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TRANSIENT BEHAVIOUR BY SEQUENCE CHAIN OF A FLUID QUEUES

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ABSTRACT

In this paper, we obtain the transient solution in closed form of fluid queue driven by a birth and death process on an infinite-state space whose birth and death rates are suggested by a chain sequence. The probability with which the buffer content becomes empty at an arbitrary time is also determined. Numerical illustrations are added to capture the variations in the behaviour of this performance measure against time.

1. MODEL DESCRIPTION

Consider a fluid queue driven by a birth and death process, $\{X(t), t \geq 0\}$ with rates suggested by a chain sequence, viz, the birth and death parameter satisfy

$$\lambda_{n_1} + \mu_{n_1} = 1,\ \lambda_{n_1 - 1}\mu_{n_1} = \beta,\ i.e., (1 - \mu_{n_1 - 1})\mu_{n_1} = \beta,\ n_1 = 1,2,3,\ldots, \tag{1}$$

with $\lambda_0 = 1$ and $\mu_0 = 0$ so that $\{\mu_{n_1}\}$ is the minimal parameter sequence for the constant term chain sequence $\{\beta, \beta, \beta, \ldots\},\ 0 < \beta \leq 1/4$, so that λ_{n_1} and μ_{n_1} are positive, given by

$$\lambda_{n_1} = \frac{\alpha U_{n+1}(1/\alpha)}{2U_n(1/\alpha)},\ n_1 = 1,2,3,\ldots, \tag{2}$$

$$\mu_{n_1} = \frac{\alpha U_{n-1}(1/\alpha)}{2U_n(1/\alpha)},\ n_1 = 1,2,3,\ldots, \tag{3}$$

where $U_n(.)$ is the Chebyshev polynomial of second kind of order n and $\alpha = 2\sqrt{\beta}$. Note that

$$\mu_1\mu_2\ldots\mu_j = \left(\frac{\alpha}{2}\right)^j \frac{1}{U_j(1/\alpha)} = \frac{\left(\sqrt{\beta}\right)^j}{U_j\left(1/2\sqrt{\beta}\right)} \tag{4}$$

The transition probability for the process $\{X(t), t \geq 0\}$, whose birth and death rates are governed by (1), with $X(0) = 0$, are

$$P_n(t) = 2(n+1)U_n\left(\frac{1}{\alpha}\right)\frac{e^{-t}I_{n+1}(\alpha t)}{\alpha t} \tag{5}$$

(Lenin and Parthasarathy [3].)

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It can easily be show that the sequence $\{\lambda_{n_1}\}$ is decreasing with n and tends towards $(1+\sqrt{1-4\beta})/2$, so that it could represent a queue with discouraged arrivals. The sequence $\{\mu_{n_1}\}$ is thus increasing with n towards $(1-\sqrt{1-4\beta})/2$, which means that the service rate of the queue can be dynamically adapted in the function of the number of customer in the queue, until a fixed limit. This kind of model is mathematically interesting because it is indeed rare and has closed- form solution.

If $C(t)$ denotes the content of the buffer at time t, the 2-dimentional process $\{X(t), C(t), t \geq 0\}$ constitutes a Markov process. When the process $X(t)$ is positive, the fluid level in the buffer increases at a constant rate $r > 0$ and when $X(t) = 0$, the fluid level in the buffer decreases at a constant rate $r_0 < 0$. We suppose that $X(0) = 0$ and $C(0) = 0$. Fluid models of this type find application in the field of telecommunication for modeling the network traffic and in the approximation of discrete stochastic queueing networks. For practical design and performance evaluation, it is essential to obtain information about the buffer occupancy distribution.

If $G_i(t,x) \equiv P(X(t) = i, C(t) \leq x), i \in \varphi, t, x \geq 0$, the Kolmogorov forward equations for the Markov process $\{X(t), C(t)\}$ are given by

$$\frac{\partial G_0(t,x)}{\partial t} = -r_0 \frac{\partial G_0(t,x)}{\partial x} = G_0(t,x) + \mu_1 G_1(t,x),$$

$$\frac{\partial G_i(t,x)}{\partial t} = -r \frac{\partial G_i(t,x)}{\partial x} + \lambda_{i-1} G_{i-1}(t,x) - G_i(t,x) + \mu_{i+1} G_{i+1}(t,x), i \in \varphi \mid \{0\}, t, x \geq 0, \tag{6}$$

subject to the initial condition

$$G_0(0,x) = 1, \; G_i(0,x) = 0 \quad for \quad i = 1,2,3,\ldots \tag{7}$$

and boundary condition

$$G_i(t,0) = q_i(t) \quad for \quad i = 0,1,2,,\ldots \tag{8}$$

Here $q_i(t)$ represents the probability that at time t the buffer is empty and the state of the background Markov process is i. The content of the buffer decreases and thereby becomes empty only when the net input rate of the fluid into the buffer is negative. Therefore, when the buffer becomes empty at any time t, the background process should necessarily be in state zero corresponding to which the effective input rate is $r_0 < 0$. Hence we have $q_i(t) = 0$ for $i = 1,2,3,\ldots$ as $r > 0$ when $X(t) = i$ for $i = 1,2,3,\ldots$

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The transient distribution of the buffer content is given by

$$P_r(C(t) > x) = 1 - \sum_{i=0}^{\infty} G_i(t,x).$$

In this sequence let $G_i^*(s,x)$ and $G_i^{**}(s,w)$ denote the single Laplace transform (with respect to t) and double Laplace transform (with respect to t and x) of $G_i(t,x)$,respectively.

2. TRANSIENT SOLUTION

The expression for the joint distribution of the buffer content of the fluid queue model under consideration using an approach similar to F.Gullemin [2] is given by

$$G_i(t,x) = \sum_{n=0}^{\infty} e^{-t} \frac{t^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{x}{rt}\right)^k \left(1 - \frac{x}{rt}\right)^{n-k} b_i(n,k), \; i = 0,1,2,\ldots \tag{9}$$

for every $t \ge 0$ and $x \in [0, rt)$ where the coefficient $b_i(n,k)$ are given by the following recursive expressions

(i) For $i = 0$

$$b_0(n,n) = \begin{cases} \dfrac{1}{k+1}\dbinom{2k}{k}\beta^k & if \;\; n = 2k, \\ 0 & if \;\; n = 2k+1, \end{cases} \tag{10}$$

$$b_0(n,k) = \frac{-r_0}{r-r_0} b_0(n,k+1) + \frac{r\beta}{r-r_0} b_1(n-1,k) \;\; for \;\; n \ge 1,\; 0 \le k \le n-1. \tag{11}$$

(ii) For $i \ge 0$

$$b_i(n,0) = 0 \quad for \quad n \ge 0,$$

$$b_i(n,k) = \lambda_{i-1} b_{i-1}(n-1,k-1) + \mu_{i+1} b_{i+1}(n-1,k-1) \;\; for \quad n \ge 1,\; 1 \le k \le n. \tag{12}$$

From (9), the probability that the buffer is empty at time t is given by

$$G_0(t,0) = q_0(t) = e^{-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} b_0(n,0), \tag{13}$$

where $b_0(n,0)$ for all $n \ge 1$ are obtained from the recurrence relations (10) and (11). The following theorem presents an alternate formula for the evaluation of $b_0(n,0)$.

Theorem: 2.1 *For all* $n \ge 1$,

$$b_0(n,0) = \frac{r\beta}{r-r_0} \sum_{i=1}^{n-1} \left(\frac{-r_0}{r-r_0} \right)^{i} \sum_{l=0}^{[(i-1)/2]} \binom{2l}{l} \frac{\beta^l}{l+1} b_0(n-2l-2, i-2l-l) + \left(\frac{-r_0}{r-r_0} \right)^{n} b_0(n,n). \tag{14}$$

Proof: The following propositions and lemma presents a simplified formula for evaluating the various terms involved in the determinations of $b_0(n,0)$ thereby reducing the computational complexity

Proposition: 2.2 *For all* $n \ge 1,\; 0 \le k \le n-1$,

$$b_0(n,k) - \left(\frac{-r_0}{r-r_0} \right)^{n-k} b_0(n,n) = \left(\frac{r\beta}{r-r_0} \right) \sum_{i=k}^{n-1} \left(\frac{-r_0}{r-r_0} \right)^{i-k} b_1(n-1,i). \tag{15}$$

Proof: Recall (10),

$$b_0(n,k) = \frac{-r_0}{r-r_0} b_0(n,k+1) + \frac{r\beta}{r-r_0} b_1(n-1,k). \tag{16}$$

Multiplying the above equation by $(-r_0/(r-r_0))^{i-k}$ and summing over all i from k to $n-1$,we get

$$\sum_{i=k}^{n-1} \left(\frac{-r_0}{r-r_0} \right)^{i-k} b_0(n,i) - \sum_{i=k}^{n-1} \left(\frac{-r_0}{r-r_0} \right)^{i-k+1} b_0(n,i+1) = \left(\frac{r\beta}{r-r_0} \right) \sum_{i=k}^{n-1} \left(\frac{-r_0}{r-r_0} \right)^{i-k} b_1(n-1,i). \tag{17}$$

Hence we have

$$b_0(n,k) - \left(\frac{-r_0}{r-r_0} \right)^{n-k} b_0(n,n) = \left(\frac{r\beta}{r-r_0} \right) \sum_{i=k}^{n-1} \left(\frac{-r_0}{r-r_0} \right)^{i-k} b_1(n-1,i). \tag{18}$$

Lemma: 2.3 *For* $i \ge 1,\ b_i(n,k) = 0$ *for* $0 \le n < i$ *and*

$$b_i(n,k) = \begin{cases} 0 & if\ 0 \le k < i, \\ \left(\sqrt{\beta}\right)^i U_i\left(\dfrac{1}{2\sqrt{\beta}}\right)^{[(k-i)/2]} \displaystyle\sum_{l=0} s(i,l)\beta^l b_0(n-2l-i, k-2l-i) & if\ i \le k \le n, \\ 0 & if\ k > n, \end{cases} \tag{19}$$

where the number $s(i,l)$ are referred to as the ballot numbers given by

$$s(i,l) = i\frac{(2l+i-l)!}{l!(l+i)!} \tag{20}$$

Proof: Recall (12)

$$b_i(n,k) = \lambda_{i-1}b_{i-1}(n-1,k-1) + \mu_{i+1}b_{i+1}(n-1,k-1)\ for\ \ n \ge 1,\ 1 \le k \le n. \tag{21}$$

For all $n \ge 1,\ 1 \le k \le n,$ define

$$B_0(n,k) = b_0(n,k), \tag{22}$$

$$B_i(n,k) = (\mu_1\mu_2 \dots \mu_i)b_i(n,k), i \ge 1 \tag{23}$$

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Then (12) becomes,

$$B_i(n,k) = \lambda_{i-1}\mu_i B_{i-1}(n-1,k-1) + B_{i+1}(n-1,k-1). \tag{24}$$

From (1), $\lambda_{i-1}\mu_i = \beta,$ hence we have

$$B_i(n,k) = \beta B_{i-1}(n-1,k-1) + B_{i+1}(n-1,k-1). \tag{25}$$

Now, define

$$H_i(n,v) = \sum_{k=0}^{n} v^k B_i(n,k), \tag{26}$$

then (26) reduces to

$$H_i(n,v) = v\beta\, H_{i-1}(n-1,v) + v\, H_{i+1}(n-1,v), \quad i \ge 1, n \ge 1 \tag{27}$$

Again define

$$\phi_i(u,v) = \sum_{n=0}^{\infty} \frac{u^n H_i(n,v)}{n!}, \tag{28}$$

then (28) reduces to

$$\phi_i^{'}(u,v) = v\beta\phi_{i-1}(u,v) + v\phi_{i+1}(u,v) \quad for\ \ i \ge 1. \tag{29}$$

Laplace transform of the above equation with respect to u yields

$$z\phi_i^*(z,v) = v\beta\phi_{i-1}^*(z,v) + v\phi_{i+1}^*(z,v). \tag{30}$$

Writing in the form of continued fraction, we get

$$\frac{\phi_i^*(z,v)}{\phi_{i-1}^*(z,v)} = \frac{v\beta}{z - v(\phi_{i+1}^*(z,v)/\phi_i^*(z,v))} = \frac{v\beta v^2 \beta v^2 \beta}{z - z - z -}\dots \tag{31}$$

Solving the above continued fraction, we get

$$\frac{\phi_i^*(z,v)}{\phi_{i-1}^*(z,v)} = \frac{z-\sqrt{z^2-4v^2\beta}}{2v}, \quad i=1,2,3,\ldots, \tag{32}$$

$$\frac{\phi_i^*(z,v)}{\phi_{i-1}^*(z,v)} = \frac{v\beta}{z}\left(\frac{1-\sqrt{1-4(v^2\beta/z^2)}}{2(v^2\beta/z^2)}\right) = \frac{v\beta}{z}C\left(\frac{v^2\beta}{z^2}\right). \tag{33}$$

Before we proceed further, we give a brief discussion on the function $C(z)$ below.

Let $C(z)$ be the complex function define by

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}. \tag{34}$$

For $|z| \le 1/4,$ we have

$$C(z) = \sum_{n=0}^{\infty} c_n z^n, \tag{35}$$

Where the number c_n are referred to as the Catalan number given by

$$c_n = \binom{2n}{n}\frac{1}{n+1}. \tag{36}$$

More generally, for $k \ge 1$ and $|z| \le 1/4,$ we have

$$C^k(z) = \sum_{n=0}^{\infty} s(k,n) z^n, \tag{37}$$

where the numbers $s(k,n)$ are given by (20).

Continuing our discussion from (33), we easily get, for $i \ge 1$ and $\left|v\beta/z^2\right| < 1/4,$

$$\begin{aligned}\phi_i^*(z,v) &= \frac{v\beta}{z}C\left(\frac{v^2\beta}{z^2}\right)\phi_{i-1}^*(z,v) \\ &= \frac{v^i\beta^i}{z^i}C^i\left(\frac{v^2\beta}{z^2}\right)\phi_0^*(z,v) = \frac{v^i\beta^i}{z^i}\sum_{l=0}^{\infty} s(i,l)\left(\frac{v^2\beta}{z^2}\right)^l \phi_0^*(z,v).\end{aligned} \tag{38}$$

We thus have, for $i \ge 1$ and $\left|v\beta/z^2\right| < 1/4,$

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{H_i(n,v)}{z^{n+1}} &= \frac{v^i\beta^i}{z^i}\sum_{l=0}^{\infty} s(i,l)\left(\frac{v^2\beta}{z^2}\right)^l \sum_{n=0}^{\infty}\frac{H_0(n,v)}{z^{n+1}} \\ &= \beta^i \sum_{l=0}^{\infty} s(i,l)\beta^l v^{2l+i}\sum_{n=0}^{\infty}\frac{H_0(n,v)}{z^{2l+i+n+1}} \\ &= \beta^i \sum_{l=0}^{\infty} s(i,l)\beta^l v^{2l+i}\sum_{n=2l+i}^{\infty}\frac{H_0(n-2l-i,v)}{z^{n+1}} \\ &= \beta^i \sum_{i=0}^{\infty}\frac{1}{z^{n+1}}\sum_{l=0}^{[(n-i)/2]} s(i,l)\beta^l v^{2l+i} H_0(n-2l-i,v),\end{aligned} \tag{39}$$

where the last equality is obtained by exchanging the order of summation. This leads, for $i \geq 1$,to the following expression of $H_i(n,v)$:

$$H_i(n,v) = \begin{cases} 0 & if\ \ 0 \leq n < i, \\ \beta^i \displaystyle\sum_{l=0}^{[(n-i)/2]} s(i,l)\beta^l v^{2l+i} H_0(n-2l-i,v) & if\ \ n \geq i. \end{cases} \tag{40}$$

This means in particular, that $b_i(n,k) = 0$ for $i \geq 1$ and $0 \leq n < i.$

CONCLUSION

We conclude that a fluid queue driven by an infinite-state BDP whose birth and death rates are suggested by a chain sequence. The stationary solution for the background BDP suggested by a chain sequence dose not exist and hence the stationary distribution for fluid queue driven by such BDSs also does not exit. However their transient probabilities yield a simple closed form solution.

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