International Journal of Mathematical Archive-5(3), 2014, 119-123
IJMA Available online through www.ijma.info ISSN 2229 – 5046

A NOTE ON RECTANGULAR MATRICES

Md. Shahidul Islam Khan*

Department of Mathematics, Nabajyoti College, Kalgachia, PIN-781319, India.

(Received on: 27-02-14; Revised & Accepted on: 15-03-14)

ABSTRACT

In this paper, we study about determinants, adjoint, inverse, α- characteristic equation, α-minimal polynomial and α-eigenvalues of rectangular matrices. We also prove some enlightening results.

1. INTRODUCTION

Let V be a $\Gamma-$Banach algebra and m, n be positive integers. Denote by $V_{m,n}$ and $\Gamma_{n,m}$ the sets of $m\times n$ matrices with entries from V and $n\times m$ matrices with entries from Γ respectively. Let $(x_{ij}),(y_{ij})\in V_{m,n}$ and $(\gamma_{ji})\in\Gamma_{n,m}$, we define $(z_{ij})=(x_{ij})(\gamma_{ji})(y_{ij})$ where $z_{ij}=\sum_p\sum_q x_{ip}\gamma_{pq}y_{qj}$ and $\|A\|=\max\{|x_{ij}|: i=1,2,....m;\ j=1,2,.....n\}$, where $A=(x_{ij})\in V_{m,n}$. Then $V_{m,n}$ is a $\Gamma_{m,n}-$Banach algebra with respect to matrix addition, scalar multiplication, matrix multiplication and norm defined as above. The concept of $\alpha-$ characteristic equations has been generalized by several authors [1], [2], [3], [5].In [3], define the product of rectangular matrices A and B of order $m\times n$ by $A.B=A\alpha B$, for a fixed rectangular matrix $\alpha_{n\times m}$. With this product, we have $A^2=A\alpha A$, $A^3=A^2(\alpha A)$, $A^4=A^3(\alpha A),……..,A^n=A^{n-1}(\alpha A)$.

2. DEFINITIONS AND EXAMPLES

Definition 2.1: A determinant $|A|$ for a rectangular matrix $A_{m\times n}$ $(m\le n)$

Let J_n be the set of integers $\{1,2,3,......,n\}$. Let the integers $m,\ k_{p^{1}},\ k_{p^2},\ ....k_{p^m}$ be such that

i. $m\le n$
ii. $k_{p^i}\in J_n$ for all $i\in J_m$ and $p=1,\ 2,\ 3,\ ....$
iii. $k_{p^1}<k_{p^2}<.......<k_{p^m}$

For an integer *d*, $1\le d\le(n-m+1)$, define a set S_d such that
$S_d=\{e_p^d=(d,\ k_{p^2},\ k_{p^3},\ ....,\ k_{p^m})\}$

If $N_d={}^{n-d}C_{m-1}$, then the cardinal number of S_d is N_d .

The set S_d , $1\le d\le(n-m+1)$ will be ordered as follows. A set $S_u<S_v$ whenever $u<v$. Moreover, the elements e_p^d and e_q^d will be placed in the order $e_p^d<e_q^d$ whenever $k_{p^s}<k_{q^s}$ for s=2, 3… m.

All the m-tuples, therefore, admit of the following order; namely
$e_1^1<e_2^1<.......<e_{N_1}^1<e_2^2<.......<e_{N_2}^2<..........<e_1^{n-m+1}<...........<e_{n-m+1}^{n-m+1}$.

Corresponding author: Md. Shahidul Islam Khan*
Department of Mathematics, Nabajyoti College, Kalgachia, PIN-781319, India.
E-mail: shahidul_islamkhan786@yahoo.com

Consider the matrix $A = (a_{ij})_{m\times n}$, $m \le n$. Let A_p^d be a sub-matrix of order $m \times m$ of *A* whose columns conform to the ordering of integers in e_p^d; $1 \le d \le (n-m+1)$, $1 \le p \le N_d$.

For an $m \times n$ $(m \le n)$, matrix *A* with real elements. Let A_p^d be defined as above, then the number $\sum_{d=1}^{n-m+1} \sum_{p=1}^{N_d} \det(A_p^d)$ will be defined as the determinant of *A* and will be denoted by $|A|$.

For an $m \times n$ $(m \ge n)$, matrix *A* with real elements, $|A|$ will be defined as $|A^T|$ where $|A^T|$ denotes the transpose of matrix of *A*.

Example: Let $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 7 \end{pmatrix}$. Here $m = 2$ and $n = 5$, so $1 \le d \le (5-2+1)$
i.e. $d = 1, 2, 3, 4$ and $N_1 = 4,\ N_2 = 3,\ N_3 = 2,\ N_4 = 1$. The sets S_1, S_2, S_3 and S_4 contains the following elements, namely

$S_1 = \{(1,2), (1,3), (1,4), (1,5)\}$
$S_2 = \{(2,3), (2,4), (2,5)\}$
$S_3 = \{(3,4), (3,5)\}$
$S_4 = \{(4,5)\}$

There by the above definition, we have

$$|A| = \begin{vmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 & 7 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} + \begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} + \begin{vmatrix} 1 & 5 \\ 3 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} + \begin{vmatrix} 2 & 4 \\ 4 & 6 \end{vmatrix} + \begin{vmatrix} 2 & 5 \\ 4 & 7 \end{vmatrix} + \begin{vmatrix} 3 & 4 \\ 5 & 6 \end{vmatrix} + \begin{vmatrix} 3 & 5 \\ 5 & 7 \end{vmatrix} + \begin{vmatrix} 4 & 5 \\ 6 & 7 \end{vmatrix}$$
$$= -42$$

Definition 2.2: Cofactors, Adjoints and Inverse of rectangular matrix

Let A be a $m \times n$ $(m \le n)$ rectangular matrix of order $m \times n$; then we have by definition that $|A|$ is a linear homogeneous function of the entries in the ith row of *A*. If c_{ij} denotes the coefficient of a_{ij}, $j = 1, 2, 3, \ldots\ldots., n$, then we get the expression

$$|A| = a_{i1}c_{i1} + a_{i2}c_{i2} + a_{i3}c_{i3} + \ldots\ldots.. + a_{in}c_{in}$$

The coefficient c_{ij} of a_{ij} of the above expression is called the cofactor of a_{ij}. Let *E, F, G* and *H* be the sub-matrices of *A* of the order $(i-1)\times(j-1)$, $(i-1)\times(n-j)$, $(m-i)\times(n-j)$ and $(m-i)\times(j-1)$ respectively such that $A = \begin{bmatrix} E & : & F \\ \ldots\ldots\ldots\ldots. & a_{ij} & \ldots\ldots\ldots\ldots \\ H & : & G \end{bmatrix}$ then the determinant of the sub-matrix M_{ij} of the order $(m-1)\times(n-1)$ corresponds to the $M_{ij} = \begin{vmatrix} E & -F \\ -H & G \end{vmatrix}$, the cofactor of a_{ij}, that is $c_{ij} = |M_{ij}|$.

A rectangular matrix A of order $m \times n$ $(m \le n)$ is said to be non-singular if $|A| \ne 0$; otherwise it is said to be singular.

If *A* is non-singular, then its inverse A^{-1} is defined by $A^{-1} = \dfrac{Adj(A)}{|A|}$.

Definition 2.3: $\alpha-$**characteristic Equation,** $\alpha-$**eigenvalue,** $\alpha-$**eigenvector**

Let us consider a rectangular matrix $A_{m\times n}$ $(m\neq n)$. Then we consider a fixed rectangular matrix $\alpha_{n\times m}$ of the opposite order of A. Then αA and $A\alpha$ are both square matrices of order n and m respectively. If $m<n$, then αA is the highest n^{th} order singular square matrix and $A\alpha$ is the lowest m^{th} order square matrix forming their product. Then the matrix $\alpha A-\lambda I_n$ and $A\alpha-\lambda I_m$ are called the left $\alpha-$characteristic matrix and right $\alpha-$characteristic matrix of A respectively, where λ is an indeterminate. Also the determinant $|\alpha A-\lambda I_n|$ is a polynomial in λ of degree n, called the left $\alpha-$characteristic polynomial of A and $|A\alpha-\lambda I_m|$ is a polynomial in λ of degree m, called the right $\alpha-$characteristic polynomial of A. That is, the characteristic polynomial of singular square matrix αA is called the left $\alpha-$characteristic polynomial of A and the characteristic polynomial of the square matrix $A\alpha$ is called the right $\alpha-$characteristic polynomial of A. The equations $|\alpha A-\lambda I_n|=0$ and $|A\alpha-\lambda I_m|=0$ are called the left $\alpha-$characteristic equation and right $\alpha-$characteristic equation of A respectively. Then the rectangular matrix A satisfies the left $\alpha-$characteristic equation, and the left $\alpha-$characteristic equation of A is called the $\alpha-$characteristic equation of A.

For, m>n the rectangular matrix $A_{m\times n}$ satisfies the right $\alpha-$characteristic equation of A. So in this case, the equation $|A\alpha-\lambda I_m|=0$ is called the $\alpha-$characteristic equation of A. The roots of the $\alpha-$characteristic equation of a rectangular matrix A are called the $\alpha-$eigenvalues of A

If λ is an $\alpha-$eigenvalue of rectangular matrix A of order $m\times n$ $(m<n)$, then the matrix $\alpha A-\lambda I_n$ is singular. The equation $(\alpha A-\lambda I_n)X=0$ then possesses a non-zero solution i.e. there exists a non-zero column vector X such that $\alpha AX=\lambda X$. A non-zero vector X satisfying this equation is called a $\alpha-$characteristic vector or $\alpha-$eigenvector of A corresponding to the $\alpha-$eigenvalue λ.

Definition 2.4: $\alpha-$**minimal polynomial**

For a rectangular matrix $A_{m\times n}$ $(m\neq n)$ over a field K, let $J(A)$ denote the collection of all polynomial $f(\lambda)$ for which $f(A)=0$ (Note that $J(A)$ is non empty, since the $\alpha-$characteristic polynomial of A belongs to $J(A)$). Let $m_\alpha(\lambda)$ be the monic polynomial of minimal degree in $J(A)$. Then $m_\alpha(\lambda)$ is called the $\alpha-$minimal polynomial of A.

3. MAIN RESULTS

Theorem 3.1: Let *A* be a rectangular matrix of order $m\times n$ $(m\leq n)$.

i. If $m=1$, then $|A|=a_{11}+a_{12}+a_{13}+........+a_{1n}$.
ii. If $m=n$, then $|A|=\det(A)=\det(A_1^1)$
iii. If any row of *A* is multiplied by c, then $|A|$ is multiplied by c.
iv. If any two rows are identical, then $|A|=0.$
v. The values of a determinant changes in sign only, if any two rows are interchanged.
vi. If each element in a row is an algebraic sum of two or more quantities, then the determinant can be expressed as an algebraic sum of two or more determinants.

Proof: Proofs are straightforward and so omitted.

Theorem 3.2: If *A* is a rectangular matrix of order $m\times n$ $(m\leq n)$, then

i. $A.Adj(A)=|A|I_m$ where I_m is the unit matrix of order m.
ii. $Adj(A).A=$ a singular matrix of order n.

Proof: Proofs are straightforward and so omitted.

Theorem 3.3: If *A* is an $m \times n$ $(m < n)$ rectangular matrix and α is an $n \times m$ rectangular matrix, then *A* is a zero of its $\alpha -$characteristic polynomial.

Proof: Since *A* is an $m \times n$ $(m < n)$ rectangular matrix and α is an $n \times m$ rectangular matrix, so the $\alpha -$ characteristic polynomial of *A* is of the form
$$\Delta(\lambda) = |\lambda I - \alpha A| = a_m \lambda^n + a_{m-1}\lambda^{n-1} + a_{m-2}\lambda^{n-2} + ... + a_0 \lambda^{n-m}.$$

Let $B(\lambda)$ denote the classical adjoint of the matrix $\lambda I - \alpha A$. The elements of $B(\lambda)$ are cofactors of the matrix $\lambda I - \alpha A$ and hence are polynomials in λ of degree not exceeding $n-1$.

Thus $B(\lambda) = B_{m-1}\lambda^{n-1} + B_{m-2}\lambda^{n-2} + ... + B_1^{n-m+1} + B_0 \lambda^{n-m}$, where the B_i are n-square matrices over K which are independent on λ. By the fundamental property of the adjoint

$$(\lambda I - \alpha A)B(\lambda) = |\lambda I - \alpha A| I$$
$$(\lambda I - \alpha A)\left(B_{m-1}\lambda^{n-1} + B_{m-2}\lambda^{n-2} + ... + B_1^{n-m+1} + B_0 \lambda^{n-m}\right)$$
$$= \left(a_m \lambda^n + a_{m-1}\lambda^{n-1} + a_{m-2}\lambda^{n-2} + ... + a_0 \lambda^{n-m}\right) I$$

Removing parentheses and equating the coefficients of corresponding of λ,
$$B_{m-1} = a_m I$$
$$B_{m-2} - (\alpha A)B_{m-1} = a_{m-1} I$$
$$B_{m-3} - (\alpha A)B_{m-2} = a_{m-2} I$$
……………………..
……………………….
$$B_0 - (\alpha A)B_1 = a_1 I$$
$$-(\alpha A)B_0 = a_0$$

Multiplying the above matrix equations by $A^n,\ A^{n-1},\ A^{n-2},\ ......,\ A^{n-m+1},\ A^{n-m}$ respectively.
$$A^n B_{m-1} = a_m A^n$$
$$A^{n-1} B_{m-2} - A^n B_{m-1} = a_{m-1} A^{n-1}$$
$$A^{n-2} B_{m-3} - A^{n-1} B_{m-2} = a_{m-1} A^{n-2}$$
…………………………
…………………………
$$A^{n-m+1} B_0 - A^{n-m+2} B_1 = a_1 A^{n-m+1}$$
$$-A^{n-m+1} B_0 = a_0 A^{n-m}$$

Adding the above matrix equations, we get

$$0 = a_m A^n + a_{m-1} A^{n-1} + a_{m-2} A^{n-2} + ... + a_0 A^{n-m}$$

In other words, $\Delta(A) = 0$. That *A* is a zero of its characteristic polynomial.

Theorem 3.4: Let *A* and α be two rectangular matrices of order $m \times n$ and $n \times m$ $(m < n)$ respectively. If $m_\alpha(\lambda) = \lambda^r + c_1\lambda^{r-1} + c_2\lambda^{r-2} + ... + c_{r-1}\lambda$ is the $\alpha -$minimal polynomial of *A* and $B_r = (\alpha A)^r + c_1(\alpha A)^{r-1} + c_2(\alpha A)^{r-2} + c_3(\alpha A)^{r-3} + ... + c_r I$, then $-\alpha A B_{r-1} = 0$.

Proof: Suppose $m_\alpha(\lambda) = \lambda^r + c_1\lambda^{r-1} + c_2\lambda^{r-2} + ... + c_{r-1}\lambda$. Consider the following matrices:
$$B_0 = I$$
$$B_1 = \alpha A + c_1 I$$
$$B_2 = (\alpha A)^2 + c_1 \alpha A + c_2 I$$

$B_3 = (\alpha A)^3 + c_1(\alpha A)^2 + c_2\alpha A + c_3 I$

………………………………………

…………………………………………….

$B_{r-1} = (\alpha A)^{r-1} + c_1(\alpha A)^{r-2} + c_2(\alpha A)^{r-3} + ... + c_{r-1} I$

$B_r = (\alpha A)^r + c_1(\alpha A)^{r-1} + c_2(\alpha A)^{r-2} + c_3(\alpha A)^{r-3} + ... + c_r I$

Then $B_0 = I$

$B_1 - \alpha AB_0 = c_1 I$

$B_2 - \alpha AB_1 = c_2 I$

$B_3 - \alpha AB_2 = c_3 I$

…………………………..

…………………………..

$B_{r-1} - \alpha AB_{r-2} = c_{r-1} I$

Thus $-\alpha AB_{r-1} = c_r I - B_r$

$$
\begin{aligned}
&= c_r I - [(\alpha A)^r + c_1(\alpha A)^{r-1} + c_2(\alpha A)^{r-2} + c_3(\alpha A)^{r-3} + ...c_r I] \\
&= -[\alpha A^r + c_1\alpha A^{r-1} + c_2\alpha A^{r-2} + c_3\alpha A^{r-3} + ...c_{r-1}\alpha A] \\
&= -\alpha(A^r + c_1 A^{r-1} + c_2 A^{r-2} + c_3 A^{r-3} + ...c_{r-1} A) \\
&= -\alpha.m_\alpha(A) \\
&= -\alpha.0 \\
&= 0
\end{aligned}
$$

REFERENCES

[1]. G. L. Booth, “Radicals of matrix $\Gamma -$rings”, Math. Japonica, Vol.33, 1988, pp. 325-334.

[2]. H.K. Nath, “A study of Gamma-Banach algebras”, Ph.D. Thesis, (Gauhati University), 2001.

[3]. Jinn Miao Chen, “Von Neumann regularity of two matrix $\Gamma_{n,m} -$ring $M_{m,.n}$ over $\Gamma -$ring”, M. J. Xinjiang Univ. Nat. Sci., Vol. 4, 1987, pp. 37-42.

[4]. P. Rajkhowa and Md. Shahidul Islam Khan, “On α-Characteristic Equations and α-Minimal Polynomial of Rectangular Matrices”, IOSR J. of Mathematics, Vol. 6, Issue 3, 2013, pp. 42- 46.

[5]. T. K. Dutta and H. K. Nath, “On the Gamma ring of rectangular Matrices”, Bulletin of Pure and Applied Science, Vol. 16E, 1997, pp. 207-216.

Source of support: Nil, Conflict of interest: None Declared

