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GENERALISED GAUSSIAN QUADRATURE OVER A ELLIPTICAL REGION $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

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ABSTRACT

This paper presents a Generalised Gaussian quadrature method for the evaluated double integral over a arbitrary function $f(x, y)$ over a elliptical region. Using the transformation of variables new Gaussian points and corresponding weights are calculated. The performances of the proposed method is demonstrated by means of a numerical examples.

Keywords- *Finite element method, Generalised Gaussian quadrature, Elliptical region, extended numerical integration.*

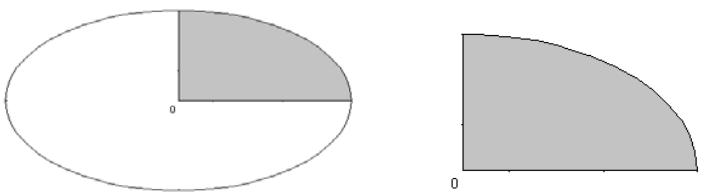
1. INTRODUCTION

The finite element method has become a powerful tool for the numerical solution of a wide range of engineering problem, particularly when analytical solutions are not available or very difficult to arriving the results. Numerical methods for integration approximate a definite integral of a given function by a weighted sum of function values at specified points. There are many quadrature methods available for approximating integrals. Surface integrals are used in multiple areas of physics and engineering. In particular, they are used for Problems involving calculations of mass of a shell, center of mass and moments of inertia of a shell, fluid flow and mass flow across a surface, electric charge distributed over a surface, plate bending, plane strain, heat conduction over a plate, and similar problems in other areas of engineering which are very difficult to analyse using analytical techniques, These problems can be solved using the finite element method. The method proposed here is termed as Generalized Gaussian rules, since the Generalized Gaussian quadrature nodes and weights for products of polynomial and logarithmic function given in [9] by Ma *et al.* are used in this paper

From the literature review we may realize that several works in numerical integration using Gaussian quadrature over various region have been carried out [1-5], Generalized Gaussian quadrature rules over regions with parabolic edges given in [5], evaluation of surface area of an ellipsoid and related integrals of elliptic integrals is given in [6-8]. In this paper we use Generalised Gaussian quadrature method to evaluate the surface integral over the arbitrary function in elliptical region

The paper is organized as follows. In Section 2 we will introduce the Generalized Gaussian quadrature formula over a elliptical region of various values of a and b. In Section 3 we compare the numerical results with some illustrative examples.

2. GENERALISED GAUSSIAN QUADRATURE OVER A ELLIPTICAL REGION



(a) **(b)**

Fig. (a) Elliptical region (b) Elliptical quarter region

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Generalized Gaussian quadrature rule for integrating function bounded by the elliptical region by

$\left\{(x,\ y)/-a\ \le\ x\le\ a, -b\sqrt{1-\frac{x^2}{a^2}}\le\ y\ \le b\sqrt{1-\frac{x^2}{a^2}}\right\}$ with various values of a and b

2.1 Formulation of integrals over a Elliptical region

The Numerical integration of an arbitrary function f over the elliptical region E is given by

$$\mathrm{I}=\iint_{\mathrm{E}} \mathrm{f(x,y)dx\,dy}=\int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} f(x,y)dy\,dx=4\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} f(x,y)dx\,dy \tag{1}$$

The double integral over the elliptical surface of eqn.(1) can be transformed to the standard square $\{(\ \xi\ ,\eta\ )\ /\ \ 0\le\ \xi\le 1,\ 0\le\eta\le 1\}$ Transformation is

$$x=\ a\,\xi \text{ and } y=b\,\eta\sqrt{1-\ \xi^2} \tag{2}$$

We have

$$I=\int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} f(x,y)dy\,dx=\int_0^1\int_0^1 f\big(x(\xi\ ,\eta),y(\xi\ ,\eta)\big)\,J\,d\xi\,d\eta \tag{3}$$

Where $J\,(\xi\ ,\eta)$ is the Jacobians of the transformation

$$J(\xi\ ,\eta)=\begin{vmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi}\\ \frac{\partial x}{\partial\eta} & \frac{\partial y}{\partial\eta}\end{vmatrix}=ab\sqrt{1-\ \xi^2}$$

From eqn.(3) , we can write as

$$\begin{aligned} I&=\int_0^1\int_0^1 f\left(a\,\xi\ ,b\,\eta\sqrt{1-\ \xi^2}\right)\ ab\sqrt{1-\ \xi^2}\ d\xi\ d\eta\\ &=\sum_{i=1}^n\sum_{j=1}^n ab\sqrt{1-\ \xi^2}\ \ w_i\ w_j\ f\big(x(\xi_i\ ,\eta_j)\ ,y(\xi_i\ ,\eta_j)\big)\end{aligned} \tag{4}$$

Where $\xi_{i\ ,}\,\eta_j$ are sampling points and corresponding to its weight coefficients w_i, w_j . We can rewrite eqn. (4) as

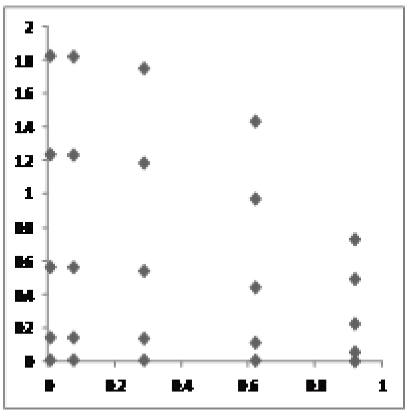
$$\mathrm{I}=\sum_k^{N=n\times n}\quad W_k f(x_k,y_k) \tag{5}$$

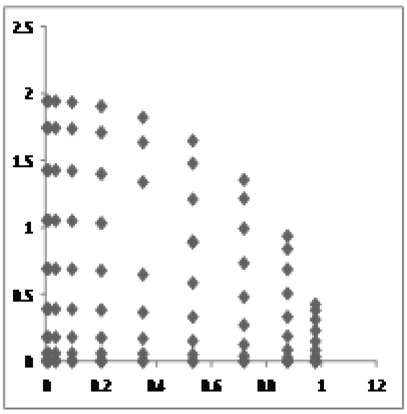
Where $\quad W_k=ab\sqrt{1-\ \xi^2}\ \ w_i\ w_j$ (5a)

$x_k=\ a\,\xi$ and $y_k=b\,\eta\sqrt{1-\ \xi^2}$ (5b)

if $k,i,j=1,2,3,\ldots\ldots\ldots$

We find out new sampling points x_k , y_k and weights coefficients W_k of various order N =5, 10, 15, 20 by using eqn. (5a) and (5b)





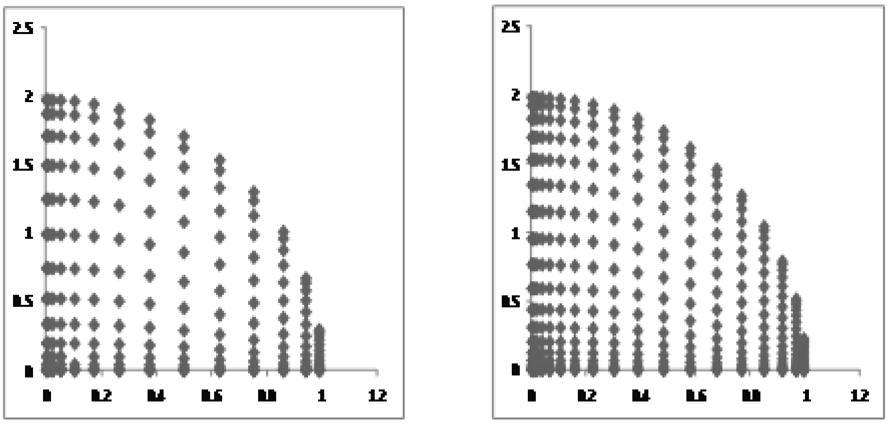


Fig. 1: Distribution of Gaussian points (x_k, y_k) values for the elliptical region with a= 1 and b=2 of order N= 5, 10, 15, 20

a =1, b = 2			a = 2, b = 3		
x_k	y_k	W_k	x_k	y_k	W_k
0.0056522282	0.0113042758	0.0008859337	0.0113044564	0.0169564137	0.0026578011
0.0734303717	0.0112739382	0.0054870516	0.1468607434	0.0169109074	0.0164611545
0.2849574044	0.0108357756	0.0116891073	0.5699148089	0.0162536635	0.0350673219
0.6194822640	0.0088741187	0.0115727355	1.2389645281	0.0133111780	0.0347182067
0.9157580830	0.0045413412	0.0035228535	1.8315161660	0.0068120118	0.0105685607
0.0056522282	0.1468583975	0.0055018169	0.0113044564	0.2202875963	0.0165054509
0.0734303717	0.1464642703	0.0340756353	0.1468607434	0.2196964054	0.1022269059
0.2849574044	0.1407719235	0.0725915821	0.5699148089	0.2111578853	0.2177747464
0.6194822640	0.1152872480	0.0718688915	1.2389645281	0.1729308720	0.2156066746
0.9157580830	0.0589983921	0.0218775914	1.8315161660	0.0884975882	0.0656327743
0.0056522282	0.5699057051	0.0121945024	0.0113044564	0.8548585576	0.0365835074
0.0734303717	0.5683762362	0.0755269434	0.1468607434	0.8525643543	0.2265808303
0.2849574044	0.5462862438	0.1608956155	0.5699148089	0.8194293657	0.4826868465
0.6194822640	0.4473891960	0.1592938078	1.2389645281	0.6710837940	0.4778814236
0.9157580830	0.2289519758	0.0484905885	1.8315161660	0.3434279637	0.1454717657
0.0056522282	1.2389447369	0.0147419027	0.0113044564	1.8584171054	0.0442257083
0.0734303717	1.2356197528	0.0913043283	0.1468607434	1.8534296292	0.2739129851
0.2849574044	1.1875972824	0.1945062973	0.5699148089	1.7813959237	0.5835188919
0.6194822640	0.9726003526	0.1925698761	1.2389645281	1.4589005289	0.5777096284
0.9157580830	0.4977294365	0.0586201482	1.8315161660	0.7465941548	0.1758604448
0.0056522282	1.8314869094	0.0087690632	0.0113044564	2.7472303641	0.0263071896
0.0734303717	1.8265717064	0.0543114032	0.1468607434	2.7398575596	0.1629342098
0.2849574044	1.7555818363	0.1156999907	0.5699148089	2.6333727545	0.3470999723
0.6194822640	1.4377597004	0.1145481313	1.2389645281	2.1566395507	0.3436443947
0.9157580830	0.7357753096	0.0348695682	1.8315161660	1.1036629645	0.1046087048

Table - 1. sampling points and weights coefficient over the elliptical region for N = 5

a = 3, b = 1			a = 3, b = 5		
x_k	y_k	W_k	x_k	y_k	W_k
0.0169566846	0.0056521379	0.0013289005	0.0169566846	0.0282606895	0.0066445027
0.2202911152	0.0056369691	0.0082305774	0.2202911152	0.0281848457	0.0411528871
0.8548722133	0.0054178878	0.0175336609	0.8548722133	0.0270894392	0.0876683049
1.8584467922	0.0044370593	0.0173591033	1.8584467922	0.0221852963	0.0867955169
2.7472742490	0.0022706706	0.0052842803	2.7472742490	0.0113533531	0.0264214018
0.0169566846	0.0734291987	0.0082527254	0.0169566846	0.3671459937	0.0412636271
0.2202911152	0.0732321351	0.0511134529	0.2202911152	0.3661606757	0.2555672649
0.8548722133	0.0703859617	0.1088873732	0.8548722133	0.3519298089	0.5444368661
1.8584467922	0.0576436240	0.1078033373	1.8584467922	0.2882181201	0.5390166865
2.7472742490	0.0294991960	0.0328163871	2.7472742490	0.1474959803	0.1640819357
0.0169566846	0.2849528525	0.0182917537	0.0169566846	1.4247642628	0.0914587686
0.2202911152	0.2841881181	0.1132904151	0.2202911152	1.4209405905	0.5664520758
0.8548722133	0.2731431219	0.2413434232	0.8548722133	1.3657156095	1.2067171164
1.8584467922	0.2236945980	0.2389407118	1.8584467922	1.1184729900	1.1947035595
2.7472742490	0.1144759872	0.0727358828	2.7472742490	0.5723799396	0.3636794143
0.0169566846	0.6194723684	0.0221128541	0.0169566846	3.0973618424	0.1105642709
0.2202911152	0.6178098764	0.1369564923	0.2202911152	3.0890493821	0.6847824629
0.8548722133	0.5937986412	0.2917594459	0.8548722133	2.9689932061	1.4587972295
1.8584467922	0.4863001763	0.2888548142	1.8584467922	2.4315008816	1.4442740711
2.7472742490	0.2488647182	0.0879302224	2.7472742490	1.2443235914	0.4396511120
0.0169566846	0.9157434547	0.0131535948	0.0169566846	4.5787172735	0.0657679741
0.2202911152	0.9132858532	0.0814671049	0.2202911152	4.5664292660	0.4073355245
0.8548722133	0.8777909181	0.1735499861	0.8548722133	4.3889545909	0.8677499307
1.8584467922	0.7188798502	0.1718221970	1.8584467922	3.5943992512	0.8591109851
2.7472742490	0.3678876548	0.0523043524	2.7472742490	1.8394382741	0.2615217620

Table - 2. Sampling points and weights coefficient over the elliptical region for N = 5

3. NUMERICAL RESULT

Exact value	Order N	a =1, b =2	a =2, b =3	a =3, b =1	a =3, b =5
$\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{1}{\sqrt{x+y}}\,dydx$	N=5	1.54488928	3.56038672	2.03995207	7.05500991
	N=10	1.53947504	3.54636869	2.03703985	7.03240668
	N=15	1.54203191	3.55492725	2.03549616	7.03538355
	N=20	1.54049858	3.55085967	2.03598996	7.03571281
	Exact value	**1.54040620**	**3.55066361**	**2.03590987**	**7.03531291**
$\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sin(x+y+1)\,dydx$	N=5	1.04804220	−0.03523476	0.74277592	−4.32827142
	N=10	1.04347500	−0.03446897	0.74630196	−4.28230492
	N=15	1.04375735	−0.03522616	0.74825321	−4.27581001
	N=20	1.04372970	−0.03493799	0.74867129	−4.27467008
	Exact value	**1.04364491**	**-0.03497595**	**0.74877868**	**-4.27412110**
$\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{x^4+y^4}{1+x^2y}\,dydx$	N=5	2.89331325	33.49180566	11.86992952	395.72857198
	N=10	2.87048251	33.43557949	11.55554547	394.41943253
	N=15	2.88801076	33.43560078	11.54756727	394.31848828
	N=20	2.88788791	33.43151140	11.54605581	394.31572930
	Exact value	**2.88779547**	**33.42785294**	**11.5460881**	**394.31023323**
$\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{\sin(x+y)}{\sqrt{xy}}\,dydx$	N=5	3.36921995	5.61641028	3.98116315	3.03166239
	N=10	3.39811433	5.67025502	4.01465050	3.13225067
	N=15	3.42142641	5.75526794	4.06468746	3.21107630
	N=20	3.42944302	5.75255621	4.06276473	3.23798255
	Exact value	**3.42943845**	**5.75346999**	**4.06942919**	**3.23202882**
$\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{\log(x+y)}{\sqrt{1+x+y}}\,dydx$	N=5	0.062995191	1.56409741	0.45121347	5.96783010
	N=10	0.064813274	1.56557558	0.46019627	5.95896007
	N=15	0.061348657	1.55402140	0.44390503	5.94940189
	N=20	0.062592889	1.55807248	0.44737598	5.94703520
	Exact value	**0.062592156**	**1.55796005**	**0.447299301**	**5.947649568**

$\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{\sqrt{x}-\sqrt{y}}{1+x^2+y^2}\,dxdy$	N=5	-0.12333075	-0.12556115	0.22942798	-0.21441799
	N=10	-0.11934112	-0.12494023	0.23034524	-0.22198552
	N=15	-0.12515784	-0.12960183	0.22733422	-0.22479545
	N=20	-0.12480982	-0.12806459	0.22818610	-0.22023419
	Exact value	**-0.12484804**	**-0.12813150**	**0.22816399**	**-0.22033550**

4. CONCLUSIONS

In this paper we derived Generalised Gaussian quadrature method for calculating integral over a elliptical region $\left\{(x,\ y)/\ 0\ \le\ x \le\ a,\ 0 \le\ y\ \le b\sqrt{1-\frac{x^2}{a^2}}\right\}$ with various values of a and b. New sampling points and its weights are calculated of various order N = 5, 10, 15, 20. We have then evaluate the typical integrals Governed by the proposed method. The results obtained are in excellent agreement with the exact value.

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