G and is denoted by $\gamma_t(G)$.

Equitable domination has interesting application in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society persons with nearly equal status, tend to be friendly.

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Let $D \subseteq V(G)$ and u be any vertex in D. The out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V - D)|$. D is called a near equitable dominating set of G if for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_{ne} and is called the near equitable domination number of G [1]. A near equitable dominating set D is said to be a total near equitable dominating set if every vertex $w \in V$ is adjacent to an element of D. The minimum cardinality of total near equitable dominating set of G is called the total near equitable domination number of G and is denoted by $\gamma_{tne}(G)$ [2].

We need the following Definition.

Definition: 1.1[1] Let $G = (V, E)$ be a graph and D be a near equitable dominating set of G. Then $u \in D$ is a near equitable pendant vertex if $od_D(u) = 1$. A set D is called a near equitable pendant dominating set if every vertex in D is an equitable pendant vertex.

2. STRONG TOTAL NEAR EQUITABLE DOMINATION IN GRAPHS

Definition: A near equitable dominating set D of a graph G is said to be a strong total near equitable dominating set (stned-set) if for every vertex $v \in D$ there exists $u \in D$ such that u is adjacent to v and $|od_D(u) - od_D(v)| \leq 1$. A stned-set D is said to be minimal if no proper subset of D is a stned-set. The minimum cardinality of stned-set of G is called the strong total near equitable domination number of G and is denoted by $\gamma_{stne}(G)$.

We note that this parameter is only defined for graphs without isolated vertices and, since each total near equitable dominating set is a near equitable dominating set and each strong total near equitable dominating set is a total near equitable dominating set, we have $\gamma_{ne}(G) \leq \gamma_{tne}(G) \leq \gamma_{stne}(G)$.

Proposition: 2.1 *Let G be any graph, and let $D \subset V(G)$ be a strong total near equitable dominating set of G. Then for every component C of G, $D \cap V(C)$ is a strong total near equitable dominating set in C.*

A vertex of a graph is said to be pendant if its neighborhood contains exactly one vertex. The vertex which is adjacent to the pendant vertex is called support vertex.

Proposition: 2.2 *Let T be a tree. Then a total near equitable dominating set is a strong total near equitable dominating set if every support vertex is adjacent to at most two pendant vertices.*

Theorem: 2.3 *Let T be a tree. Then $\gamma_{stne}(T) = 2n - m - 2$ if and only if T is a star.*

Proof: Let T be a tree of order n. Since T is a star, $\gamma_{stne}(T) = n - 1$. Since for any tree, $m = n - 1$, we have $\gamma_{stne}(T) = n - 1 = 2n - (n - 1) - 2 = 2n - m - 2$.

Conversely, suppose that T is not star, then T contains more than one support vertex. Therefore $\gamma_{stne}(T) \leq n - 2$. Thus $\gamma_{stne}(T) < 2n - m - 2$.

Theorem: 2.4 *For any cycle C_n $\gamma_{stne}(C_n) = n - 2$ if and only if $n = 4, 5, 6$.*

Proof: Let $G \cong C_n$ If $n = 4, 5, 6$. Clearly, $\gamma_{stne}(C_n) = n - 2$.
Conversely, suppose that $\gamma_{stne}(C_n) = n - 2$. Since $G \cong C_n$. Assume that $n \neq 4, 5, 6$. If $n = 3$, then $\gamma_{stne}(C_3) = 2 \neq 1$. If $n = 7$, then $\gamma_{stne}(C_7) = 4 < 5$. Similarly, for $n \geq 8$, $\gamma_{stne}(C_n) < n - 2$.

Definition: A graph G is a near equitably balanced graph if for any near equitable dominating set D of G, $od_D(u) = od_D(v)$, for all $u, v \in D$.

Example: A path C_4 is a near equitably balanced graph. But a path P_5 is not near equitably balanced graph.

Remark: 2.5 *Let G be a graph such that any near equitable dominating set of G is a near equitable pendant dominating set. Then G is a near equitably balanced graph.*

Theorem: 2.6 *Let G be a near equitably balanced graph. Then D is a strong total near equitable dominating set of G if and only if D is total near equitable dominating set.*

Proof: Let G be a near equitably balanced graph. Then for any near equitable dominating set D of G, $od_D(u) = od_D(v)$ for all $u, v \in D$. Also, for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. Since D is a total near equitable dominating set, for any $u \in D$ there exists $v \in D$ such that u is adjacent to v. Therefore D is a strong near equitable dominating set.

Theorem: 2.7 *Let G be a near equitably balanced graph and let D be a near equitable dominating set of G. Then for any $w, w' \in V - D$, $|od_{V-D}(w) - od_{V-D}(w')| \leq 2$.*

Proof: Let D be a near equitable dominating set of a near equitably balanced graph G. Suppose that w, w' are any two vertices of $V - D$ such that $od_{V-D}(w) \leq od_{V-D}(w')$. Since D is a near equitable dominating set of G, for any $u \in D$, $od_{V-D}(w) \leq od_D(u) \leq od_{V-D}(w')$ such that $|od_D(u) - od_{V-D}(w)| \leq 1$ and $|od_D(u) - od_{V-D}(w')| \leq 1$. Therefore $|od_{V-D}(w) - od_{V-D}(w')| \leq 2$.

A graph G is called k-regular if every vertex of G has degree k. A graph is said to be regular if it is k-regular for some nonnegative integer k. Analogous to this definition we can define the near equitably regular graph as follows.

Definition: Let G be a near equitably balanced graph and let D be a near equitable dominating set of G. Then G is a near equitably regular graph if for any $u, \in D$ and $v \in V - D$, $od_D(u) = od_{V-D}(v)$.

Example: A cycle C_4 is a near equitably regular graph.

Theorem: 2.8 *Let $G(n, m)$ be a near equitably regular graph, $m \geq 1$ and let D be a near equitable dominating set of G such that the subgraph $\langle V - D \rangle$ induced by $V - D$ is connected. Then $V - D$ is a strong total near equitable dominating set.*

Proof: Let $G(n, m)$ be a near equitably regular graph, $m \geq 1$. Then for any stned- set D, $od_D(u) = od_D(v) = od_{V-D}(w) = od_{V-D}(w') \geq 1$ for all $u, v \in D$ and for all $w, w' \in V - D$. Therefore for any $u \in D$, $od_D(u) \geq 1$. Since the subgraph $\langle V - D \rangle$ induced by $V - D$ is connected, it follows that $V - D$ is a strong total near equitable dominating set.

Definition: A near equitably regular graph with vertices of out degree k is called a k - near equitably regular graph or near equitably regular graph of out degree k.

Theorem: 2.9 *A k-regular graph is a k - near equitably regular graph if and only if it is a k-regular bipartite graph or a totaly disconnected.*

Proof: Let $G(n,m)$ be a k-regular graph. Then $deg_G(u)=k$ for all $u \in V(G)$. Suppose that G is a k- near equitably regular graph, it follows that $od_D(u)=od_D(v)=od_{V-D}(w)=k$ for all $u,v \in D$ and $w \in V-D$. Therefore both subgraphs $\langle D\rangle$ and $\langle V-D\rangle$ induced by D and $V-D$, respectively are totally disconnected. Thus G is totaly disconnected for $k=0$ and k-regular bipartite graph for $k \geq 1$. The converse is obvious.

Theorem: 2.10 *Let G be a near equitably regular graph and let D be a total near equitable dominating set of G. Then D is a strong total near equitable set.*

Proof: Let G be a near equitably regular graph, Suppose that D is a total near equitable dominating set, then for any $v \in V$ there exists $u \in D$ such that v is adjacent to u and $od_D(u)=od_D(v)$ or $od_D(u)=od_{V-D}(v)$. Therefore D is a strong total near equitable set.

Definition: Let G be a graph and let D be a near equitable dominating set of G. Then G is a near equitably bi-regular graph if for any $u,v \in D$ and $w \in V-D$, $od_D(u)=od_D(v)=od_{V-D}(w)\pm 1$.

Proposition 2.11 *Any complete graph K_n is a near equitably bi-regular graph.*

Proposition: 2.12 *Any near equitably bi-regular graph is a near equitably balanced graph.*

Proposition: 2.13 *Let G be a graph and let D be a near equitable pendant dominating set of G. Then for any $u \in D$ and $v \in V-D$, $od_D(u) \leq od_{V-D}(v) \leq 2$.*

Theorem: 2.14 *Let G be a graph and let D be a near equitable pendant dominating set of G. Then*

(i) G is a near equitably regular graph if and only if $od_{V-D}(v)=1$

(ii) G is a near equitably bi-regular graph if and only if $od_{V-D}(v)=2$.

Theorem: 2.15 *Let G be a graph and let D be a near equitable dominating set of G such that the subgraphs $\langle D\rangle$ and $\langle V-D\rangle$ induced by D and $V-D$, respectively are a bipartite graph. Then for any $u \in D$,* $\sum_{u\in D} od_D(u)=m$.

Proof: Suppose that D is a near equitable dominating set of G such that the subgraphs $\langle D\rangle$ and $\langle V-D\rangle$ induced by D and $V-D$, respectively are bipartite graphs, then $od_D(u)=deg(u)$ and $od_{V-D}(v)=deg(v)$, for all $u \in D$ and $v \in V-D$. Since $\sum_{w\in V} deg(w)=2m$, we have $\sum_{u\in D} od_D(u)=\frac{1}{2}\sum_{w\in V} deg(w)=m$.

3. BOUNDS

In this section, we present sharp bounds for $\gamma_{stne}(G)$.

Theorem: 3.1 *Let G be a connected graph of order n, $n \geq 4$. Then $\gamma_{stne}(G) \leq n-1$. Further equality holds for $k_{1,n}$.*

Proof: It is enough to show that for any minimum strong total near equitable dominating set D of G, $|V-D| \geq 1$. Since G is a connected graph of order n, $n \geq 4$, it follows that $\delta(G) \geq 1$. Suppose $|V-D|=0$, it follows that $|D|=n$. Therefore G is totally disconnected, a contradiction.

Theorem: 3.2 *Let $G(n,m)$ be a near equitably regular graph, $m \geq 1$ and let D be a near equitable dominating set of G such that the subgraph $\langle V-D \rangle$ induced by $V-D$ is connected. Then $\gamma_{stne}(G) \leq n-\gamma(G)$. Further equality holds for C_4.*

Proof: Let G be a near equitably regular graph. By Theorem 2.8, $V-D$ is a strong total near equitable dominating set. Therefore, $\gamma_{stne}(G) \leq |V-D| \leq n-\gamma(G)$.

4. MINIMAL STRONG TOTAL NEAR EQUITABLE DOMINATING SETS

We now proceed to obtain a characterization of minimal stned-sets.

Theorem: 4.1 *Let D be a dominating set of a graph G. If D is a stned- set, then D is a minimal stned- set if and only if one of the following holds:*

1. D is minimal near equitable dominating set.
2. For any two adjacent vertices $x, y \in D$, $od_D(x) > od_D(y)$ and for any vertex $v \in D$ deferent from x and y, the set U_v is nonempty, where $U_v = \{x, y \in D, od_D(x) - od_D(y) = 1$, and v is adjacent to x but not adjacent to $y\}$.

Proof: Suppose that D is a minimal strong total near equitable dominating set of G. Then for any $v \in D$, $D-\{v\}$ is not strong total near equitable dominating set. If D is a minimal near equitable dominating set, then we are done. If not, then for any $v \in D$, let $U_v = \{x, y \in D, od_D(x) - od_D(y) = 1$, and v is adjacent to x but not adjacent to $y\}$. There exist $x, y \in D-\{v\}$ such that $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| > 1$. If both x, y are adjacent to v, then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \leq 1$, a contradiction. If both x, y are not adjacent to v, then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \leq 1$, a contradiction. So, v is adjacent to precisely one vertex of $\{x, y\}$. Without loss of generality, assume that v is adjacent to x and v not adjacent to y.

Then,

$$1 < |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) + 1 - od_D(y)| \leq |od_D(x) - od_D(y)| + 1$$

So, $|od_D(x) - od_D(y)| > 0$. But $|od_D(x) - od_D(y)| \leq 1$.

So, $|od_D(x) - od_D(y)| = 1$. Therefore $od_D(x) - od_D(y) = 1$. Hence U_v is not empty.

Conversely, let D be a strong total near equitable dominating set. Suppose to the contrary D is not a minimal strong total near equitable dominating set. Then for every $v \in D$, $D-\{v\}$ is a strong total near equitable dominating set. So, D is not a minimal near dominating set, a contradiction. Next, suppose that D is a strong total near equitable dominating set and (2) holds. Then for every $v \in D$, U_v is not empty. So, for every $v \in D$, there exist $x, y \in D$ such that v is adjacent to precisely one vertex of $\{x, y\}$, and $od_D(x) - od_D(y) = 1$. Suppose to the contrary D is not a minimal strong total near equitable dominating set. Then for every $v \in D$, $D-\{v\}$ is a strong total near equitable dominating set. So, $1 < |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| \leq 1$ and thus we have $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = 1$. Then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)|$, either $\{x, y\} \subseteq N(v)$, or $\{x, y\} \cap N(v) = \varphi$, a contradiction.

5. ON CORONA OF GRAPHS

The corona $G \circ H$ of graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of

H, and then joining the ith vertex of G to every vertex in the ith copy of H. It is customary to denote by H_v that copy of H whose vertices are adjoined with the vertex v of G. In effect, $G \circ H$ is composed of the subgraphs $H_v + v$ joined together by the edges of G. Moreover, $V(G \circ H) = \bigcup_{v \in V(G)} V(H_v + v)$.

Theorem: 5.1 *Let G and H be any two graphs and let D be a near equitable dominating set of H such that for any adjacent vertices $u \in D$ and $v \in V(H) - D$, $od_D(u) \le od_{V(H)-D}(v)$, D is near equitable dominating set of $G \circ H$.*

Proof: Let D be a near equitable dominating set of H. Since for any adjacent vertices $u \in D$ and $v \in V(H) - D$, $od_D(u) \le od_{V(H)-D}(v)$, we have for any adjacent vertices $u \in D$ and $v \in V(G \circ H) - D$, $|od_D(u) - od_{(G \circ H)-D}(v)| \le 1$. Hence D is a near equitable dominating set of $G \circ H$.

Corollary: 5.2 *In Theorem 5.1, if D is a stned- set of H, then D is a stned- set of $G \circ H$.*

Theorem: 5.3 *Let G and H be any two graphs such that G is a connected graph. Then $V(G)$ is a stned- set of $G \circ H$ if and only if $H \cong nK_1$, $n = 1,2$ or $H \cong K_2$.*

Proof: Let $V(G \circ H) = (V_1, V_2)$, where $V_1 = V(G)$ and $V_2 = V(H)$. Suppose that $V(G)$ is a stned- set of $G \circ H$. Since $V(G \circ H) = \bigcup_{v \in V_1} V(H_v + v)$, it follows that for any $u \in V_2$, $od_{V_2}(u) = 1$. Since $V(G)$ is a stned- set of $G \circ H$, it follows that for any $v \in V_1$, $od_{V_1}(v) \le 2$.

Therefore $|V(H)| \le 2$. Hence $H \cong nK_1$, $n = 1,2$ or $H \cong K_2$.

Conversely, suppose that $H \cong nK_1$, $n = 1,2$ or $H \cong K_2$, then for any $v \in V_1$, $od_{V_1}(v) \le 2$ and for any $u \in V_2$, $od_{V_2}(u) = 1$. Therefore $|od_{V_1}(v) - od_{V_2}(u)| \le 1$. Since G is a connected graph, for any $w \in V_1$ there exists $v \in V_1$ such that $|od_{V_1}(v) - od_{V_1}(w)| \le 1$. Thus $V(G)$ is a stned- set of $G \circ H$.

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REFERENCES

1. Ali Mohammed Sahal and Veena Mathad, On near equitable domination in graphs, Asian Journal of Current Engineering and Maths, communicated.
2. Ali Mohammed Sahal and Veena Mathad, Total near equitable domination in graphs, International Journal of Mathematical Combinatorics, communicated.
3. Anitha, S. Arumugam and Mustapha Chellali, Equitable domination in graphs, Discrete Mathematics, Algorithms and Applications, 3(2011), 311-321.
4. G. Chartrand and L. Lesnaik, Graphs and Digraphs, Chapman and Hall. CRC, 4th edition, 2005.
5. E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi, Total domination in graphs, Networks 10 (1980), 211-219.
6. F. Harary and T.W Haynes, Double domination in graphs, Ars Combin, 55,(2000), 201-213.
7. T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of domination in graphs, Marcel Dekker, New York, 1998.
8. T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Domination in graphs, Advanced Topics,Marcel Dekker, New York, 1998.
9. Venkatasubramanian Swaminathan and Kuppusamy Markandan Dharmalingam, Degree equitable domination on graphs, Kragujevac J. Math, 35 (2011), 191-197.

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