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A STUDY OF SOME INEQUALITY CONTAINING THE BETA FUNCTION

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ABSTRACT

In this paper a new method is presented in order to check the monotonicity of the beta function using the Euler gamma function.

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1. INTRODUCTION:

The Euler Gamma function Γ is defined for $x > 0$ by

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt. \tag{1.1}$$

The digamma function ψ is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)},$$

and has the representation

$$\psi(x) = -\gamma + (x-1)\sum_{k=0}^{\infty} \frac{1}{(k+1)(x+k)}. \tag{1.2}$$

In [2], J.Sandor proved the following inequality

$$\frac{1}{\Gamma(1+a)} \le \frac{\Gamma(1+x)^a}{\Gamma(1+ax)} \le 1, \quad x \in [0,1], \quad a \ge 1. \tag{1.3}$$

Lator on, L. Bougoffa [1] in his roll generalized inequality (1.3) by giving the following

Theorem: 1.1. *Let f be a function defined by*

$$f(x) = \frac{\Gamma(1+bx)^a}{\Gamma(1+ax)^b}, \quad \forall x \ge 0,$$

in which $1+ax > 0$ *and* $1+bx > 0$, *then for all* $a \ge b > 0$ *or* $0 > a \ge b$ $(a > 0$ *and* $b < 0)$, f *is decreasing (increasing) respectively on* $[0,\infty)$.

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The aim of this paper is to present a new result concerning the beta function. In fact, we prove the following:

2. RESULT:

Theorem: 2.1. *Let f be a function defined by*

$$f(x) = \beta(1+ax, 1+bx),$$

when $1+ax>0$, $1+bx>0$. *Define*

$$\lambda = \min\{aB+bA,\ aB^2+bA^2\},\ \mu = \max\{aB+bA,\ aB^2+bA^2\}.$$

Then

Case A. when a, b have the same signs.

1. f is decreasing whenever a, b are positive.
2. f is increasing whenever a, b are negative.

Case B. when a, b have different signs.

1. f is decreasing if $\lambda \geq 0$.
2. f is increasing if $\mu \leq 0$.

Proof: Let

$$f(x) = \beta(A, B) = \frac{\Gamma(A)\Gamma(B)}{\Gamma(A+B)},$$

where $A = 1+ax$, $B = 1+bx$. The above implies

$$\ln f(x) = \ln\Gamma(A) + \ln\Gamma(B) - \ln\Gamma(A+B).$$

Differentiating, we obtain

$$\frac{f'(x)}{f(x)} = a\frac{\Gamma'(A)}{\Gamma(A)} + b\frac{\Gamma'(B)}{\Gamma(B)} - (a+b)\frac{\Gamma'(A+B)}{\Gamma(A+B)}$$

$$= a\psi(A) + b\psi(B) - (a+b)\psi(A+B)$$

$$= a(A-1)\sum_{k=0}^{\infty}\frac{1}{(1+k)(A+k)} + b(B-1)\sum_{k=0}^{\infty}\frac{1}{(1+k)(B+k)}$$

$$-(a+b)(A+B-1)\sum_{k=0}^{\infty}\frac{1}{(1+k)(A+B+k)}$$

$$= \sum_{k=0}^{\infty}\frac{[a(A-1)(B+k+b(A+k)(B-1))](A+B+k)-(a+b)(A+k)(B+k)(A+B-1)}{(1+k)(A+k)(B+k)(A+B+k)}.$$

Simplifying, we have

$$\frac{f'(x)}{f(x)} = -\sum_{k=0}^{\infty}\frac{(aB^2+bA^2)k^0 + (aB+bA+aB^2+bA^2)k + (aB+bA)k^2}{(1+k)(A+k)(B+k)(A+B+k)}$$

Case: 1 if $a, b \geq 0$, then $\frac{f'(x)}{f(x)} \leq 0$, which implies $f'(x) \leq 0$.

That is f is non-increasing

if $a, b \le 0$, then $\frac{f'(x)}{f(x)} \ge 0$, and hence $f'(x) \ge 0$.

which implies f is non-decreasing.

Case: 2 when a, b have different signs

If $\lambda \ge 0$, then $\min\{aB + bA, \; aB^2 + bA^2\} \ge 0$, which implies each of the two terms $aB + bA$, $aB^2 + bA^2$ non-negative and therefore f is non-increasing as in case-1.

If $\mu \le 0$, then each of the two terms $aB + bA$, $aB^2 + bA^2$ non-positive which implies f is non-decreasing as in case-1.

3. APPLICATIONS:

Corollary: 3.1 Let $1 + ax > 0$, $1 + x > 0$ .Then

(**a**) The function $\beta(1 + ax, 1 + x)$ is decreasing for $a \ge 1$, $x \in \left(\frac{-1}{a}, \infty\right)$.

(**b**) The function $\beta(1 + ax, 1 + x)$ is decreasing for $0 < a \le 1$, $x \in (-1, \infty)$.

Proof: The proof follows from Th.2.1 by putting $b = 1$.

Corollary: 3.2. Let $1 + ax > 0$, $1 + x > 0$ .Then

(**a**) The function $\beta(1 - ax, 1 - x)$ is decreasing for $a \ge 1$, $x \in \left(0, \frac{1}{a}\right)$.

(**b**) The function $\beta(1 - ax, 1 - x)$ is decreasing for $0 < a \le 1$, $x \in (0, 1)$.

Proof: The proof follows from Theorem.2.1 by putting $b = 1$.

Corollary: 3.3. Let λ, μ, A, B be as defined in Theorem 2.1, let $1 + ax > 0$, $1 + x > 0$ .Then the function $\beta(1 - ax, 1 + x)$ decreasing if $\lambda \ge 0$ and increasing if $\mu \le 0$.

Proof: The proof follows from Theorem.2.1 by putting $b = 1$.

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