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NUMERICAL METHOD FOR SOLVING SINGULARLY PERTURBED INITIAL BOUNDARY VALUE PROBLEM FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

Adel Y. Elmekkawy* and Sayed A. Zaki

Department of Mathematics and Statistics, Faculty of Sciences, Taif University, Taif, Saudi Arabia.

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ABSTRACT

In this paper, we shall develop a new approach to an implicit method for solving convection–diffusion equation by small parameter with the time derivative term. The suggested method gives highly accurate result whatever the exact solution is too large. The stability condition and the advantages of the considered method compared with the classical methods as Crank-Nicolson method are discussed.

Keywords: *Pade approximation, Restrictive Pade` approximation, finite difference and parabolic partial differential equations.*

1. INTRODUCTION

Consider the singularly perturbed convection – diffusion equation

$$\delta \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = b\frac{\partial^2 u}{\partial x^2} \tag{1}$$

Where $\delta > 0$ small, b is is the thermal diffusivity and u (x, t) is given continuous function satisfies the initial and boundary conditions:

$$\left.\begin{array}{l} u(x,0) = u_0(x), \quad 0 \le x \le 1 \\ u(0,t) = g_0(t), \quad u(1,t) = g_1(t), \quad t \ge 0 \end{array}\right\} \tag{2}$$

In this paper we define an implicit method for solving the singularly perturbed convection – diffusion parabolic partial differential equation produces very high accuracy compared with the other classical method, i.e. the numerical solution produced by the considered method is almost identical to the exact solution. We use the restrictive Pade` approximation as done in [6],[7],[8],[9] and [10] to approximate the exponential function.

2. RESTRICTIVE PADE` APPROXIMATION (RPA)

The restrictive Pade` approximation can be written as done in [6] in the form

$$RPA\left[M+\alpha/N\right]_{f(x)}(x) = \frac{\sum_{i=0}^{M} a_i\, x^i + \sum_{i=1}^{\alpha} \varepsilon_i\, x^{M+i}}{1+\sum_{i=1}^{N} b_i\, x^i} \tag{3}$$

Where α is a positive integer dose not exceeding the degree of the denominator N, i.e. α =1(1) N, such that

$$f(x) - RPA[M+\alpha/N]_{f(x)}(x) = o(x^{M+N+1})\,. \tag{4}$$

Corresponding author: **Adel Y. Elmekkawy***
Department of Mathematics and Statistics, Faculty of Sciences, Taif University, Taif, Saudi Arabia.
E-mail: Adel_Elmekkawy@yahoo.com

Let $f(x)$ has a Maclaurin series $f(x)=\sum_{i=0}^{\infty} c_i x^i$, then from equations (3) and (4) we have

$$\left(\sum_{i=0}^{\infty} c_i x^i\right)\left(1+\sum_{i=1}^{N} b_i x^i\right)-\left(\sum_{i=0}^{M} a_i x^i\right)-\left(\sum_{i=1}^{\alpha} \varepsilon_i x^{i+M}\right)=o(x^{M+N+1}). \tag{5}$$

The vanishing of the first $(M+N+1)$ powers of x on the left hand side of (5) implies a system of $(M+N+1)$ equations.

$$\left.\begin{array}{ll} a_r = c_r + \sum_{i=1}^{r} c_{r-i}\, b_i, & r = 0(1)M, \\ & (b_i = 0 \;\; if \;\; i > M) \\ c_{M+N-s} + \sum_{i=1}^{N} c_{M+N-i-s}\, b_i = \varepsilon_{N-s}, & s = 0(1)N-1, \\ & (c_i = 0 \;\; if \;\; i < 0) \end{array}\right\}. \tag{6}$$

Hence we can determine the coefficient, a_i and b_i as a function of ε_i, $i=1(1)\alpha$, where the parameters ε_i are to be determined, such that

$$f(x_i) = RPA[M+\alpha/N]_{f(x)}(x_i),\;\; i=1(1)\alpha \;\; . \tag{7}$$

It means that the considered approximation is exact at $(\alpha+1)$ points.

Consider the function $f(x)=\left(\dfrac{1+0.5x+0.25x^2}{1+5x}\right)^{0.5}$.

It's Pade` approximation and restrictive Pade` approximation takes the forms:

$$PA\;[2/1]_{f(x)}(x)=\frac{1+1.9311x-0.563724x^2}{1+4.1811x}.$$

$$RPA\,[2/1]_{f(x)}(x)=\frac{1+1.73134\;x-0.114257x^2}{1+3.98134\;x}$$

$$where\;\alpha=1\;and\;x_\alpha=0.6$$

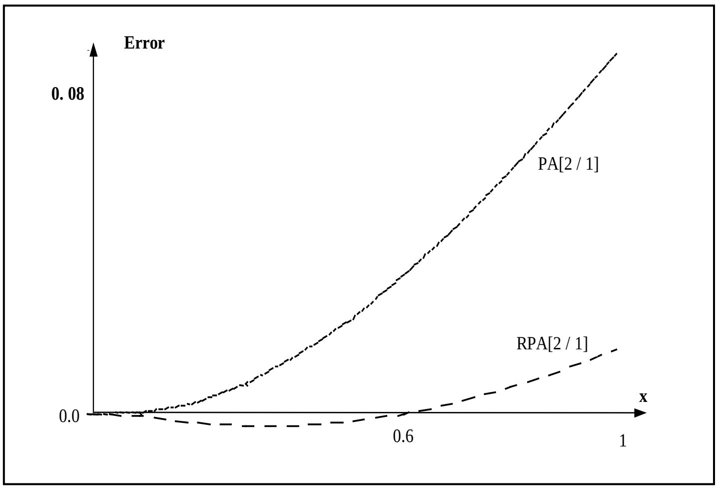

Fig. 1: Comparison of the errors between PA [2 / 1] and RPA [2 / 1]

3. RESTRICTIVE PADE` APPROXIMATION (RPA) FOR SOLVING SINGULARLY PERTURBED CONVECTION – DIFFUSION EQUATION

Consider the singularly perturbed convection–diffusion equation (1). The exact solution of grid representation of equations (1) is:

$$u_{i,j+1} = \exp\left(k\frac{\partial}{\partial t}\right)u_{i,j} = \exp\left(k\left(\frac{b}{\delta}D_X^2 - \frac{a}{\delta}D_X\right)\right)u_{i,j} \ . \tag{8}$$

Where $D_x^2\,u_{i,j} \cong b(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$ *and* $D_x\,u_{i,j} \cong \frac{ch}{2}(u_{i+1,j} - u_{i-1,j})$

Then equation (8) can take the form

$$u_{i,j+1} = \exp(r(D_X^2 - r_1 D_X))u_{i,j} \tag{9}$$

Where $r = \frac{k}{\delta h^2}$ *and* $r_1 = \frac{ah}{2}$

Then the restrictive Pade` approximation [1/1] can take the form:

$$RPA[1/1]_{\exp(r(D_X^2 - r_1 D_X))}(r) = \left(1 + (\varepsilon_{i,j} + \frac{1}{2}(D_X^2 - r_1 D_X))\,r\right)^{-1}\left(1 + (\varepsilon_{i,j} - \frac{1}{2}(D_X^2 - r_1 D_X))\,r\right) \tag{10}$$

Then we can approximate equation (9) as:

$$u_{i,j+1} = \left(1 + (\varepsilon_{i,j} + \frac{1}{2}(D_X^2 - r_1 D_X))\,r\right)^{-1}\left(1 + (\varepsilon_{i,j} - \frac{1}{2}(D_X^2 - r_1 D_X))\,r\right)u_{i,j} \tag{11}$$

Which can take the equivalent scalar form:

$$\begin{aligned} -0.5(b + r_1)u_{i-1,j+1} + (1 + r(\varepsilon_{i,j} + b))u_{i,j+1} - 0.5r(b - r_1)u_{i+1,j+1} \\ = 0.5(b + r_1)u_{i-1,j} + (1 + r(\varepsilon_{i,j} - b))u_{i,j} + 0.5(b - r_1)u_{i+1,j} \end{aligned} \tag{12}$$

To determine the restrictive parameters $\varepsilon_{i,j}$ we must have the exact solution at the first level, this enables the value of u(x, t) at the grid point.

4. THE STABILITY ANALYSIS

A Von Neumann stability analysis must considered the finite difference equations (12). This is accomplished by substituting the Fourier components of $u^n_{i,j,k}$ *as* $u^n_{i,j,k} = U^n\, e^{I\alpha hi}\, e^{I\beta hj}\, e^{I\gamma hk}$, *where* $I = \sqrt{-1}$, U^n is the amplitude at time level n, and α, β, γ are the wave numbers in the x, y, z directions respectively. If a phase angles $\theta = \alpha h,\ \phi = \beta h,\ \psi = \gamma h$ are defined, then $u^n_{i,j,k} = U^n\, e^{I\theta i}\, e^{I\phi j} e^{I\psi k}$. The amplification factor is

$$G = \frac{(1 + r\varepsilon_{i,j} + rb\cos\theta - rb) - Irr_1\sin\theta}{(1 + r\varepsilon_{i,j} - rb\cos\theta + rb) + Irr_1\sin\theta} \ .$$

Consequently the considered method will be stable when $|G| \le 1$, i.e. $-1 \le r\varepsilon_{i,j} \le 1$

5. NUMERICAL RESULTS

We present some numerical examples to compare the considered method (12) with Crank-Nicolson method (C.N.) as done in [14], and we consider two cases. We apply our method on the examples 1and 2 such that the exact solution is given at the first level to determine the restrictive parameters $\varepsilon_{i,j}$, and hence we use it for another levels for calculation. In general the exact solution at the first level is unknown, so we can use the Crank-Nicolson method, to evaluate the solutions at the first time level by large number of very small time step length k to determine the restrictive parameters $\varepsilon_{i,j}$, then we can use large time step length k to evaluate the solution at another levels.

Example: 1

$$0.1\,\frac{\partial u}{\partial t}+0.1\,\frac{\partial u}{\partial x}=0.2\,\frac{\partial^2 u}{\partial x^2},$$

$$\textit{with the initial condition}\; u(x,0)=\exp(x),$$

$$\textit{and the boundary conditions}:\; u(0,t)=\exp(t),\quad u(1,t)=\exp(1+t),\quad 0\le t\le T$$

Its exact solution is given by: $u(x,\ t)=\exp(x+t)$

Example: 2

$$0.01\frac{\partial u}{\partial t}+0.2\,\frac{\partial u}{\partial x}=0.19\,\frac{\partial^2 u}{\partial x^2},$$

$$\textit{with the initial condition}\; u(x,0)=\exp(x),$$

$$\textit{and the boundary conditions}:\; u(0,t)=\exp(-t),\quad u(1,t)=\exp(1-t),\quad 0\le t\le T$$

Its exact solution is given by: $u(x,\ t)=\exp(x-t)$

t	x	Crank-Nicolson method	The considered method
		A. E.	A. E.
5	0.2	1.7×10^{-3}	5.7×10^{-14}
	0.5	3.0×10^{-3}	2.0×10^{-14}
	0.9	1.4×10^{-3}	5.7×10^{-14}
10	0.2	2.5×10^{-1}	5.5×10^{-11}
	0.5	4.5×10^{-1}	3.6×10^{-11}
	0.9	2.0×10^{-1}	7.2×10^{-12}
20	0.2	400.0	1.8×10^{-6}
	0.5	9931.4	2.6×10^{-6}
	0.9	4425.4	3.1×10^{-6}

Table: 1. Comparison of the absolute errors (A.E.) between Crank-Nicolson method and the considered method for h=0.1 and k=0.1, for example 1, where $u(0.2,20)=5.9\times10^{8}$.

t	x	Crank-Nicolson method	The considered method
		A. E.	A. E.
5	0.2	5.9×10^{-7}	3.5×10^{-17}
	0.5	1.3×10^{-6}	9.0×10^{-17}
	0.9	1.1×10^{-6}	2.9×10^{-16}
10	0.2	2.2×10^{-8}	1.0×10^{-17}
	0.5	1.1×10^{-8}	3.6×10^{-17}
	0.9	8.8×10^{-8}	4.6×10^{-17}
20	0.2	2.1×10^{-9}	8.0×10^{-19}
	0.5	4.3×10^{-10}	4.0×10^{-18}
	0.9	4.3×10^{-9}	4.2×10^{-19}

Table: 2. Comparison of the absolute errors (A.E.) between Crank-Nicolson method and the considered method for h=0.1 and k=0.0.05, for example 2.

6. CONCLUSION

The numerical results presented tables (1), and (2) shows that the absolute errors obtained by the considered methods is almost of order 10^{-10} of that absolute errors obtained by Crank-Nicolson method.

In the case of too large solution for example 1, it is clear from the given data in table (1) that the absolute errors associated with Crank-Nicolson method is too large compared with that of the considered method.

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