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A NOTE ON X_d -FRAMES IN BANACH SPACES

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ABSTRACT

Frames with respect to some sequence space, namely X_d -frames were introduced and studied by Casazza et al [1]. We further study X_d -frames and proved if a Banach space has a X_d -frames then it also has the Parseval and an exact Parseval X_d -frames. A necessary and sufficient condition and also a sufficient condition for the stability of X_d -frame have been given. Also, X_d -Bessel sequences, for a sequence to be X_d -frame has been obtained.

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Key Words: *X_d -frame, X_d -Bessel sequence, Banach frame, Stability, an exact X_d -frame.*

1. INTRODUCTION

Fourier transform has been a major tool in analysis for over a century. It has a lacking for signal analysis in which it hides in its phases information concerning the moment of emission and duration of a signal. What was needed was localized time frequency representation which has this information encoded in it. In 1946 Dennis Gabor [8], filled this gap and formulated a fundamental approach to signal decomposition in terms of elementary signals. On the basis of this development, in 1952 the notion of frame was determined by Duffin and Schaeffer [6] in Hilbert spaces in the following way: Let X be a Separable Hilbert space, the system of non-zero elements $\{x_n\}_{n\in\mathbb{N}} \subset X$ be called a frame in X if there exist the constants $0 < A \le B < \infty$ such that for each $x \in X$, it is valid

$$A \| x \|_X^2 \le \sum_{n=1}^{\infty} | (x, x_n) |^2 \le B \| x \|_X^2 \tag{1.1}$$

where $\| \bullet \|_X$ and $(\bullet,\bullet)$ is a norm and scalar product in X respectively. The constants A and B in (1.1) are called lower and upper frame bounds, the number $K = A / B$ is called condition coefficient of the frame $\{x_n\}_{n\in\mathbb{N}}$. In the case, when $K = 1$, $\{x_n\}_{n\in\mathbb{N}}$ is a tight frame. Development in theory of frames in Hilbert space reduced to obtaining the analogues of the known results for the Banach case. By the theory of frames [3, 6, 7, 9] we have their Banach extensions in [1, 2, 4, 5, 10]. Frames have many properties of bases but lacks a very important one, namely, uniqueness. This property of frames make them very useful in the study of function spaces, signal and image processing, filter banks, wireless and communications etc.

The notion of X_d -frame generalizing the notion of p -frame studied in [4] was introduced in [1]. In the present paper, we further study X_d -frames and obtain a necessary and sufficient condition. Also, a sufficient condition for stability of X_d -frame has been given and proved if a Banach spaces has a X_d -frame then it is also has a Parseval and an exact Parseval X_d -frames.

Further, X_d -Bessel sequence has been studied and a sufficient condition in terms of X_d -Bessel sequence, for a sequence to be a X_d -frame has been given.

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2. PRELIMINARIES

Throughout this paper X will denote an infinite dimensional Banach space over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$). Let X^* and X^{**} denote the first and second conjugate space of X respectively, $L(X, X)$ Banach space of all continuous linear mapping from X into X , and X_d be an associated Banach space of scalar valued sequences index by $\mathbb{N}$. Let $[f_n]$ the closed linear span of $\{f_n\}$ and $[\widetilde{f_n}]$ the closed linear span of $\{f_n\}$ in $\sigma(X^*, X)$ -topology. A sequence $\{f_n\} \subset X^*$ is said to be complete if $[f_n] = X^*$ and total if $\{x \in X : f_n(x) = 0, n \in \mathbb{N}\} = \{0\}$.

Definition: 2.1 ([1]) A sequence space X_d is called a BK-space, if it is a Banach space and the coordinate functional are continuous on X_d , i.e., the relations $x_n = \left\{\alpha_j^{(n)}\right\}, x = \left\{\alpha_j\right\} \in X_d$, $\lim_{n\to\infty} x_n = x$ imply

$\lim_{n\to\infty} \alpha_j^{(n)} = \alpha_j$ $(j = 1, 2, ...)$.

Definition: 2.2 ([1]) Let X_d be a BK-space and $\{f_i\}_{i=1}^{\infty} \subset X^*$. The sequence $\{f_i\}_{i=1}^{\infty}$ is called a X_d -frame for X with lower bound A and upper bound B if $0 < A \le B < \infty$ and for every $x \in X$ one has

(i) $\{f_i(x)\}_{i=1}^{\infty} \in X_d$;

(ii) $A\|x\|_X \le \left\|\{f_i(x)\}_{i=1}^{\infty}\right\|_{X_d} \le B\|x\|_X$. (2.1)

When (i) and the upper inequality in (ii) hold for every $x \in X$, $\{f_i\}_{i=1}^{\infty}$ is called a X_d -Bessel sequence for X with bound B . The positive constants A and B , respectively, are called lower and upper frame bounds of the X_d -frame. The inequality (2.1) is called the frame inequality. It is easy to observe that frame bounds need not be unique. Further, the X_d -frame is called tight frame if it is possible to choose A and B , satisfying (2.1) with $A = B$ and normalized tight (or Parseval) if $A = B = 1$. If removal of one f_n renders the collection $\{f_n\} \subset X^*$ no longer a X_d -frame for X , then $\{f_n\}$ is called can an exact X_d -frame.

Let $\{f_i\}_{i=1}^{\infty} \subset X^*$. The operators U and T given by $Ux = \{f_i(x)\}$, $x \in X$, and $T\{d_i\} = \sum_{i=0}^{\infty} d_i f_i$, are called the analysis operator for $\{f_i\}_{i=1}^{\infty}$ and the synthesis operator for $\{f_i\}_{i=1}^{\infty}$, respectively.

3. MAIN RESULTS

The following result which is referred in this paper is listed in the form of a lemma.

Lemma: 3.1 ([11]) If $\{f_n\} \subset X^*$ is a X_d -frame for X with respect to X_d . Then $\{f_n\}$ is an exact iff $f_n \notin [\widetilde{f_i}]_{i \ne n}$, for all n .

Proof: It is straight forward on the line of totalness of $\{f_n\}$ obtained from frame inequality of X_d -frame.

Theorem: 3.2 If X be a Banach space having a X_d -frame. Then X has a Parseval X_d - frame as well as exact Parseval X_d -frame.

Proof: Let $\{f_n\} \subset X^*$ be a X_d -frame for X with respect to X_d . Then by frame inequality, $\{f_n\}$ is total over X. Therefore by Remark 7.1, in [12], there exists an associated Banach space $\{\{f_n(x)\} : x \in X\}$ with norm given by

$$\|\{f_n(x)\}\|_{X_d} = \|x\|_X ,\ x \in X .$$

Therefore X is a Parseval X_d -frame.

Further, we may assume, without loss of generality that $\{f_n\}$ is finitely linear independent. (In case $\{f_n\}$ is not finitely linear independent, we can derive a subsequence $\{g_i\} \subset \{f_n\}$ which is finitely linear independent and total over X .) Then, for each $n \in \mathbb{N}$, there exists an element $x_n \in X$, such that $f_i(x_n) = 0$, $i = 1, 2, ..., n-1$, and $f_n(x_n) = 1$, indeed if $f_i(x) = 0$ $(i = 1, 2, ... n-1)$ would imply $f_n(x) = 0$, then we would have by [4], $g_n \in [g_1, g_2, ..., g_n]$, contradicting our assumption. Define $\{h_n\} \subset X^*$ by $h_1 = f_1$ and

$$h_n = f_n - \sum_{i=1}^{n-1} f_n(x_i) h_i, \quad n = 2, 3, 4.... \tag{3.1}$$

Then $\{h_n\}$ is total over X such that $h_i(x_j) = \delta_{ij}$, $i, j \in \mathbb{N}$. Therefore, there exists an associated Banach space $X_{d_1} = \{\{h_n(x)\} : x \in X\}$ equipped with norm $\| \{h_n(x)\} \|_{X_{d_1}} = \| x \|_X$, $x \in X$. Hence $\{h_n\}$ is a Parseval X_d -frame for X. Further, $h_n \notin [\widetilde{h_i}]_{i \neq n}$, for all $n \in \mathbb{N}$. Hence by Lemma 3.1, $\{h_n\}$ is an exact Parseval X_d -frame for X .

Remark: 3.3 Since $\{f_n\}$ is total on X if and only if it's finite linear combinations with rational coefficients are w^*-dense in X^* and since the conjugate space of every separable Banach space is w^*-separable. Therefore, in particular, every separable Banach space X has an exact Parseval X_d -frame.

Regarding consequence of the above theorem for X_d -frames, we prove the following theorem.

Theorem: 3.4 A X_d -frame $\{f_n\} \subset X^*$ is an exact if there exists a sequence of non-zero operator $\{v_n\} \subset L(X, X)$ such that

$$v_n^*(f) = f, \; f \in [f_i]_{i=1}^n \text{ and } v_n^*(f) = 0, \; f \in [f_i]_{i=n+1}^\infty.$$

Proof: Let us assume that above statement is true. Then $v_1^*(f_1) = f_1$. Let x_1 be in the range of v_1 and $y_1 \in X$ be such that $x_1 = v_1(y_1)$ and $f_1(x_1) = 1$. Then, for $i = 2, 3, ...$, we have $f_i(x_1) = 0$. Again, let x_2 be in the range of $v_2 - v_1$ and $y_2 \in X$ be such that $x_2 = (v_2 - v_1)(y_2)$ and $f_i(x_2) = 0$. Then $f_1(x_2) = 0$. Further for $i = 3, 4, ...$, we have $f_i(x_2) = 0$. Continuing like this, let x_n be the range of $v_n - v_{n-1}$ and $y_n \in X$ be such that $x_2 = (v_n - v_{n-1})(y_n)$ and $f_n(x_n) = 1$. Then, for $i \neq n$, $(i \in \mathbb{N})$, we have

$$f_i(x_n) = \begin{cases} f_i(y_n) - f_i(y_n) = 0, & i < n \\ 0, & i > n, \end{cases}$$

Thus we obtain a sequence $\{x_n\} \subset X^*$ such that $f_i(x_j) = \delta_{ij}$, $i, j \in \mathbb{N}$. Therefore, $f_n \notin [\widetilde{f_i}]_{i \neq n}$, for all n. Hence by Lemma 3.1, $\{f_n\}$ is an exact X_d -frame for X .

Next, we prove that following result, regarding stability of X_d -frames.

Theorem: 3.5 If $\{f_n\} \subset X^*$ be a X_d -frames for X and let $\{g_n\} \subset X^*$ be such that $\{g_n(x)\} \subset X_d$, for all $x \in X$. Then, $\{g_n\}$ is a X_d -frames for X if and only if there exist a constant $K > 0$, such that

$$\| \{(f_n - g_n)(x)\} \|_{X_d} \leq K \min \left\{ \| \{f_n(x)\} \|_{X_d}, \| \{g_n(x)\} \|_{X_d} \right\}, \; x \in X.$$

Proof: Let $\{f_n\}$ and $\{g_n\}$ are X_d -frames with frame bounds are $A_f, B_f; A_g, B_g$, respectively. Then by applying frame inequalities for these frames, we get

$$\| \{(f_n - g_n)(x)\} \|_{X_d} \le \left(1 + \frac{B_g}{A_f}\right) \| \{f_n(x)\} \|_{X_d}, \text{ for all } x \in X. \tag{3.1}$$

Similarly, we obtain

$$\| \{(f_n - g_n)(x)\} \|_{X_d} \le \left(1 + \frac{B_f}{A_g}\right) \| \{g_n(x)\} \|_{X_d}, \text{ for all } x \in X. \tag{3.2}$$

Choosing $K = \left(1 + \frac{B_g}{A_f}\right)$ or $\left(1 + \frac{B_f}{A_g}\right)$ according as
$min\{\| \{f_n(x)\} \|_{X_d}, \| \{g_n(x)\} \|_{X_d}\}$ is $\| \{f_n(x)\} \|_{X_d}$ or $\| \{g_n(x)\} \|_{X_d}$.

Conversely, let C_f and D_f are bounds for the X_d-frame $\{f_n\}$ in X^*. Then, for all $x \in X$, we have

$$\begin{aligned} C_f \| x \|_X &\le \| \{f_n(x)\} \|_{X_d} \\ &\le \| \{(f_n - g_n)(x)\} \|_{X_d} + \| \{g_n(x)\} \|_{X_d} \\ &\le (1+K) \| \{g_n(x)\} \|_{X_d} \\ &\le (1+K)(\| \{(f_n - g_n)(x)\} \|_{X_d} + \| \{f_n(x)\} \|_{X_d}) . \\ &\le (1+K)^2 \| \{f_n(x)\} \|_{X_d} \\ &\le (1+K)^2 D_f \| x \|_X \end{aligned}$$

Therefore, $\frac{C_f}{(1+K)} \| x \|_X \le \| \{g_n(x)\} \|_{X_d} \le (1+K) D_f \| x \|_X$ for all $x \in X$.

Hence, $\{g_n\}$ is a X_d-frame for X with bounds $\frac{C_f}{(1+K)}$ and $D_f(1+K)$.

In the following theorem, we give a sufficient condition for a sequence to be a X_d-frame for X.

Theorem: 3.6 If $\{f_n\} \subset X^*$ be a X_d-frames for X with frame bounds A and B. Let $\{g_n\} \subset X^*$ be such that $\{g_n(x)\} \subset X_d$, for all $x \in X$. If there exists non-negative constants α, β, γ and δ such that

(i) $\frac{\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}{(1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}} < A$, where $\max\{\alpha,\beta,\gamma,\delta\} < 1$.

(ii) $\left\|\{(f_n - g_n)(x)\}\right\|^2_{X_d} \le \alpha \left\|\{f_n(x)\}\right\|^2_{X_d} + 2\beta \left\|\{f_n(x)\}\right\|_{X_d} \left\|\{g_n(x)\}\right\|_{X_d} + \gamma \left\|\{g_n(x)\}\right\|^2_{X_d} + \delta \|x\|^2_X$, $x \in X$.

Then $\{g_n\}$ is a X_d-frame for X with frame bounds $\frac{A-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+A)}{1+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}$ and $\frac{B+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+B)}{1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}$.

Proof: Let $\zeta = \max\{\alpha,\beta,\gamma,\delta\}$. Then (ii) may be reproduced as:

$$\| \{(f_n - g_n)(x)\} \|_{X_d} \le \sqrt{\zeta}\{\| \{f_n(x)\} \|_{X_d} + \| \{g_n(x)\} \|_{X_d} + \| x \|_X\}, \; x \in X. \tag{3.3}$$

Now,

$$\begin{aligned} \| \{g_n(x)\} \|_{X_d} &\le \| \{f_n(x)\} \|_{X_d} + \| \{f_n - g_n)(x)\} \|_{X_d} \\ &\le \| \{f_n(x)\} \|_{X_d} + \sqrt{\zeta}\{\| \{f_n(x)\} \|_{X_d} + \| \{g_n(x)\} \|_{X_d} + \| x \|_X\} \text{ (using (3.3))} \end{aligned}$$

This gives,

$$(1-\sqrt{\zeta})\,\|\{g_n(x)\}\|_{X_d} \le \{(1+\sqrt{\zeta})B+\sqrt{\zeta}\}\,\|x\|_X,\ x\in X.$$

Similarly,

$$(1+\sqrt{\zeta})\,\|\{g_n(x)\}\|_{X_d} \ge \{(1-\sqrt{\zeta})A+\sqrt{\zeta}\}\,\|x\|_X,\ x\in X.$$

Therefore

$$\left\{\frac{(1-\sqrt{\zeta})A-\sqrt{\zeta}}{1+\sqrt{\zeta}}\right\}\|x\|_X \le \|\{g_n(x)\}\|_{X_d} \le \left\{\frac{(1+\sqrt{\zeta})B+\sqrt{\zeta}}{1-\sqrt{\zeta}}\right\}\|x\|_X,\ x\in X.$$

Hence, $\{g_n\}$ is a X_d -frame for X with the frame bounds $\frac{A-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+A)}{1+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}$ and $\frac{B+\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}(1+B)}{1-\sqrt{\max\{\alpha,\beta,\gamma,\delta\}}}$.

The next two theorems also gives a sufficient condition for a sequence to be a X_d -frame for X .

Theorem: 3.7 Let $\{f_n\}\subset X^*$ be a X_d -frame for X with frame bounds A and B and let $\{g_n\}\subset X^*$ be a X_d -Bessel sequence for X with bounds $K<A$, then, $\{f_n\pm g_n\}$ is a X_d -frame for X .

Proof: Suppose that R_T , R_Q are analysis operators of X_d -Bessel sequences $\{f_n\},\{g_n\}$ for X , respectively. For any $x\in X$, we have

$$\begin{aligned}\left\|\{(f_n\pm g_n)(x)\}\right\|_{X_d} &= \left\|R_T(x)\pm R_Q(x)\right\|_{X_d}\\ &\le \left\|\{R_T(x)\}\right\|_{X_d}+\left\|\{R_Q(x)\}\right\|_{X_d}\\ &\le (B+K)\|x\|_X,\ x\in X.\end{aligned}$$

Thus, $\{f_n\pm g_n\}$ is Bessel sequence for X . We also have

$$\begin{aligned}\left\|\{(f_n\pm g_n)(x)\}\right\|_{X_d} &= \left\|R_T(x)\pm R_Q(x)\right\|_{X_d}\\ &\ge \left\|\{R_T(x)\}\right\|_{X_d}-\left\|\{R_Q(x)\}\right\|_{X_d}\\ &\ge (A-K)\|x\|_X,\ x\in X.\end{aligned}$$

Hence, $\{f_n\pm g_n\}$ is a X_d -frame for X with respect to X_d .

Theorem: 3.8 Let $\{f_n\}\subset X^*$ be a X_d -frame for X with frame bounds A and B. Let $\{g_n\}\subset X^*$ be such that $\{g_n(x)\}\subset X_d$, for all $x\in X$ and let $\{f_n+g_n\}$ be a X_d -Bessel sequence for X and with bounds $K<A$. Then $\{g_n\}$ is a X_d -frame for X with bounds $A-K$ and $B+K$.

Proof: In the light of the frame inequality for the X_d -frame $\{f_n\}$ and the fact that K is a X_d -Bessel bounds for the X_d -Bessel sequence $\{f_n+g_n\}$, we have

$$\begin{aligned}(A-K)\,\|x\|_X &\le \|\{f_n(x)\}\|_{X_d}-\|\{(f_n+g_n)(x)\}\|_{X_d}\\ &\le \|\{g_n(x)\}\|_{X_d}\\ &\le \|\{f_n(x)\}\|_{X_d}+\|\{(f_n+g_n)(x)\}\|_{X_d}\\ &\le (B+K)\,\|x\|_X,\ x\in X\end{aligned}$$

Hence $\{g_n\}$ is a X_d -frame for X with the required frame bounds $A-K$ and $B+K$.

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