

STRONG TOTAL NEAR EQUITABLE DOMINATION IN GRAPHS

Ali Mohammed Sahal* and Veena Mathad

*Departement of Stadies in Mathematics,
University of Mysore, Manasagangotri, Mysore-570 006, India.*

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ABSTRACT

Let $G = (V, E)$ be a graph, let $D \subseteq V$ and u be any vertex in D . Then the out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V - D)|$. A subset $D \subseteq V(G)$ is called a near equitable dominating set of G if for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. A near equitable dominating set is called a strong total near equitable dominating set (stned-set) if for every vertex $v \in D$ there exists $u \in D$ such that u is adjacent to v and $|od_D(u) - od_D(v)| \leq 1$. The minimum cardinality of stned-set of G is called the strong total near equitable domination number of G and is denoted by $\gamma_{stne}(G)$. In this paper, we initiate a study of this parameter.

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Keywords: Equitable Domination Number, Near Equitable Domination Number, Total Near Equitable Domination Number, Strong Total Near Equitable Domination Number.

1. INTRODUCTION

By a graph $G = (V, E)$ we mean a finite, undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m , respectively. For graph theoretic terminology we refer to Chartrand and Lesnaik [4].

Let $G = (V, E)$ be a graph and let $v \in V$. The open neighborhood and the closed neighborhood of v are denoted by $N(v) = \{u \in V : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. If $S \subseteq V$ then $N(S) = \cup_{v \in S} N(v)$ and $N[S] = N(S) \cup S$.

A subset S of V is called a dominating set if $N[S] = V$. The minimum (maximum) cardinality of a minimal dominating set of G is called the domination number (upper domination number) of G and is denoted by $\gamma(G)$ ($\Gamma(G)$). An excellent treatment of the fundamentals of domination is given in the book by Haynes et al. [7]. A survey of several advanced topics in domination is given in the book edited by Haynes et al. [8]. Various types of domination have been defined and studied by several authors and more than 75 models of domination are listed in the appendix of Haynes et al. [7]. E.J. Cockayne, R.M. Dawes and S.T. Hedetniemi [5] introduced the concept of total domination in graphs. A dominating set D of a graph G is a total dominating set if every vertex of V is adjacent to some vertex of D . The cardinality of a smallest total dominating set in a graph G is called the total domination number of G and is denoted by $\gamma_t(G)$.

Equitable domination has interesting application in the context of social networks. In a network, nodes with nearly equal capacity may interact with each other in a better way. In the society persons with nearly equal status, tend to be friendly.

Corresponding author: Ali Mohammed Sahal
Departement of Stadies in Mathematics,
University of Mysore, Manasagangotri, Mysore-570 006, India.*

Let $D \subseteq V(G)$ and u be any vertex in D . The out degree of u with respect to D denoted by $od_D(u)$, is defined as $od_D(u) = |N(u) \cap (V - D)|$. D is called a near equitable dominating set of G if for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_{ne} and is called the near equitable domination number of G [1]. A near equitable dominating set D is said to be a total near equitable dominating set if every vertex $w \in V$ is adjacent to an element of D . The minimum cardinality of total near equitable dominating set of G is called the total near equitable domination number of G and is denoted by $\gamma_{me}(G)$ [2].

We need the following Definition.

Definition: 1.1[1] Let $G = (V, E)$ be a graph and D be a near equitable dominating set of G . Then $u \in D$ is a near equitable pendant vertex if $od_D(u) = 1$. A set D is called a near equitable pendant dominating set if every vertex in D is an equitable pendant vertex.

2. STRONG TOTAL NEAR EQUITABLE DOMINATION IN GRAPHS

Definition: A near equitable dominating set D of a graph G is said to be a strong total near equitable dominating set (stned-set) if for every vertex $v \in D$ there exists $u \in D$ such that u is adjacent to v and $|od_D(u) - od_D(v)| \leq 1$. A stned-set D is said to be minimal if no proper subset of D is a stned-set. The minimum cardinality of stned-set of G is called the strong total near equitable domination number of G and is denoted by $\gamma_{sne}(G)$.

We note that this parameter is only defined for graphs without isolated vertices and, since each total near equitable dominating set is a near equitable dominating set and each strong total near equitable dominating set is a total near equitable dominating set, we have $\gamma_{ne}(G) \leq \gamma_{me}(G) \leq \gamma_{sne}(G)$.

Proposition: 2.1 Let G be any graph, and let $D \subset V(G)$ be a strong total near equitable dominating set of G . Then for every component C of G , $D \cap V(C)$ is a strong total near equitable dominating set in C .

A vertex of a graph is said to be pendant if its neighborhood contains exactly one vertex. The vertex which is adjacent to the pendant vertex is called support vertex.

Proposition: 2.2 Let T be a tree. Then a total near equitable dominating set is a strong total near equitable dominating set if every support vertex is adjacent to at most two pendant vertices.

Theorem: 2.3 Let T be a tree. Then $\gamma_{sne}(T) = 2n - m - 2$ if and only if T is a star.

Proof: Let T be a tree of order n . Since T is a star, $\gamma_{sne}(T) = n - 1$. Since for any tree, $m = n - 1$, we have $\gamma_{sne}(T) = n - 1 = 2n - (n - 1) - 2 = 2n - m - 2$.

Conversely, suppose that T is not star, then T contains more than one support vertex. Therefore $\gamma_{sne}(T) \leq n - 2$. Thus $\gamma_{sne}(T) < 2n - m - 2$.

Theorem: 2.4 For any cycle C_n $\gamma_{sne}(C_n) = n - 2$ if and only if $n = 4, 5, 6$.

Proof: Let $G \cong C_n$ If $n = 4, 5, 6$. Clearly, $\gamma_{sne}(C_n) = n - 2$.

Conversely, suppose that $\gamma_{sne}(C_n) = n - 2$. Since $G \cong C_n$. Assume that $n \neq 4, 5, 6$. If $n = 3$, then $\gamma_{sne}(C_3) = 2 \neq 1$. If $n = 7$, then $\gamma_{sne}(C_7) = 4 < 5$. Similarly, for $n \geq 8$, $\gamma_{sne}(C_n) < n - 2$.

Definition: A graph G is a near equitably balanced graph if for any near equitable dominating set D of G , $od_D(u) = od_D(v)$, for all $u, v \in D$.

Example: A path C_4 is a near equitably balanced graph. But a path P_5 is not near equitably balanced graph.

Remark: 2.5 Let G be a graph such that any near equitable dominating set of G is a near equitable pendant dominating set. Then G is a near equitably balanced graph.

Theorem: 2.6 Let G be a near equitably balanced graph. Then D is a strong total near equitable dominating set of G if and only if D is total near equitable dominating set.

Proof: Let G be a near equitably balanced graph. Then for any near equitable dominating set D of G , $od_D(u) = od_D(v)$ for all $u, v \in D$. Also, for every $v \in V - D$ there exists a vertex $u \in D$ such that u is adjacent to v and $|od_D(u) - od_{V-D}(v)| \leq 1$. Since D is a total near equitable dominating set, for any $u \in D$ there exists $v \in D$ such that u is adjacent to v . Therefore D is a strong near equitable dominating set.

Theorem: 2.7 Let G be a near equitably balanced graph and let D be a near equitable dominating set of G . Then for any $w, w' \in V - D$, $|od_{V-D}(w) - od_{V-D}(w')| \leq 2$.

Proof: Let D be a near equitable dominating set of a near equitably balanced graph G . Suppose that w, w' are any two vertices of $V - D$ such that $od_{V-D}(w) \leq od_{V-D}(w')$. Since D is a near equitable dominating set of G , for any $u \in D$, $od_{V-D}(w) \leq od_D(u) \leq od_{V-D}(w')$ such that $|od_D(u) - od_{V-D}(w)| \leq 1$ and $|od_D(u) - od_{V-D}(w')| \leq 1$. Therefore $|od_{V-D}(w) - od_{V-D}(w')| \leq 2$.

A graph G is called k -regular if every vertex of G has degree k . A graph is said to be regular if it is k -regular for some nonnegative integer k . Analogous to this definition we can define the near equitably regular graph as follows.

Definition: Let G be a near equitably balanced graph and let D be a near equitable dominating set of G . Then G is a near equitably regular graph if for any $u, v \in D$ and $w, w' \in V - D$, $od_D(u) = od_{V-D}(v)$.

Example: A cycle C_4 is a near equitably regular graph.

Theorem: 2.8 Let $G(n, m)$ be a near equitably regular graph, $m \geq 1$ and let D be a near equitable dominating set of G such that the subgraph $\langle V - D \rangle$ induced by $V - D$ is connected. Then $V - D$ is a strong total near equitable dominating set.

Proof: Let $G(n, m)$ be a near equitably regular graph, $m \geq 1$. Then for any set D , $od_D(u) = od_D(v) = od_{V-D}(w) = od_{V-D}(w') \geq 1$ for all $u, v \in D$ and for all $w, w' \in V - D$. Therefore for any $u \in D$, $od_D(u) \geq 1$. Since the subgraph $\langle V - D \rangle$ induced by $V - D$ is connected, it follows that $V - D$ is a strong total near equitable dominating set.

Definition: A near equitably regular graph with vertices of out degree k is called a k -near equitably regular graph or near equitably regular graph of out degree k .

Theorem: 2.9 A k -regular graph is a k -near equitably regular graph if and only if it is a k -regular bipartite graph or a totally disconnected.

Proof: Let $G(n, m)$ be a k -regular graph. Then $deg_G(u) = k$ for all $u \in V(G)$. Suppose that G is a k -near equitably regular graph, it follows that $od_D(u) = od_D(v) = od_{V-D}(w) = k$ for all $u, v \in D$ and $w \in V - D$. Therefore both subgraphs $\langle D \rangle$ and $\langle V - D \rangle$ induced by D and $V - D$, respectively are totally disconnected. Thus G is totally disconnected for $k = 0$ and k -regular bipartite graph for $k \geq 1$. The converse is obvious.

Theorem: 2.10 Let G be a near equitably regular graph and let D be a total near equitable dominating set of G . Then D is a strong total near equitable set.

Proof: Let G be a near equitably regular graph, Suppose that D is a total near equitable dominating set, then for any $v \in V$ there exists $u \in D$ such that v is adjacent to u and $od_D(u) = od_D(v)$ or $od_D(u) = od_{V-D}(v)$. Therefore D is a strong total near equitable set.

Definition: Let G be a graph and let D be a near equitable dominating set of G . Then G is a near equitably bi-regular graph if for any $u, v \in D$ and $w \in V - D$, $od_D(u) = od_D(v) = od_{V-D}(w) \pm 1$.

Proposition 2.11 Any complete graph K_n is a near equitably bi-regular graph.

Proposition: 2.12 Any near equitably bi-regular graph is a near equitably balanced graph.

Proposition: 2.13 Let G be a graph and let D be a near equitable pendant dominating set of G . Then for any $u \in D$ and $v \in V - D$, $od_D(u) \leq od_{V-D}(v) \leq 2$.

Theorem: 2.14 Let G be a graph and let D be a near equitable pendant dominating set of G . Then

- (i) G is a near equitably regular graph if and only if $od_{V-D}(v) = 1$
- (ii) G is a near equitably bi-regular graph if and only if $od_{V-D}(v) = 2$.

Theorem: 2.15 Let G be a graph and let D be a near equitable dominating set of G such that the subgraphs $\langle D \rangle$ and $\langle V - D \rangle$ induced by D and $V - D$, respectively are a bipartite graph. Then for any $u \in D$,

$$\sum_{u \in D} od_D(u) = m.$$

Proof: Suppose that D is a near equitable dominating set of G such that the subgraphs $\langle D \rangle$ and $\langle V - D \rangle$ induced by D and $V - D$, respectively are bipartite graphs, then $od_D(u) = deg(u)$ and $od_{V-D}(v) = deg(v)$, for all

$u \in D$ and $v \in V - D$. Since $\sum_{w \in V} deg(w) = 2m$, we have $\sum_{u \in D} od_D(u) = \frac{1}{2} \sum_{w \in V} deg(w) = m$.

3. BOUNDS

In this section, we present sharp bounds for $\gamma_{stne}(G)$.

Theorem: 3.1 Let G be a connected graph of order n , $n \geq 4$. Then $\gamma_{stne}(G) \leq n - 1$. Further equality holds for $k_{1,n}$.

Proof: It is enough to show that for any minimum strong total near equitable dominating set D of G , $|V - D| \geq 1$. Since G is a connected graph of order n , $n \geq 4$, it follows that $\delta(G) \geq 1$. Suppose $|V - D| = 0$, it follows that $|D| = n$. Therefore G is totally disconnected, a contradiction.

Theorem: 3.2 Let $G(n, m)$ be a near equitably regular graph, $m \geq 1$ and let D be a near equitable dominating set of G such that the subgraph $\langle V - D \rangle$ induced by $V - D$ is connected. Then $\gamma_{stne}(G) \leq n - \gamma(G)$. Further equality holds for C_4 .

Proof: Let G be a near equitably regular graph. By Theorem 2.8, $V - D$ is a strong total near equitable dominating set. Therefore, $\gamma_{stne}(G) \leq |V - D| \leq n - \gamma(G)$.

4. MINIMAL STRONG TOTAL NEAR EQUITABLE DOMINATING SETS

We now proceed to obtain a characterization of minimal stned-sets.

Theorem: 4.1 Let D be a dominating set of a graph G . If D is a stned- set, then D is a minimal stned- set if and only if one of the following holds:

1. D is minimal near equitable dominating set.
2. For any two adjacent vertices $x, y \in D$, $od_D(x) > od_D(y)$ and for any vertex $v \in D$ deferent from x and y , the set U_v is nonempty, where $U_v = \{x, y \in D, od_D(x) - od_D(y) = 1, \text{ and } v \text{ is adjacent to } x \text{ but not adjacent to } y\}$.

Proof: Suppose that D is a minimal strong total near equitable dominating set of G . Then for any $v \in D$, $D - \{v\}$ is not strong total near equitable dominating set. If D is a minimal near equitable dominating set, then we are done. If not, then for any $v \in D$, let $U_v = \{x, y \in D, od_D(x) - od_D(y) = 1, \text{ and } v \text{ is adjacent to } x \text{ but not adjacent to } y\}$. There exist $x, y \in D - \{v\}$ such that $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| > 1$. If both x, y are adjacent to v , then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \leq 1$, a contradiction. If both x, y are not adjacent to v , then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)| \leq 1$, a contradiction. So, v is adjacent to precisely one vertex of $\{x, y\}$. Without loss of generality, assume that v is adjacent to x and v not adjacent to y .

Then,

$$1 < |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) + 1 - od_D(y)| \leq |od_D(x) - od_D(y)| + 1$$

So, $|od_D(x) - od_D(y)| > 0$. But $|od_D(x) - od_D(y)| \leq 1$.

So, $|od_D(x) - od_D(y)| = 1$. Therefore $od_D(x) - od_D(y) = 1$. Hence U_v is not empty.

Conversely, let D be a strong total near equitable dominating set. Suppose to the contrary D is not a minimal strong total near equitable dominating set. Then for every $v \in D$, $D - \{v\}$ is a strong total near equitable dominating set. So, D is not a minimal near dominating set, a contradiction. Next, suppose that D is a strong total near equitable dominating set and (2) holds. Then for every $v \in D$, U_v is not empty. So, for every $v \in D$, there exist $x, y \in D$ such that v is adjacent to precisely one vertex of $\{x, y\}$, and $od_D(x) - od_D(y) = 1$. Suppose to the contrary D is not a minimal strong total near equitable dominating set. Then for every $v \in D$, $D - \{v\}$ is a strong total near equitable dominating set. So, $1 < |od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| \leq 1$ and thus we have $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = 1$. Then $|od_{D-\{v\}}(x) - od_{D-\{v\}}(y)| = |od_D(x) - od_D(y)|$, either $\{x, y\} \subseteq N(v)$, or $\{x, y\} \cap N(v) = \emptyset$, a contradiction.

5. ON CORONA OF GRAPHS

The corona $G \circ H$ of graphs G and H is the graph obtained by taking one copy of G and $|V(G)|$ copies of

H , and then joining the i th vertex of G to every vertex in the i th copy of H . It is customary to denote by H_v that copy of H whose vertices are adjoined with the vertex v of G . In effect, $G \circ H$ is composed of the subgraphs $H_v + v$ joined together by the edges of G . Moreover, $V(G \circ H) = \bigcup_{v \in V(G)} V(H_v + v)$.

Theorem: 5.1 Let G and H be any two graphs and let D be a near equitable dominating set of H such that for any adjacent vertices $u \in D$ and $v \in V(H) - D$, $od_D(u) \leq od_{V(H)-D}(v)$, D is near equitable dominating set of $G \circ H$.

Proof: Let D be a near equitable dominating set of H . Since for any adjacent vertices $u \in D$ and $v \in V(H) - D$, $od_D(u) \leq od_{V(H)-D}(v)$, we have for any adjacent vertices $u \in D$ and $v \in V(G \circ H) - D$, $|od_D(u) - od_{(G \circ H)-D}(v)| \leq 1$. Hence D is a near equitable dominating set of $G \circ H$.

Corollary: 5.2 In Theorem 5.1, if D is a stned- set of H , then D is a stned- set of $G \circ H$.

Theorem: 5.3 Let G and H be any two graphs such that G is a connected graph. Then $V(G)$ is a stned- set of $G \circ H$ if and only if $H \cong nK_1$, $n = 1, 2$ or $H \cong K_2$.

Proof: Let $V(G \circ H) = (V_1, V_2)$, where $V_1 = V(G)$ and $V_2 = V(H)$. Suppose that $V(G)$ is a stned- set of $G \circ H$. Since $V(G \circ H) = \bigcup_{v \in V_1} V(H_v + v)$, it follows that for any $u \in V_2$, $od_{V_2}(u) = 1$. Since $V(G)$ is a stned- set of $G \circ H$, it follows that for any $v \in V_1$, $od_{V_1}(v) \leq 2$.

Therefore $|V(H)| \leq 2$. Hence $H \cong nK_1$, $n = 1, 2$ or $H \cong K_2$.

Conversely, suppose that $H \cong nK_1$, $n = 1, 2$ or $H \cong K_2$, then for any $v \in V_1$, $od_{V_1}(v) \leq 2$ and for any $u \in V_2$, $od_{V_2}(u) = 1$. Therefore $|od_{V_1}(v) - od_{V_2}(u)| \leq 1$. Since G is a connected graph, for any $w \in V_1$ there exists $v \in V_1$ such that $|od_{V_1}(v) - od_{V_1}(w)| \leq 1$. Thus $V(G)$ is a stned- set of $G \circ H$.

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