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A NOTE ON RECTANGULAR MATRICES

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ABSTRACT

In this paper, we study about determinants, adjoint, inverse, α- characteristic equation, α-minimal polynomial and α-eigenvalues of rectangular matrices. We also prove some enlightening results.

1. INTRODUCTION

Let V be a $\Gamma-$Banach algebra and m, n be positive integers. Denote by $V_{m,n}$ and $\Gamma_{n,m}$ the sets of $m\times n$ matrices with entries from V and $n\times m$ matrices with entries from Γ respectively. Let $(x_{ij}),(y_{ij})\in V_{m,n}$ and $(\gamma_{ji})\in\Gamma_{n,m}$, we define $(z_{ij})=(x_{ij})(\gamma_{ji})(y_{ij})$ where $z_{ij}=\sum_p\sum_q x_{ip}\gamma_{pq}y_{qj}$ and $\|A\|=\max\{|x_{ij}|:i=1,2,....m;\ j=1,2,.....n\}$, where $A=(x_{ij})\in V_{m,n}$. Then $V_{m,n}$ is a $\Gamma_{m,n}-$Banach algebra with respect to matrix addition, scalar multiplication, matrix multiplication and norm defined as above. The concept of $\alpha-$ characteristic equations has been generalized by several authors [1], [2], [3], [5].In [3], define the product of rectangular matrices A and B of order $m\times n$ by $A.B=A\alpha B$, for a fixed rectangular matrix $\alpha_{n\times m}$. With this product, we have $A^2=A\alpha A$, $A^3=A^2(\alpha A)$, $A^4=A^3(\alpha A)$,........, $A^n=A^{n-1}(\alpha A)$.

2. DEFINITIONS AND EXAMPLES

Definition 2.1: A determinant $|A|$ for a rectangular matrix $A_{m\times n}$ $(m\le n)$

Let J_n be the set of integers $\{1,2,3,......,n\}$. Let the integers $m,\ k_{p^1},\ k_{p^2},\ ....k_{p^m}$ be such that

i. $m\le n$
ii. $k_{p^i}\in J_n$ for all $i\in J_m$ and $p=1,\ 2,\ 3,\ ....$
iii. $k_{p^1}<k_{p^2}<.......<k_{p^m}$

For an integer *d*, $1\le d\le(n-m+1)$, define a set S_d such that

$$S_d=\{e_p^d=(d,\ k_{p^2},\ k_{p^3},\ ....,\ k_{p^m})\}$$

If $N_d={}^{n-d}C_{m-1}$, then the cardinal number of S_d is N_d.

The set S_d, $1\le d\le(n-m+1)$ will be ordered as follows. A set $S_u<S_v$ whenever $u<v$. Moreover, the elements e_p^d and e_q^d will be placed in the order $e_p^d<e_q^d$ whenever $k_{p^s}<k_{q^s}$ for s=2, 3… m.

All the m-tuples, therefore, admit of the following order; namely

$$e_1^1<e_2^1<.......<e_{N_1}^1<e_2^2<.......<e_{N_2}^2<..........<e_1^{n-m+1}<...........<e_{n-m+1}^{n-m+1}.$$

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Consider the matrix $A=(a_{ij})_{m\times n}$, $m\le n$. Let A_p^d be a sub-matrix of order $m\times m$ of *A* whose columns conform to the ordering of integers in e_p^d; $1\le d\le (n-m+1)$, $1\le p\le N_d$.

For an $m\times n$ $(m\le n)$, matrix *A* with real elements. Let A_p^d be defined as above, then the number $\sum_{d=1}^{n-m+1}\sum_{p=1}^{N_d}\det(A_p^d)$ will be defined as the determinant of *A* and will be denoted by $|A|$.

For an $m\times n$ $(m\ge n)$, matrix *A* with real elements, $|A|$ will be defined as $|A^T|$ where $|A^T|$ denotes the transpose of matrix of *A*.

Example: Let $A=\begin{pmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 6 & 7\end{pmatrix}$. Here $m=2$ and $n=5$, so $1\le d\le (5-2+1)$
i.e. $d=1,\ 2,\ 3,\ 4$ and $N_1=4,\ N_2=3,\ N_3=2,\ N_4=1$. The sets $S_1,\ S_2,\ S_3$ and S_4 contains the following elements, namely

$S_1=\{(1,2),\ (1,3),\ (1,4),\ (1,5)\}$
$S_2=\{(2,3),\ (2,4),\ (2,5)\}$
$S_3=\{(3,4),\ (3,5)\}$
$S_4=\{(4,5)\}$

There by the above definition, we have

$$|A|=\begin{vmatrix}1 & 2 & 3 & 4 & 5\\ 3 & 4 & 5 & 6 & 7\end{vmatrix}$$
$$=\begin{vmatrix}1 & 2\\ 3 & 4\end{vmatrix}+\begin{vmatrix}1 & 3\\ 3 & 5\end{vmatrix}+\begin{vmatrix}1 & 4\\ 3 & 6\end{vmatrix}+\begin{vmatrix}1 & 5\\ 3 & 7\end{vmatrix}+\begin{vmatrix}2 & 3\\ 4 & 5\end{vmatrix}+\begin{vmatrix}2 & 4\\ 4 & 6\end{vmatrix}+\begin{vmatrix}2 & 5\\ 4 & 7\end{vmatrix}+\begin{vmatrix}3 & 4\\ 5 & 6\end{vmatrix}+\begin{vmatrix}3 & 5\\ 5 & 7\end{vmatrix}+\begin{vmatrix}4 & 5\\ 6 & 7\end{vmatrix}$$
$$=-42$$

Definition 2.2: Cofactors, Adjoints and Inverse of rectangular matrix

Let A be a $m\times n$ $(m\le n)$ rectangular matrix of order $m\times n$; then we have by definition that $|A|$ is a linear homogeneous function of the entries in the ith row of *A*. If c_{ij} denotes the coefficient of a_{ij}, $j=1,\ 2,\ 3,.......,n$, then we get the expression

$$|A|=a_{i1}c_{i1}+a_{i2}c_{i2}+a_{i3}c_{i3}+........+a_{in}c_{in}$$

The coefficient c_{ij} of a_{ij} of the above expression is called the cofactor of a_{ij}. Let *E, F, G* and *H* be the sub-matrices of *A* of the order $(i-1)\times(j-1),\ (i-1)\times(n-j),\ (m-i)\times(n-j)$ and $(m-i)\times(j-1)$ respectively such that $A=\begin{bmatrix}E & : & F\\ \multicolumn{3}{c}{.............a_{ij}.............}\\ H & : & G\end{bmatrix}$ then the determinant of the sub-matrix M_{ij} of the order $(m-1)\times(n-1)$ corresponds to the $M_{ij}=\begin{vmatrix}E & -F\\ -H & G\end{vmatrix}$, the cofactor of a_{ij}, that is $c_{ij}=|M_{ij}|$.

A rectangular matrix A of order $m\times n$ $(m\le n)$ is said to be non-singular if $|A|\ne 0$; otherwise it is said to be singular.

If *A* is non-singular, then its inverse A^{-1} is defined by $A^{-1}=\dfrac{Adj(A)}{|A|}$.

Definition 2.3: $\alpha-$characteristic Equation, $\alpha-$eigenvalue, $\alpha-$eigenvector

Let us consider a rectangular matrix $A_{m\times n}$ $(m\neq n)$. Then we consider a fixed rectangular matrix $\alpha_{n\times m}$ of the opposite order of A. Then αA and $A\alpha$ are both square matrices of order n and m respectively. If $m<n$, then αA is the highest n^{th} order singular square matrix and $A\alpha$ is the lowest m^{th} order square matrix forming their product. Then the matrix $\alpha A-\lambda I_n$ and $A\alpha-\lambda I_m$ are called the left $\alpha-$characteristic matrix and right $\alpha-$characteristic matrix of A respectively, where λ is an indeterminate. Also the determinant $|\alpha A-\lambda I_n|$ is a polynomial in λ of degree n, called the left $\alpha-$characteristic polynomial of A and $|A\alpha-\lambda I_m|$ is a polynomial in λ of degree m, called the right $\alpha-$characteristic polynomial of A. That is, the characteristic polynomial of singular square matrix αA is called the left $\alpha-$characteristic polynomial of A and the characteristic polynomial of the square matrix $A\alpha$ is called the right $\alpha-$characteristic polynomial of A. The equations $|\alpha A-\lambda I_n|=0$ and $|A\alpha-\lambda I_m|=0$ are called the left $\alpha-$characteristic equation and right $\alpha-$characteristic equation of A respectively. Then the rectangular matrix A satisfies the left $\alpha-$characteristic equation, and the left $\alpha-$characteristic equation of A is called the $\alpha-$characteristic equation of A.

For, m>n the rectangular matrix $A_{m\times n}$ satisfies the right $\alpha-$characteristic equation of A. So in this case, the equation $|A\alpha-\lambda I_m|=0$ is called the $\alpha-$characteristic equation of A. The roots of the $\alpha-$characteristic equation of a rectangular matrix A are called the $\alpha-$eigenvalues of A

If λ is an $\alpha-$eigenvalue of rectangular matrix A of order $m\times n$ $(m<n)$, then the matrix $\alpha A-\lambda I_n$ is singular. The equation $(\alpha A-\lambda I_n)X=0$ then possesses a non-zero solution i.e. there exists a non-zero column vector X such that $\alpha AX=\lambda X$. A non-zero vector X satisfying this equation is called a $\alpha-$characteristic vector or $\alpha-$eigenvector of A corresponding to the $\alpha-$eigenvalue λ.

Definition 2.4: $\alpha-$minimal polynomial

For a rectangular matrix $A_{m\times n}$ $(m\neq n)$ over a field K, let $J(A)$ denote the collection of all polynomial $f(\lambda)$ for which $f(A)=0$ (Note that $J(A)$ is non empty, since the $\alpha-$characteristic polynomial of A belongs to $J(A)$). Let $m_\alpha(\lambda)$ be the monic polynomial of minimal degree in $J(A)$. Then $m_\alpha(\lambda)$ is called the $\alpha-$minimal polynomial of A.

3. MAIN RESULTS

Theorem 3.1: Let *A* be a rectangular matrix of order $m\times n$ $(m\leq n)$.

i. If $m=1$, then $|A|=a_{11}+a_{12}+a_{13}+........+a_{1n}$.
ii. If $m=n$, then $|A|=\det(A)=\det(A_1^1)$
iii. If any row of *A* is multiplied by c, then $|A|$ is multiplied by c.
iv. If any two rows are identical, then $|A|=0.$
v. The values of a determinant changes in sign only, if any two rows are interchanged.
vi. If each element in a row is an algebraic sum of two or more quantities, then the determinant can be expressed as an algebraic sum of two or more determinants.

Proof: Proofs are straightforward and so omitted.

Theorem 3.2: If *A* is a rectangular matrix of order $m\times n$ $(m\leq n)$, then

i. $A.Adj(A)=|A|I_m$ where I_m is the unit matrix of order m.
ii. $Adj(A).A=$ a singular matrix of order n.

Proof: Proofs are straightforward and so omitted.

Theorem 3.3: If *A* is an $m\times n$ $(m<n)$ rectangular matrix and α is an $n\times m$ rectangular matrix, then *A* is a zero of its $\alpha-$characteristic polynomial.

Proof: Since *A* is an $m\times n$ $(m<n)$ rectangular matrix and α is an $n\times m$ rectangular matrix, so the $\alpha-$ characteristic polynomial of *A* is of the form

$$\Delta(\lambda)=|\lambda I-\alpha A|=a_m\lambda^n+a_{m-1}\lambda^{n-1}+a_{m-2}\lambda^{n-2}+.....+a_0\lambda^{n-m}.$$

Let $B(\lambda)$ denote the classical adjoint of the matrix $\lambda I-\alpha A$. The elements of $B(\lambda)$ are cofactors of the matrix $\lambda I-\alpha A$ and hence are polynomials in λ of degree not exceeding $n-1$.

Thus $B(\lambda)=B_{m-1}\lambda^{n-1}+B_{m-2}\lambda^{n-2}+.....+B_1^{n-m+1}+B_0\lambda^{n-m}$, where the B_i are n-square matrices over K which are independent on λ. By the fundamental property of the adjoint

$$(\lambda I-\alpha A)B(\lambda)=|\lambda I-\alpha A|I$$

$$(\lambda I-\alpha A)\left(B_{m-1}\lambda^{n-1}+B_{m-2}\lambda^{n-2}+.....+B_1^{n-m+1}+B_0\lambda^{n-m}\right)$$
$$=\left(a_m\lambda^n+a_{m-1}\lambda^{n-1}+a_{m-2}\lambda^{n-2}+.....+a_0\lambda^{n-m}\right)I$$

Removing parentheses and equating the coefficients of corresponding of λ,

$$B_{m-1}=a_mI$$
$$B_{m-2}-(\alpha A)B_{m-1}=a_{m-1}I$$
$$B_{m-3}-(\alpha A)B_{m-2}=a_{m-2}I$$

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$$B_0-(\alpha A)B_1=a_1I$$
$$-(\alpha A)B_0=a_0$$

Multiplying the above matrix equations by $A^n,\ A^{n-1},\ A^{n-2},\ ......,\ A^{n-m+1},\ A^{n-m}$ respectively.

$$A^nB_{m-1}=a_mA^n$$
$$A^{n-1}B_{m-2}-A^nB_{m-1}=a_{m-1}A^{n-1}$$
$$A^{n-2}B_{m-3}-A^{n-1}B_{m-2}=a_{m-1}A^{n-2}$$

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$$A^{n-m+1}B_0-A^{n-m+2}B_1=a_1A^{n-m+1}$$
$$-A^{n-m+1}B_0=a_0A^{n-m}$$

Adding the above matrix equations, we get

$$0=a_mA^n+a_{m-1}A^{n-1}+a_{m-2}A^{n-2}+.....+a_0A^{n-m}$$

In other words, $\Delta(A)=0$. That *A* is a zero of its characteristic polynomial.

Theorem 3.4: Let *A* and α be two rectangular matrices of order $m\times n$ and $n\times m$ $(m<n)$ respectively. If $m_\alpha(\lambda)=\lambda^r+c_1\lambda^{r-1}+c_2\lambda^{r-2}+.....+c_{r-1}\lambda$ is the $\alpha-$minimal polynomial of *A* and $B_r=(\alpha A)^r+c_1(\alpha A)^{r-1}+c_2(\alpha A)^{r-2}+c_3(\alpha A)^{r-3}+.....+c_rI$, then $-\alpha AB_{r-1}=0$.

Proof: Suppose $m_\alpha(\lambda)=\lambda^r+c_1\lambda^{r-1}+c_2\lambda^{r-2}+.....+c_{r-1}\lambda$. Consider the following matrices:

$$B_0=I$$
$$B_1=\alpha A+c_1I$$
$$B_2=(\alpha A)^2+c_1\alpha A+c_2I$$

$B_3 = (\alpha A)^3 + c_1(\alpha A)^2 + c_2\alpha A + c_3 I$

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$B_{r-1} = (\alpha A)^{r-1} + c_1(\alpha A)^{r-2} + c_2(\alpha A)^{r-3} + ... + c_{r-1} I$

$B_r = (\alpha A)^r + c_1(\alpha A)^{r-1} + c_2(\alpha A)^{r-2} + c_3(\alpha A)^{r-3} + ... + c_r I$

Then $B_0 = I$

$B_1 - \alpha A B_0 = c_1 I$

$B_2 - \alpha A B_1 = c_2 I$

$B_3 - \alpha A B_2 = c_3 I$

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$B_{r-1} - \alpha A B_{r-2} = c_{r-1} I$

Thus

$$\begin{aligned} -\alpha A B_{r-1} &= c_r I - B_r \\ &= c_r I - [(\alpha A)^r + c_1(\alpha A)^{r-1} + c_2(\alpha A)^{r-2} + c_3(\alpha A)^{r-3} + ...c_r I] \\ &= -[\alpha A^r + c_1\alpha A^{r-1} + c_2\alpha A^{r-2} + c_3\alpha A^{r-3} + ...c_{r-1}\alpha A] \\ &= -\alpha(A^r + c_1 A^{r-1} + c_2 A^{r-2} + c_3 A^{r-3} + ...c_{r-1} A) \\ &= -\alpha.m_\alpha(A) \\ &= -\alpha.0 \\ &= 0 \end{aligned}$$

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