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GOLDEN GRAPHS-II

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ABSTRACT

In this paper, we have introduced new definition of golden graphs and hence almost pure golden graphs and characterized graphs as almost pure golden graphs .Also generated an infinite class of C_n, a cycle on n nodes as golden graphs.

Keywords: *Trees, Paths, Cycle, adjacency matrix, characteristic polynomial of a graph G and Golden ratio.*

I. INTRODUCTION

The golden ratio has fascinated western philosophers, mathematicians, artists, sculptors, scientists and almost all intellectuals in all walks of life for at least 2,400 years. Many westerns intellectuals have come across the golden ratio in their chosen field specialization and they have used names such as golden ratio, extreme and mean ratio, medical section, divine section, golden number, golden proportion and mean of Phidias. The historic evidence of the appearance of Golden ratio (GR) in graph theory, first time GR appeared in graph theory in connection with chromatic polynomials. W.TTutte(1970), Michel O Alberston(1973), Saeid Alikhani and Yee-hock(2009) all dealt with GR in connection with chromatic polynomials. Pavel Chebotarev (2008), deal with GR in connection with spanning forest. We are giving an account of GR in graph which we came across while studying the spectral properties of graphs. While studying the spectra of C_5, a cycle on 5 nodes ,we find that its Eigen values are 2, $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$, and with multiplicities as 1,2,2 respectively which are nothing but Golden ratio (Divine ratio).Interestingly, we asked the question which graphs have Eigen values as Golden ratio and also which graphs has Eigen values as λ, $\frac{-1+\sqrt{5}}{2}$, $\frac{-1-\sqrt{5}}{2}$,where $\lambda \neq \frac{-1+\sqrt{5}}{2}, \frac{-1-\sqrt{5}}{2}$ which we defined as almost pure golden graphs .In this paper we have proved logically that, there are infinite class of C_n, a cycle on n nodes which have GR as eigen value and cycle C_5 is the only almost pure golden graph.

II. PRELIMINARIES

Let G be a graph without loops or multiple links having n nodes. Then the adjacency matrix of G, $A(G) = A$, is a square matrix, symmetric matrix of order n ,whose elements A_{ij} are ones or zeros if the corresponding nodes are adjacent or not, respectively. This matrix has (not necessarily distinct) real-valued Eigen values ,which are denoted by $\lambda_1, \lambda_2, ..., \lambda_n$. The set of Eigen values of A together with their multiplicities form the spectrum of G, which will be represented here as Spec $(G) = \{[\lambda_1]^{m_1}, [\lambda_2]^{m_2} ....[\lambda_n]^{m_n}\}$, where λ_i is the i^{th} Eigen value with m_i multiplicity. Here the Eigen values are assumed to be labelled in a non-increasing manner.

$$\lambda_1 \geq \lambda_2 \geq ...... \geq \lambda_n.$$

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Let $P_{n,} C_n, K_n$ be the path graph .the cycle graph and the complete graph on n nodes, respectively. The path P_n is a tree with two nodes of degree 1and the other two nodes with degree 2. A tree is acyclic graph (with out cycles). A cycle C_n is a graph on n nodes containing a single cycle through all nodes.

Theorem: 2.1 [1] Let $\phi(G,x) = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + ...C_n$ be the characteristic polynomial of a graph G, then the co-efficient of $\phi(G,x)$ satisfy:

a) $C_0 = 1$

b) $C_1 = o$

c) $-C_2$ is the number of edges in G.

d) $-C_3$ is twice the triangles in G.

Theorem: 2.2[1] If G is graph without isolated nodes having components $G_{1,} G_2, ..., G_k$, then $\phi(G,\lambda) = \prod_{i=1}^{k} \phi(G_i, \lambda)$

Theorem: 2.3[1] If G has $\lambda_1 > \lambda_2 > \lambda_3$ as distinct eigen values and $\lambda_2 + \lambda_3 = r - 3$, then $G \cong C_5$

Theorem: 2.4[1] For a graph G with adjacency matrix A their exists a polynomial $P(x)$, such that $P(A) = J$, if and only if G is regular and connected. In this case we have $P(x)\dfrac{n(x-\lambda^{(2)})..........(x-\lambda^{(m)}}{(r-\lambda^{(2)})...................(r-\lambda^{(m)})}$, where n is the number of nodes, r is the index, and $\lambda^{(1)} = r, \lambda^{(2)} ....\lambda^{(m)}$ are all distinct eigen values of G.

Theorem: 2.4[1] A regular connected graph of G of degree r is strongly regular if and only if it has exactly three distinct eigen values $\lambda^{(1)} = r, \lambda^{(2)}, \lambda^{(3)}$. If G is strongly regular, then $e = r + \lambda^{(2)}\lambda^{(3)} + \lambda^{(2)} + \lambda^{(3)}$ and $f = r + \lambda^{(2)}\lambda^{(3)}$.

III. MAIN RESULTS

We defined almost pure golden graph and Golden graph as follows,

Definition: 3.1 A graph G is said to be almost Pure golden graph, if all the eigen values of G are $\lambda, \lambda_1 = \dfrac{-1+\sqrt{5}}{2}$, $\lambda_2 = \dfrac{-1-\sqrt{5}}{2}$, where $\lambda \neq \dfrac{-1+\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}$.

Definition: 3.2 A graph G is said to be golden graph, if at least one of the eigen values of G are Golden ratios.

Lemma: 3.3 A connected graph G is almost pure golden graph if and only if $G = C_5$, a cycle on five nodes.

Proof: Necessity condition**:** Let G be a almost pure golden graph of order n ,size m and only Eigen values of G are

$\lambda, \lambda_1 = \dfrac{-1+\sqrt{5}}{2}$, $\lambda_2 = \dfrac{-1-\sqrt{5}}{2}$, where $\lambda \neq \dfrac{-1+\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}$,with multiplicities as r, l, k respectively.

Thus, the characteristic polynomial of G can be expressed as

$$\phi(G,x) = (x-\lambda)^r (x-\lambda_1)^l (x-\lambda_2)^k \tag{1}$$

By expanding and collecting like powers of x, we have

$$\phi(G,x) = x^{r+l+k} - \left[{}^kC_1\lambda_2 + {}^lC_1\lambda_1 + {}^rC_1\lambda\right]x^{r+l+k-1} + \ldots..$$

By the properties of characteristic polynomial of graph [Theorm: 2.1], we have,

$$\left[{}^kC_1\lambda_2 + {}^lC_1\lambda_1 + {}^rC_1\lambda\right] = 0$$

$$\Rightarrow \quad k\lambda_2 + l\lambda_1 + r\lambda = 0$$

$$\Rightarrow \quad k\left[\frac{-1-\sqrt{5}}{2}\right] + l\left[\frac{-1+\sqrt{5}}{2}\right] + r\lambda = 0$$

$$\Rightarrow \quad k\left[-1-\sqrt{5}\right] + l\left[-1+\sqrt{5}\right] + 2r\lambda = 0$$

$$\Rightarrow \quad -k - k\sqrt{5} - l + l\sqrt{5} + 2r\lambda = 0$$

$\Rightarrow \quad \sqrt{5}[l-k] - [l+k] + 2r\lambda = 0$,holds when $l = k \Rightarrow l = k$.

$$\Rightarrow \quad -2l + 2r\lambda = 0$$

$$\Rightarrow \quad -2l = -2r\lambda$$

$\Rightarrow \quad l = r\lambda$, Since l & r are integers.

$\therefore \quad \lambda$ is also an integer.

Since λ is an integer, the graph G should be regular of degree λ .

We claim that $\lambda = 2$.

Now G have exactly three distinct Eigen values $\lambda, \lambda_1, \lambda_2$. Then obviously $\lambda > 0$ and G is not complete graph and accordingly $aA^2 + bA + cI = J, a \neq 0$ holds. (Theorem 2.3&2.4) (2)

Since λ_1, λ_2 are roots of the equation $a\lambda^2 + b\lambda + c = 0$, (here $a = 1, b = 1, c = -1$) holds .

A comparison of the diagonal elements of the left and right of equation (2) yields the equation

$$a\lambda + c = 1, \Rightarrow 1(\lambda) - 1 = 1, \Rightarrow \lambda = 2.$$

Therefore G is regular graph of degree 2 and has three distant Eigen values.

$\therefore$ G is strongly regular graph.

The only strongly regular graph with degree 2 and three distinct Eigen values in C_n is C_5 .

Hence necessary condition.

Converse is obvious.

Theorem: 3.4 An graph G is almost pure golden if and only if every component of G is C_5 .

Proof: Suppose G is a pure golden graph of order n and size m . G consists of components say $G_1, G_2, \ldots\ldots, G_k$.

Then the characteristic polynomial of G is given by [theorm2.2]

$$\phi(G,\lambda)=\prod_{i=1}^{k}\phi(G_i,\lambda)$$

Now we claim that, each component of G is C_5 .

By the Lemma [3.3], G_i is C_5

Hence the claim is established.

Converse is obvious.

Theorem: 3.5 C_n is golden graph if and only if $n=5k$.

Proof: Let C_n be a golden graph .We know that spectrum of C_n is $2\cos\left(\frac{2\pi k}{n}\right)$, where $k=1,2.......n$.

Therefore $2\cos\left(\frac{2\pi k}{n}\right)=\frac{-1+\sqrt{5}}{2}$, *for* some k

$$\Rightarrow \quad \cos\left(\frac{2\pi k}{n}\right)=\frac{-1+\sqrt{5}}{4}$$

We know that, $\cos 72^\circ=\frac{-1+\sqrt{5}}{4}$

$\therefore\quad \frac{2\pi k}{n}=2l\pi\pm\alpha$, where $\alpha\in I$

$$\Rightarrow \quad \frac{2\pi k}{n}=2l\pi\pm 72^\circ$$

$$\Rightarrow \quad k=l.n\pm\frac{n}{5}$$

$$\Rightarrow \quad 5k=n(5l\pm 1)$$

$\therefore\quad l=0$, because l being the number of full rotation and π representing only half of the rotation must be zero, the above equation becomes

$\Rightarrow n=5k$

$\therefore$ If C_n is golden graph, then $n=5k$.

For the converse, it is enough to claim that $\phi(C_{5k})$ is divisible by x^2+x-1 .

Now, $\phi(C_{5k})=x.\phi(P_{5k-1})-2\phi(P_{5k-2})-2$
$\qquad\qquad\qquad = g(x)-2(\phi(P_{5k-2})+1)$

Where (x^2+x-1) divides $g(x)$ [7]

To complete the proof, it is sufficient to prove that (x^2+x-1) divides $\phi(P_{5k-2})+1$, which we shall claim by induction.

For $k=1$, $\phi(P_3)+1 = x(x^2-2)+1$.

Now $\quad x^2+x-1)\, x^3-2x+1\,(\, x-1$

$$\begin{array}{l} x^3+x^2-x \\ \underline{-\quad -\quad +} \\ -x^2-x+1 \\ \underline{-x^2-x+1} \end{array}$$

$$\therefore \qquad x^2+x-1 \big/ \phi(P_3)+1$$

Assume that, the result is true for $k-1$, i.e. $x^2+x-1 \big/ \varphi(P_{5k-7})+1$

Now,

$$\begin{aligned} \varphi(P_{5k-7})+1 &= \varphi(P_4)\varphi(P_{5k-6})-\varphi(P_3).\varphi(P_{5k-7})+1 \\ &= \varphi(p_4).\varphi(P_{5k-6})-\varphi(P_3).\varphi(P_{5k-1})-\varphi(P_3)+\varphi(P_3)+1 \\ &= \varphi(\mathrm{P}_4).\varphi(P_{5k-6})-\varphi(P_3)\left[\varphi(P_{5k-1})+1\right]+\left[\varphi(P_3)+1\right] \end{aligned}$$

Since x^2+x-1 divides the first term, and the same thing is true for second term as $\phi(P_{5k-1})+1$ is divisible by x^2+x-1 by theorem in [7] and the last is divisible by inductive hypothesis $(k=1)$.

Hence the claim is established.

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