

CAYLEY THEOREM FOR CENTRE OF C-ALGEBRA

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ABSTRACT

In this we study the important properties of ternary operation Γ on C-Algebra, should be viewed as a conditional “if x , then p , else q ” and established Cayley’s theorem on centre of C-Algebra.

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INTRODUCTION

In [1], Guzman and Squier introduced the concept of C-algebras as the variety generated by the three element algebra $C = \{T, F, U\}$ which the algebraic form of the three valued conditional logic. They proved that C and the two element Boolean algebra $B = \{T, F\}$ are the only sub-directly irreducible C-algebras and that the variety of C-algebras is minimal cover of the variety of Boolean algebras. They also define ternary operation $\Gamma_a(p, q) = (a \wedge p) \vee (a' \wedge q)$ on C-Algebra, should be viewed as a conditional “if x , then p , else q ” and derived important properties of Γ . In [2], Swamy, Rao and Ravi kumar introduced the concept of the centre $B(A)$ of C-algebra A with T as the set of all elements x in A with $x \vee x' = T$ and proved that the centre is a Boolean algebra under the induced operations, which is isomorphic to the Boolean centre of the algebra A .

1. C-ALGEBRA

In this section we recall the definition of C- algebra and some results from [1], [2] and [3]. Let us start with the definition of a C-Algebra.

Definition 1.1: [1] By a C-algebra we mean an algebra of type $(2, 2, 1)$ with binary operations \wedge, \vee and unary operation $'$ satisfying the following identities.

- | | |
|--|---|
| a) $x'' = x$ | b) $(x \wedge y)' = x' \vee y'$ |
| c) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$ | d) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ |
| d) $(x \vee y) \wedge z = (x \wedge z) \vee (x' \wedge y \wedge z)$ | f) $x \vee (x \wedge y) = x$ |
| g) $(x \wedge y) \vee (y \wedge x) = (y \wedge x) \vee (x \wedge y)$ | |

Example 1.2: The three element algebra $C = \{T, F, U\}$ with the operations given by the following tables is a C-algebra

\wedge	T	F	U	\vee	T	F	U	x	x^{\sim}
T	T	F	U	T	T	T	T	T	F
F	F	F	F	F	T	F	U	F	T
U	U	U	U	U	U	U	U	U	U

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Note 1.3: [1] The identities 1.1(a), 1.1(b) imply that the variety of C-algebras satisfies all the dual statements of 1.1(c) to 1.1(f). \wedge and \vee are not commutative in C. The ordinary right distributive law of \wedge over \vee fails in C. Every Boolean algebra is a C-algebra.

Lemma 1.4: The following properties of a C-algebra can be verified directly [1], [2], [3], and [4]

- (i) $x \wedge x = x$
- (ii) $x \wedge y = x \wedge (x' \vee y) = (x' \vee y) \wedge x$
- (iii) $x \vee (x' \wedge x) = (x' \wedge x) \vee x = x$
- (iv) $(x \vee x') \wedge y = (x \wedge y) \vee (x' \wedge y)$
- (v) $x \vee x' = x' \vee x$
- (vi) $x \vee y \vee x = x \vee y$
- (vii) $x \wedge x' \wedge y = x \wedge x'$
- (viii) $x \vee y = F$ if and only if $x = y = F$
- (ix) If $x \vee y = T$ then $x \vee x' = T$
- (x) $x \vee T = x \vee x'$
- (xi) $T \vee x = T$ and $F \wedge x = F$

Definition 1.5: [2] Let A be a C-algebra with T (T is the identity element for \wedge in A). Then the Boolean centre of A is defined as the set $B(A) = \{a \in A / a \vee a' = T\}$. $B(A)$ is known to be a Boolean algebra under the operations induced by those on A.

2. THE TERNARY OPERATION Γ

Definition 2.1: [1] Let A be a C-algebra. If $x, p, q \in A$, define $\Gamma_x(p, q) = (x \wedge p) \vee (x' \wedge q)$ ($\Gamma_x(p, q)$ should be viewed as a conditional “if x , then p , else q ”).

Lemma 2.2: [1] Every C-algebra satisfies the laws:

- (a) $\Gamma_x(p, q)' = \Gamma_x(p', q')$
- (b) $\Gamma_x(p, q) \wedge r = \Gamma_x(p \wedge r, q \wedge r)$
- (c) $\Gamma_x(p, q) \vee r = \Gamma_x(p \vee r, q \vee r)$
- (d) $\Gamma_x(\Gamma_y(p, q), \Gamma_y(r, s)) = \Gamma_y(\Gamma_x(p, r), \Gamma_x(q, s))$

Lemma 2.3: [1] Every C-algebra with the indicated constants satisfies the following laws

- (i) $\Gamma_U(p, q) = U$ (ii) $\Gamma_x(U, U) = U$ (iii) $\Gamma_T(p, q) = p$ (iv) $\Gamma_F(p, q) = q$ (v) $\Gamma_x(T, F) = x$

Lemma 2.4: Every C-algebra with the indicated constants satisfies the following laws.

- (a) $\Gamma_{x \vee y}(p, q) = \Gamma_x(p, \Gamma_y(p, q))$ (b) $\Gamma_{x \wedge y}(p, q) = \Gamma_x(\Gamma_y((p, q), q))$ (c) $\Gamma_p(p, p) = p$

Proof:

$$\begin{aligned}
 \text{(a) } \Gamma_x(p, \Gamma_y(p, q)) &= (x \wedge p) \vee (x' \wedge \Gamma_y(p, q)) \\
 &= (x \wedge p) \vee [x' \wedge \{(y \wedge p) \vee (y' \wedge q)\}] \\
 &= (x \wedge p) \vee [\{x' \wedge (y \wedge p)\} \vee \{x' \wedge (y' \wedge q)\}] \\
 &= [(x \wedge p) \vee \{x' \wedge (y \wedge p)\}] \vee \{x' \wedge (y' \wedge q)\} \\
 &= [(x \wedge p) \vee \{x' \wedge y \wedge p\}] \vee \{x' \wedge (y' \wedge q)\} \\
 &= \{(x \vee y) \wedge p\} \vee \{(x \vee y)' \wedge q\} \\
 &= \Gamma_{x \vee y}(p, q)
 \end{aligned}$$

Therefore $\Gamma_{x \vee y}(p, q) = \Gamma_x(p, \Gamma_y(p, q))$

$$\begin{aligned}
 \text{(vi) } \Gamma_x(\Gamma_y((p, q), q)) &= (x \wedge \Gamma_y(p, q)) \vee (x' \wedge q) \\
 &= [x \wedge \{(y \wedge p) \vee (y' \wedge q)\}] \vee (x' \wedge q)
 \end{aligned}$$

$$\begin{aligned}
 &= [(x \wedge (y \wedge p)) \vee (x \wedge (y' \wedge q))] \vee (x' \wedge q) \\
 &= (x \wedge (y \wedge p)) \vee [(x \wedge (y' \wedge q)) \vee (x' \wedge q)] \\
 &= (x \wedge (y \wedge p)) \vee [(x \wedge y') \wedge q] \vee (x' \wedge q) \\
 &= (x \wedge (y \wedge p)) \vee \{(x \wedge y') \vee x'\} \wedge q \\
 &= \{(x \wedge y) \wedge p\} \vee \{(x' \vee y') \wedge q\} \\
 &= \{(x \wedge y) \wedge p\} \vee \{(x \wedge y) \sim \wedge q\} \\
 &= \Gamma_{x \wedge y}(p, q)
 \end{aligned}$$

Therefore $\Gamma_{x \wedge y}(p, q) = \Gamma_x(\Gamma_y((p, q), q))$

$$(vii) \Gamma_p(p, p) = (p \wedge p) \vee (p' \wedge p) = p \vee (p' \wedge p) = p$$

Therefore $\Gamma_p(p, p) = p$.

Observe that $\Gamma_a(p, p) = (a \wedge p) \vee (a' \wedge p) = (a \vee a') \wedge p$, if $a \neq U$, then $a \vee a' = T$ and $\Gamma_a(p, p) = p$ otherwise the value is U . Hence we have the following lemma.

Lemma 2.5: Let A be a C-algebra, $a \in B(A)$, then $\Gamma_a(p, p) = p, \forall p \in A$

Proof: If $a \in B(A)$, then $\Gamma_a(p, p) = (a \wedge p) \vee (a' \wedge p)$
 $= (a \vee a') \wedge p$ (since $a \in B(A) \Rightarrow a \vee a' = T$)
 $= T \wedge p = p$

Lemma 2.6: Let A be a C-algebra, then $\Gamma_a(\Gamma_a(p, q), \Gamma_a(r, s)) = \Gamma_a(p, s)$

Proof: The result is clear, when $a = U$.

Suppose $a \neq U$ i.e., $a \vee a' = T$

$$\begin{aligned}
 \Gamma_a(\Gamma_a(p, q), \Gamma_a(r, s)) &= \Gamma_a((a \wedge p) \vee (a' \wedge q), (a \wedge r) \vee (a' \wedge s)) \\
 &= [a \wedge \{(a \wedge p) \vee (a' \wedge q)\}] \vee [a' \wedge \{(a \wedge r) \vee (a' \wedge s)\}] \\
 &= \{a \wedge (a \wedge p)\} \vee \{a \wedge (a' \wedge q)\} \vee \{a' \wedge (a \wedge r)\} \vee \{a' \wedge (a' \wedge s)\} \\
 &= \{(a \wedge p)\} \vee \{a \wedge (a' \wedge q)\} \vee \{a' \wedge (a \wedge r)\} \vee \{a' \wedge s\} \\
 &= \{(a \wedge p)\} \vee \{a \wedge a' \wedge q\} \vee \{a' \wedge a \wedge r\} \vee \{a' \wedge s\} \\
 &= \{(a \wedge p)\} \vee \{a \wedge a'\} \vee \{a' \wedge a\} \vee \{a' \wedge s\} \\
 &= \{(a \wedge p)\} \vee 0 \vee 0 \vee \{a' \wedge s\} \\
 &= (a \wedge p) \vee (a' \wedge s) \\
 &= \Gamma_a(p, s)
 \end{aligned}$$

Therefore $\Gamma_a(\Gamma_a(p, q), \Gamma_a(r, s)) = \Gamma_a(p, s)$

The property on ternary operation $\Gamma_a(\Gamma_a(p, q), \Gamma_a(r, s)) = \Gamma_a(p, s)$ is known as diagonal property hold on C-algebra.

3. CAYLEY'S THEOREM:

The well known Cayley theorem for groups may be summarized as follows: Any group is isomorphic to group of transformations on some set. Stephen L. Bloom, Zoltan Esik and E.G. Manes [5], established Cayley theorem for Boolean algebras. We establish Cayley's theorem for C-algebra in two parts. In Part-I, we construct a C-algebra $(A, \wedge, \vee, ')$, where A is a subset of $\text{Ter}(X)$ (ternary functions) on X , called Guard C-algebra on X . In Part-II, we prove that every centre of a C-algebra is isomorphic with guard C-algebra.

Cayley theorem for Groups (part I): Let X be a set and $P(X)$ be the collection of permutations of X . If G is a subset of $P(X)$ satisfying

- i) G contains identity function I .
- ii) $f, g \in G \Rightarrow f \circ g \in G$.
- iii) $f \in G \Rightarrow f^{-1} \in G$, then G is a group. Such groups are called transformation groups on X .

Cayley theorem for Groups (part II): If G is a group, there is a set X such that G is isomorphic to a transformation group on X .

Theorem 3.1: Cayley theorem for Boolean algebras (part I) 3.3: [5]

Let B be a subset of $\text{Bin}(X)$ with the following properties. B contains True and is closed under the operations $\wedge, \vee, (-)'$. Then $(B, \wedge, \vee, (-)', \text{True}, \text{False})$ is a Boolean algebra providing each function in B is idempotent and diagonal and any two such functions commute. Then B is a Boolean algebra. Such Boolean algebras on X are called "Guard Algebras" on X .

Theorem 3.2: Cayley theorem for Boolean algebras (part II): [5]

If $(B, \wedge, \vee, (-)', 1, 0)$ is any Boolean algebra, then there is a set X such that B is isomorphic to a Guard algebra on X .

In a similar way, we prove Cayley theorem for C-algebra.

Let X be a set and f be a ternary function on X , satisfying the following properties

- (i) $f(x, x) = x$
- (ii) $f(f(x, y), f(u, v)) = f(x, v)$ (Diagonal property)
- (iii) $f(g(x, y), g(u, v)) = g(f(x, u), f(y, v))$, g is a ternary function on X .

We define the operations $\wedge, \vee, '$ on the set $\text{Ter}(X)$ of all ternary functions on X as

$$(f \wedge g)(x, y) = f(g(x, y), y).$$

$$(f \vee g)(x, y) = f(x, g(x, y)).$$

$$f'(x, y) = f(y, x) \text{ and } \text{Ter}(X) \text{ having projections}$$

$$\pi_1(x, y) = x, \pi_2(x, y) = y$$

Theorem 3.3: Cayley's theorem for C-algebras (Part-I):

Let A be a subset of $\text{Ter}(X)$ with the properties, A contains projections π_1, π_2 and closed under $\wedge, \vee, '$ then $(A, \wedge, \vee, ')$ is a C-algebra provided each function in A satisfying the properties (i),(ii) and (iii) and operations $\wedge, \vee, '$ are defined above.

Proof: Let $f, g \in A$ and $x, y \in X$

$$(a) f(x, y)'' = f'(y, x) = f(x, y)$$

Therefore $f'' = f$

$$\begin{aligned} (b) (f \wedge g)'(x, y) &= (f \wedge g)(y, x) \\ &= f(g(y, x), x) \\ &= f'(x, g(y, x)) \\ &= f'(x, g'(x, y)) \\ &= (f' \vee g')(x, y) \end{aligned}$$

Therefore $(f \wedge g)' = g' \vee f'$

$$\begin{aligned}
 \text{(c) } \{f \wedge (g \wedge h)\}(x, y) &= f\{(g \wedge h)(x, y), y\} \\
 &= f\{g(h(x, y), y), y\} \\
 &= (f \wedge g)(h(x, y), y) \\
 &= \{(f \wedge g) \wedge h\}(x, y)
 \end{aligned}$$

Therefore $f \wedge (g \wedge h) = (f \wedge g) \wedge h$

$$\text{(d) To prove } (f \wedge g) \vee (f \wedge h) = f \wedge (g \vee h)$$

Consider

$$\begin{aligned}
 [(f \wedge g) \vee (f \wedge h)](x, y) &= (f \wedge g)[x, (f \wedge h)(x, y)] \\
 &= (f \wedge g)[x, f(h(x, y), y)] \\
 &= f[g\{x, f(h(x, y), y)\}, f(h(x, y), y)] \\
 &= f[g\{x, f(h(x, y), y)\}, g(f(h(x, y), y), f(h(x, y), y))] \\
 &= g[f\{x, f(h(x, y), y)\}, f(f(h(x, y), y), f(h(x, y), y))] \\
 &= g[f\{x, f(h(x, y), y)\}, f(h(x, y), y)] \\
 &= f[g\{x, h(x, y)\}, g(y, y)] \\
 &= f[g\{x, h(x, y)\}, y] \\
 &= f[(g \vee h)(x, y), y] \\
 &= \{f \wedge (g \vee h)\}(x, y)
 \end{aligned}$$

Therefore $(f \wedge g) \vee (f \wedge h) = f \wedge (g \vee h)$

$$\text{(e) } (f \vee g) \wedge h = (f \wedge h) \vee (f' \wedge g \wedge h)$$

Consider

$$\begin{aligned}
 [(f \wedge h) \vee (f' \wedge g \wedge h)](x, y) &= (f \wedge h)\{x, (f' \wedge g \wedge h)(x, y)\} \\
 &= (f \wedge h)[x, f'\{(g \wedge h)(x, y), y\}] \\
 &= (f \wedge h)[x, f\{y, (g \wedge h)(x, y)\}] \\
 &= (f \wedge h)[x, f\{y, g(h(x, y), y)\}] \\
 &= f[h\{x, f(y, g(h(x, y), y)), f(y, g(h(x, y), y))\}] \\
 &= f[h\{f(x, x), f(y, g(h(x, y), y)), f(y, g(h(x, y), y))\}] \\
 &= f[f\{h(x, y), h(x, g(h(x, y), y)), f(y, g(h(x, y), y))\}] \\
 &= f[h(x, y), g(h(x, y), y)] \\
 &= (f \vee g)(h(x, y), y) \\
 &= ((f \vee g) \wedge h)(x, y)
 \end{aligned}$$

Hence $(f \vee g) \wedge h = (f \wedge h) \vee (f' \wedge g \wedge h)$

$$\text{(f) } (f \wedge g) \vee (g \wedge f) = (g \wedge f) \vee (f \wedge g)$$

$$\begin{aligned}
 (f \wedge g) \vee (g \wedge f)(x, y) &= (f \wedge g)[x, (g \wedge f)(x, y)] \\
 &= (f \wedge g)(x, g[f(x, y), y]) \\
 &= f\{g(x, g(f(x, y), y)), g(f(x, y), y)\} \\
 &= f\{g[g(x, x), g(f(x, y), y)], g(f(x, y), y)\} \\
 &= f\{g(x, y), g(f(x, y), y)\} \\
 &= f\{g(x, y), g(f(x, y), f(y, y))\} \\
 &= f\{g(x, y), f(g(x, y), y)\}
 \end{aligned}$$

$$\begin{aligned}
 (g \wedge f) \vee (f \wedge g)(x, y) &= (g \wedge f)\{x, (f \wedge g)(x, y)\} \\
 &= (g \wedge f)\{x, (f[g(x, y), y])\} \\
 &= g\{f[x, f(g(x, y), y)], f(g(x, y), y)\} \\
 &= f\{g[x, g(x, y)], g[f(g(x, y), y), y]\} \\
 &= f\{g[g(x, x), g(x, y)], g[f(g(x, y), y), f(y, y)]\} \\
 &= f\{g[x, g(x, y)]f[g(g(x, y), y)g(y, y)]\} \\
 &= f\{g(x, g(x, y))g[f(g(x, y), y), f(y, y)]\} \\
 &= f\{g(x, y), f[g(g(x, y), y), g(y, y)]\} \\
 &= f(g(x, y), f(g(x, y), y))
 \end{aligned}$$

Therefore $(f \wedge g) \vee (g \wedge f) = (g \wedge f) \vee (f \wedge g)$

Hence $(A, \wedge, \vee, ')$ is a C-algebra, such C-algebras on A are called “Guard C-algebras” on X . Observed that on C-Algebra, $\Gamma_a(x, x) \neq x$ but if $a \in B(A)$, then $\Gamma_a(x, x) = x$ (lemma 2.5). So we are confining to Cayley’s theorem for centre of C-algebra.

Theorem3 .4: Cayley’s theorem for C-algebras (Part-II)

If $(A, \wedge, \vee, \sim 0, 1, 2)$ is a C-algebra, there is a set X such that $B(A)$ is isomorphic to guard C-algebra on X .

Proof: Define Γ_a on $B(A)$ such that $\Gamma_a(x, y) = (a \wedge x) \vee (a' \wedge y)$.

Let $\text{Ter}(B(A)) = \{\Gamma_a / a \in B(A)\}$, define $\wedge, \vee, '$ as

$$\begin{aligned}
 (\Gamma_a \wedge \Gamma_b)(x, y) &= (\Gamma_{a \wedge b})(x, y) = \Gamma_a(\Gamma_b(x, y), y) \\
 (\Gamma_a \vee \Gamma_b)(x, y) &= (\Gamma_{a \vee b})(x, y) = \Gamma_a(x, \Gamma_b(x, y)) \\
 (\Gamma_a) '(x, y) &= \Gamma_{a'}(x, y) = \Gamma_a(y, x)
 \end{aligned}$$

Now we prove that $\text{Ter}(B(A))$ is C-algebra.

$$(a) (\Gamma_a) ''(x, y) = (\Gamma_{a'}(x, y)) ' = (\Gamma_a(y, x)) ' = \Gamma_{a'}(y, x) = \Gamma_a(x, y)$$

Therefore $(\Gamma_a) '' = \Gamma_a$

Therefore $\Gamma_a \wedge \Gamma_a = \Gamma_a$

$$\begin{aligned}
 (b) (\Gamma_a \wedge \Gamma_b) '(x, y) &= (\Gamma_a \wedge \Gamma_b)(y, x) \\
 &= \Gamma_a(\Gamma_b(y, x), x) \\
 &= \Gamma_a((x, \Gamma_b(y, x))) \\
 &= \Gamma_{a'}((x, \Gamma_b(x, y))) \\
 &= (\Gamma_{a'} \vee \Gamma_b)(x, y)
 \end{aligned}$$

Therefore $(\Gamma_a \wedge \Gamma_b) ' = (\Gamma_{a'} \vee \Gamma_b)$

$$\begin{aligned}
 (c) \{(\Gamma_a \wedge \Gamma_b) \wedge \Gamma_c\}(x, y) &= (\Gamma_a \wedge \Gamma_b)\{\Gamma_c(x, y), y\} \\
 &= \Gamma_a[\Gamma_b\{\Gamma_c(x, y), y\}, y] \\
 &= \Gamma_a\{(\Gamma_b \wedge \Gamma_c)(x, y), y\} \\
 &= \{\Gamma_a \wedge (\Gamma_b \wedge \Gamma_c)\}(x, y)
 \end{aligned}$$

Therefore $(\Gamma_a \wedge \Gamma_b) \wedge \Gamma_c = \Gamma_a \wedge (\Gamma_b \wedge \Gamma_c)$

(d) To prove $\{\Gamma_a \wedge (\Gamma_b \vee \Gamma_c)\} = \{(\Gamma_a \wedge \Gamma_b) \vee (\Gamma_a \wedge \Gamma_c)\}$

Consider

$$\begin{aligned}
 & \{(\Gamma_a \wedge \Gamma_b) \vee (\Gamma_a \wedge \Gamma_c)\}(x, y) \\
 &= (\Gamma_a \wedge \Gamma_b)\{x, (\Gamma_a \wedge \Gamma_c)(x, y)\} \\
 &= (\Gamma_a \wedge \Gamma_b)[x, \Gamma_a\{\Gamma_c(x, y), y\}] \\
 &= \Gamma_a[\Gamma_b[x, \{\Gamma_a(\Gamma_c(x, y), y)\}], \Gamma_a\{\Gamma_c(x, y), y\}] \\
 &= [a \wedge \{\Gamma_b[x, \{\Gamma_a(\Gamma_c(x, y), y)\}]\} \vee [a' \wedge \{\Gamma_a\{\Gamma_c(x, y), y\}\}]] \\
 &= (a \wedge [\Gamma_b[x, a \wedge \{\Gamma_c(x, y)\} \vee (a' \wedge y)]) \vee [a' \wedge \{(a \wedge \Gamma_c(x, y)) \vee (a' \wedge y)\}] \\
 &= (a \wedge [(b \wedge x) \vee b' \wedge \{a \wedge \Gamma_c(x, y) \vee (a' \wedge y)\}]) \vee [a' \wedge a \wedge \Gamma_c(x, y) \vee (a' \wedge a' \wedge y)] \\
 &= (a \wedge [(b \wedge x) \vee (b' \wedge a \wedge \Gamma_c(x, y)) \vee (b' \wedge (a' \wedge y))]) \vee [a \wedge a' \vee (a' \wedge y)] \\
 &= (a \wedge [(b \wedge x) \vee (b' \wedge a \wedge \Gamma_c(x, y)) \vee (b' \wedge a' \wedge y)]) \vee (a' \wedge y) \\
 &= (a \wedge (b \wedge x) \vee (b' \wedge a \wedge \Gamma_c(x, y)) \vee F) \vee (a' \wedge y) \\
 &= a \wedge (b \wedge x) \vee (b' \wedge a \wedge \Gamma_c(x, y)) \vee (a' \wedge y) \\
 &= a \wedge \{(b \wedge x) \vee (b' \wedge \Gamma_c(x, y))\} \vee (a' \wedge y) \\
 &= a \wedge \{(\Gamma_b(x, \Gamma_c(x, y))) \vee (a' \wedge y)\} \\
 &= \Gamma_a\{(\Gamma_b(x, \Gamma_c(x, y))), y\} \\
 &= \Gamma_a[(\Gamma_b \vee \Gamma_c)(x, y), y] \\
 &= \{\Gamma_a \wedge (\Gamma_b \vee \Gamma_c)\}(x, y)
 \end{aligned}$$

Therefore $\{\Gamma_a \wedge (\Gamma_b \vee \Gamma_c)\} = \{(\Gamma_a \wedge \Gamma_b) \vee (\Gamma_a \wedge \Gamma_c)\}$

(e) To prove $(\Gamma_a \vee \Gamma_b) \wedge \Gamma_c = (\Gamma_a \wedge \Gamma_c) \vee (\Gamma_a' \wedge \Gamma_b \wedge \Gamma_c)$

$$\begin{aligned}
 & (\Gamma_a \wedge \Gamma_c) \vee (\Gamma_a' \wedge \Gamma_b \wedge \Gamma_c)(x, y) \\
 &= \Gamma_a \wedge \Gamma_c \{x, \Gamma_a' \wedge \Gamma_b \wedge \Gamma_c(x, y)\} \\
 &= (\Gamma_a \wedge \Gamma_c) \{x, \Gamma_a'[(\Gamma_b \wedge \Gamma_c)(x, y), y]\} \\
 &= (\Gamma_a \wedge \Gamma_c) \{x, (\Gamma_a[y, (\Gamma_b \wedge \Gamma_c)(x, y)])\} \\
 &= (\Gamma_a \wedge \Gamma_c) \{x, (\Gamma_a[y, \Gamma_b(\Gamma_c(x, y), y)])\} \\
 &= (\Gamma_a\{\Gamma_c[x, \Gamma_a[y, \Gamma_b(\Gamma_c(x, y), y)]], \Gamma_a[y, \Gamma_b(\Gamma_c(x, y), y)]\}) \\
 &= (a \wedge \{\Gamma_c[x, \Gamma_a[y, \Gamma_b(\Gamma_c(x, y), y)]\}) \vee \{a' \wedge \Gamma_a[y, \Gamma_b(\Gamma_c(x, y), y)]\} \\
 &= (a \wedge \{(c \wedge x) \vee [c' \wedge (a \wedge y) \vee (a' \wedge (\Gamma_b(\Gamma_c(x, y), y))])\}) \vee \{a' \wedge [(a \wedge y) \vee (a' \wedge [(b \wedge \Gamma_c(x, y)) \vee (b' \wedge y)])]\} \\
 &= (a \wedge c \wedge x) \vee (a \wedge c' \wedge a \wedge y) \vee (a \wedge c' \wedge a' \wedge \Gamma_b(\Gamma_c(x, y), y)) \vee (a' \wedge a \wedge y) \vee (a' \wedge a' \wedge (b \wedge \Gamma_c(x, y)) \vee (a' \wedge a' \wedge b' \wedge y)) \\
 &= (a \wedge c \wedge x) \vee (a \wedge c' \wedge y) \vee (a' \wedge (b \wedge (\Gamma_c(x, y)) \vee (a' \wedge b' \wedge y)) \\
 &= \{a \wedge [(c \wedge x) \vee (c' \wedge y)]\} \vee \{a' \wedge (b \wedge \Gamma_c(x, y)) \vee (b' \wedge y)\} \\
 &= [a \wedge (\Gamma_c(x, y))] \vee [a' \wedge (\Gamma_b(\Gamma_c(x, y), y))] \\
 &= \Gamma_a(\Gamma_c(x, y), \Gamma_b(\Gamma_c(x, y), y)) \\
 &= (\Gamma_a \vee \Gamma_b)(\Gamma_c(x, y), y) \\
 &= \{(\Gamma_a \vee \Gamma_b) \wedge \Gamma_c\}(x, y)
 \end{aligned}$$

Therefore $(\Gamma_a \vee \Gamma_b) \wedge \Gamma_c = (\Gamma_a \wedge \Gamma_c) \vee (\Gamma_a' \wedge \Gamma_b \wedge \Gamma_c)$

f) To prove $(\Gamma_a \wedge \Gamma_b) \vee (\Gamma_b \wedge \Gamma_a) = (\Gamma_b \wedge \Gamma_a) \vee (\Gamma_a \wedge \Gamma_b)$

Consider

$$\begin{aligned}
 [\Gamma_a \wedge \Gamma_b] \vee (\Gamma_b \wedge \Gamma_a)(x, y) &= (\Gamma_a \wedge \Gamma_b)[x, (\Gamma_b \wedge \Gamma_a)(x, y)] \\
 &= (\Gamma_a \wedge \Gamma_b)[x, \Gamma_b(\Gamma_a(x, y), y)] \\
 &= \Gamma_a\{\Gamma_b(x, \Gamma_b(\Gamma_a(x, y), y)), \Gamma_b(\Gamma_a(x, y), y)\} \\
 &= \Gamma_a\{\Gamma_b[\Gamma_b(x, x), \Gamma_b(\Gamma_a(x, y), y)], \Gamma_b(\Gamma_a(x, y), y)\} \\
 &= \Gamma_a\{\Gamma_b(x, y), \Gamma_b(\Gamma_a(x, y), y)\} \\
 &= a \wedge \{[(b \wedge x) \vee (b' \wedge y)] \vee [a' \wedge ((b \wedge \Gamma_a(x, y) \vee (b' \wedge y)))]\} \\
 &= (a \wedge b \wedge x) \vee (a \wedge b' \wedge y) \vee [a' \wedge b \wedge (a \wedge x)] \vee [a' \wedge b \wedge a' \wedge y] \vee [a' \wedge b' \wedge y] \\
 &= (a \wedge b \wedge x) \vee (a \wedge b' \wedge y) \vee (a' \wedge b \wedge y)
 \end{aligned}$$

Now

$$\begin{aligned}
 (\Gamma_b \wedge \Gamma_a) \vee (\Gamma_a \wedge \Gamma_b)(x, y) &= (\Gamma_b \wedge \Gamma_a)\{x, (\Gamma_a \wedge \Gamma_b)(x, y)\} \\
 &= (\Gamma_b \wedge \Gamma_a)\{x, (\Gamma_a(\Gamma_b(x, y), y))\} \\
 &= \Gamma_b\{\Gamma_a[x, \Gamma_a(\Gamma_b(x, y), y)], \Gamma_a(\Gamma_b(x, y), y)\} \\
 &= \Gamma_a\{\Gamma_b(x, \Gamma_b(x, y)), \Gamma_b(\Gamma_a(\Gamma_b(x, y), y), y)\} \\
 &= \Gamma_a\{\Gamma_b[\Gamma_b(x, x), \Gamma_b(x, y)], \Gamma_b[\Gamma_a(\Gamma_b(x, y), y), \Gamma_a(y, y)]\} \\
 &= \Gamma_a\{\Gamma_b(x, y), \Gamma_a[\Gamma_b(\Gamma_b(x, y), y), \Gamma_b(y, y)]\} \\
 &= \Gamma_a\{\Gamma_b(x, y), \Gamma_a[\Gamma_b((x, y), y)]\} \\
 &= \Gamma_a\{\Gamma_b(x, y), \Gamma_b[\Gamma_a((x, y), y)]\} \\
 &= (a \wedge b \wedge x) \vee (a \wedge b' \wedge y) \vee [(a' \wedge b \wedge a \wedge x)] \vee [a' \wedge b \wedge a' \wedge y] \vee (a' \wedge b' \wedge y) \\
 &= (a \wedge b \wedge x) \vee (a \wedge b' \wedge y) \vee (a' \wedge b' \wedge y)
 \end{aligned}$$

Therefore $(\Gamma_a \wedge \Gamma_b) \vee (\Gamma_b \wedge \Gamma_a) = (\Gamma_b \wedge \Gamma_a) \vee (\Gamma_a \wedge \Gamma_b)$

Hence $\text{Ter}(B(A))$ is a C-algebra, which is called guard C-algebra.

Define $f : B(A) \rightarrow \text{Ter}(B(A))$ by $f(a) = \Gamma_a$, $\forall a \in B(A)$

Now we prove that f is homomorphism

$$(i) f(a') = \Gamma_{a'}(x, y) = (\Gamma_a)'(x, y) = (f(a))'$$

Therefore $f(a') = (f(a))'$

$$(ii) f(a) \wedge f(b) = (\Gamma_a \wedge \Gamma_b)(x, y) = \Gamma_a(\Gamma_b(x, y), y) = \Gamma_{a \wedge b}(x, y) = f(a \wedge b)$$

Therefore $f(a \wedge b) = f(a) \wedge f(b)$

$$(iii) f(a) \vee f(b) = (\Gamma_a \vee \Gamma_b)(x, y) = \Gamma_a(y, \Gamma_b(x, y)) = \Gamma_{a \vee b}(x, y) = f(a \vee b)$$

Therefore $f(a \vee b) = f(a) \vee f(b)$.

Therefore f is homomorphism.

f is one-one:

Let $f(a) = f(b)$

$$\Rightarrow \Gamma_a(x, y) = \Gamma_b(x, y), \forall x, y$$

$$\Rightarrow \Gamma_a(1, 0) = \Gamma_b(1, 0)$$

$$\Rightarrow a = b$$

This shows that f is one-one:

Hence $B(A) \cong Ter(B(A))$

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