International Journal of Mathematical Archive-5(1), 2014, 300-307
Available online through www.ijma.info ISSN 2229 – 5046

SOME COMMON FIXED POINT THEOREMS OF COMPATIBLE MAPPINGS OF TYPE (R)

Debanand Sinha
The Global University, Nagaland.

L. Shambhu Singh*
Department of Mathematics, D. M. College of Science, Imphal-795001, Manipur (India).

(Received on: 27-12-13; Revised & Accepted on: 23-01-14)

ABSTRACT

Compatible mappings of type (R) was introduced by Rohen et.al., [5] by combining the definitions of compatible mappings and compatible mappings of type (P). In this paper, we prove some fixed point theorems of compatible mappings of type (R) satisfying contractive conditions.

Key words: *Compatible mapping, Compatible mapping of type (P), Compatible mapping of type (R), Contractive modulus function, fixed point, common fixed point.*

2000 Mathematics Subject Classification: *47H10, 54H25.*

1. INTRODUCTION

In 1986, Jungck [3] introduced the concept of compatible mappings by generalizing commuting mappings. Pathak, Chang and Cho [4] in 1994 introduced the concept of a new type of compatible mappings called compatible mappings of type (P). Rohen, Ranjit and Shambhu [5] introduced the concept of compatible mappings of type (R) by combing the definitions of compatible mappings and compatible mappings of type (P). More results on compatible mappings of type(R) can be found in [5-11]. Al-Thagafi and Shahazad [1] gave the concept of occasionally weakly compatible mappings in 2008. In 2011, H. Bouhadjera [2] proved some common fixed point theorems for three and four occasionally weakly compatible mappings satisfying different types of contractive conditions.

Now we extend the result of H. Bouhadjera [2] by employing compatible mappings of type (R) instead of occasionally weakly compatible mappings.

2. PRELIMINARIES

Definition: 2.1 Self mappings A and B of a metric space (X, d) are said to be compatible if, for all $x \in X$

$$\lim_{n\to\infty} d(ABx_n, BAx_n) = 0$$

Definition: 2.2 Self mappings A and B of a metric space (X, d) are said to be compatible of type (P) if, for all $x \in X$

$$\lim_{n\to\infty} d(AAx_n, BBx_n) = 0$$

Whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition: 2.3 Self mappings A and B of a metric space (X, d) are said to be compatible of type (R) if, for all $x \in X$

$$\lim_{n\to\infty} d(AAx_n, BBx_n) = 0 \text{ and } \lim_{n\to\infty} d(ABx_n, BAx_n) = 0$$

Whenever $\{x_n\}$ is a sequence in X and such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Bx_n = t$ for some $t \in X$.

Definition: 2.4 A function $\varphi: [0, \infty) \to [0, \infty)$ is said to be a contractive modulus if $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0$.

*Corresponding author: L. Shambhu Singh**
Department of Mathematics, D. M. College of Science, Imphal-795001, Manipur (India).

Definition: 2.5 A real valued function φ defined on $X \subset \mathbf{R}$ is said to be upper semi continuous if $\lim_{n\to\infty} \varphi(t_n) \le \varphi(t)$, for every sequence $\{t_n\}$ in X with $t_n \to t$ as $n\to\infty$.

Pathak, Murthy and Cho [6] prove the following propositions.

Proposition: 2.6: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair $\{S, T\}$ is compatible of type (R) on X and $Sz = Tz$ for $z \in X$, then

$$STz = TSz = SSz = TTz.$$

Proposition: 2.7: Let S and T be mappings from a complete metric space (X, d) into itself. If a pair $\{S, T\}$ is compatible of type (R) on X and $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$
for some $z \in X$, then we have
(i) $d(TTx_n, Sz) \to 0$ and $d(TSx_n, Sz) \to 0$ as $n \to \infty$ if S is continuous,
(ii) $d(SSx_n, Tz) \to 0$ and $d(STx_n, Tz) \to 0$ as $n \to \infty$ if T is continuous and
(iii) $STz=TSz$ and $Sz=Tz$ if S and T are continuous at z.

H. Bouhadjera [2] proved the following results.

Theorem: 2.8 Let X be a set with a symmetric d. Let f, g and h be self mapping of (X, d) and φ is a contractive modulus function satisfying:

$$d^2(fx, gy) \le \max\{\varphi(d(hx, hy))\varphi(d(hx, fx)),\ \varphi(d(hx, hy))\varphi(d(hy, fx)),\ \varphi(d(hx, hy))\varphi(d(hy, gy)),\ (d(hx, fx))\varphi(d(hy, gy)),\ \varphi(d(hx, gy))\varphi(d(hy, fx))\},$$
for all $x, y \in X$, the pair (f, h) and (g, h) is owc.

Then f, g and h have a unique common fixed point.

Theorem: 2.9 Let X be a set endowed with a symmetric d. Suppose f, g, h and k are four self mappings of (X, d) satisfying the conditions:

$$d^2(fx, gy) \le \max\{\varphi(d(hx, ky))\varphi(d(hx, fx)),\ \varphi(d(hx, ky))\varphi(d(ky, gy)),\ \varphi(d(hx, fx))\varphi(d(ky, gy)),\ \varphi(d(hx, gy))\varphi(d(ky, fx))\},$$
for all $x, y \in X$, where φ is contractive modulus, the pair (f, h) and (g, k) are owc.

Then f, g, h and k have a unique common fixed point.

3. MAIN RESULTS

Theorem: 3.1 Let A, B and T be self mapping of a complete metric space (X, d) and φ is a contractive modulus satisfying:
(*a*) $A(X) \cup B(X) \subset T(X)$.
(*b*) $d^2(Ax, By) \le \max\{\varphi(d(Tx, Ty))\varphi(d(Tx, Ax)),\ \varphi(d(Tx, Ty))\varphi(d(Ty, Ax)),\ \varphi(d(Tx, Ty))\varphi(d(Ty, By)),\ \varphi(d(Tx, Ax))\varphi(d(Ty, By)),\ \varphi(d(Tx, By))\varphi(d(Ty, Ax))\}$ for all $x, y \in X$.
(*c*) the pair (A, T) or (B, T) is compatible of type (R).
(*d*) If T is continuous, then A, B and T have a unique common fixed point.

Proof: Let $x_0 \in X$ *be* arbitrary. Consider a sequence $\{Tx_n\}$ defined as follows.

$$Tx_{2n+1} = Ax_{2n},\ Tx_{2n+2} = Bx_{2n+1} \text{ where } n = 0, 1, 2,\ldots \tag{3.1.1}$$

From condition (b) we get
$$d^2(Tx_{2n+1}, Tx_{2n+2}) = d^2(Ax_{2n}, Bx_{2n+1})$$

$$\le \max\{\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n}, Ax_{2n})),\ \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Ax_{2n})),\ \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\ \varphi(d(Tx_{2n}, Ax_{2n}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})),\ \varphi(d(Tx_{2n}, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, Ax_{2n}))\},$$

$$= \max\{\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n}, Tx_{2n+1})),\ \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+1})),\ \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+2})),\ \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+2})),\ \varphi(d(Tx_{2n}, Tx_{2n+2}))\varphi(d(Tx_{2n+1}, Tx_{2n+1}))\},$$

$$= \max\{\varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n}, Tx_{2n+1})),\ \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Tx_{2n+1}))\}.$$

This implies

$$d^2(Tx_{2n+1}, Tx_{2n+2}) \le \varphi(d(Tx_{2n}, Tx_{2n+1})) \max\{\varphi(d(Tx_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n+1}, Tx_{2n+2}))\}$$

$$\le d(Tx_{2n}, Tx_{2n+1}) \max\{d(Tx_{2n}, Tx_{2n+1}), d(Tx_{2n+1}, Tx_{2n+2})\}$$

Since φ is contractive modulus, we have

$$d(Tx_{2n+1}, Tx_{2n+2}) \le \varphi(d(Tx_{2n}, Tx_{2n+1})) \le d(Tx_{2n}, Tx_{2n+1}). \qquad (3.1.2)$$

Thus the sequence $\{d(Tx_{2n}, Tx_{2n+1})\}$ is decreasing. If $\{d(Tx_{2n}, Tx_{2n+1})\} \to \alpha$ then from (3.1.2) we have $\alpha \le \varphi(\alpha) \le \alpha$ and so we must have $\alpha = 0$, hence

$$\lim_{n\to\infty} d(Tx_{2n}, Tx_{2n+1}) = 0. \qquad (3.1.3)$$

We shall show that $\{Tx_{2n}\}$ is a Cauchy sequence. If it is not so, there exist an $\in >0$ and a sequence of integers $\{m_k\}$, $\{n_k\}$ with $m_k > n_k \ge k$, such that

$$d(Tx_{2m}, Tx_{2n}) \ge \in \qquad (3.1.4)$$

$k = 1, 2, 3,\ldots$. If m_k is the smallest integer exceeding n_k for which (3.1.4) holds, then from well-ordering principle, we have

$$d(Tx2m-1, Tx2n) < \in. \qquad (3.1.5)$$

Since

$d(Tx_{2m-1}, Tx_{2n+1}) \le d(Tx_{2m-1}, Tx_{2n}) + d(Tx_{2n}, Tx2n+1)$, and so letting $k\to\infty$, we see that

$$d(Tx_{2m-1}, Tx_{2n+1}) \le \in. \qquad (3.1.6)$$

Now

$$d(Tx_{2m}, Tx_{2n}) \le d(Tx_{2m}, Tx_{2n+1}) + d(Tx_{2n+1}, Tx_{2n}). \qquad (3.1.7)$$

But we have

$$d^2(Tx_{2m}, Tx_{2n+1}) = d^2(Ax_{2n}, Bx_{2m-1})$$

$$\le \max\{\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2n}, Ax_{2n})), \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Ax_{2n})),$$
$$\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Bx_{2m-1})), \varphi(d(Tx_{2n}, Ax_{2n}))\varphi(d(Tx_{2m-1}, Bx_{2m-1})),$$
$$\varphi(d(Tx_{2n}, Bx_{2m-1}))\varphi(d(Tx_{2m-1}, Ax_{2n}))\}$$

$$= \max\{\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Tx_{2n+1})),$$
$$\varphi(d(Tx_{2n}, Tx_{2m-1}))\varphi(d(Tx_{2m-1}, Tx_{2m})), \varphi(d(Tx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2m-1}, Tx_{2m})),$$
$$\varphi(d(Tx_{2n}, Tx_{2m}))\varphi(d(Tx_{2m-1}, Tx_{2n+1}))\} \qquad (3.1.8)$$

Using (3.1.3), (3.1.5), (3.1.6), (3.1.8) and letting $k\to\infty$, we obtain

$$\lim_k d^2(Tx_{2m}, Tx_{2n+1}) \le \max\{\varphi(\in)\varphi(\in), \varphi(\in) \lim_k d(Tx_{2m}, Tx_{2n})\}$$

$$\le \varphi(\in) \max\{\varphi(\in), \lim_k d(Tx_{2m}, Tx_{2n+1})\},$$

implies

$$\lim_k d(Tx_{2m}, Tx_{2n+1}) \le \varphi(\in) < \in.$$

Then by (3.1.7)

$$\lim_k d(Tx_{2m}, Tx_{2n}) \le \varphi(\in) + 0 < \in,$$

a contradiction. Thus $\{Tx_n\}$ is a Cauchy sequence. Since X is complete, there exist a point $z \in X$ such that $Tx_n \to z$. It follows that from (3.1.1) that the sequences $\{Ax_{2n}\}$ and $\{Bx_{2n+1}\}$ also converge to z.

Since T is continuous such that

$TTx_{2n} \rightarrow Tz$, $TAx_{2n} \rightarrow Tz$ as $n \rightarrow \infty$,

Since the pair (A, T) is compatible of type (R), we have

$AAx_{2n} \rightarrow Tz$ and $ATx_{2n} \rightarrow Tz$ as $n \rightarrow \infty$

Then from condition (b), we have

$$d^2(AAx_{2n}, Bx_{2n+1}) \le \max\{\varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(TAx_{2n}, AAx_{2n+1})), \varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, AAx_{2n})), \varphi(d(TAx_{2n}, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(TAx_{2n}, AAx_{2n}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(TAx_{2n}, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, AAx_{2n}))\}.$$

Letting $n \rightarrow \infty$ and we have

$$d^2(Tz, z) \le \max\{\varphi(d(Tz, z))\varphi(d(Tz, Tz)), \varphi(d(Tz, Tz))\varphi(d(Tz, z)), \varphi(d(Tz, z))\varphi(d(z, z)), \varphi(d(Tz, Tz))\varphi(d(z, z)), \varphi(d(Tz, z))\varphi(d(z, Tz))\}$$

$$= \varphi(d(Tz, z))\varphi(d(Tz, z)), \text{ and it implies that}$$

$$d(Tz, z) \le \varphi(d(Tz, z)) \le d(Tz, z)$$

i.e. $\varphi(d(Tz, z)) = d(Tz, z)$. Hence $Tz = z$.

Again from (b), we have

$$d^2(Az, Bx_{2n+1}) \le \max\{\varphi(d(Tz, Tx_{2n+1}))\varphi(d(Tz, Az)), \varphi(d(Tz, Tx_{2n+1}))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(Tz, Az))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(Tz, Ez))\varphi(d(Tx_{2n+1}, Bx_{2n+1})), \varphi(d(Tz, Bx_{2n+1}))\varphi(d(Tx_{2n+1}, Az))\}.$$

Letting as $n \rightarrow \infty$ and using $Tz = z$,

$$d^2(Az, z) \le \max\{\varphi(d(z, z))\varphi(d(z, Az)), \varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, Az))\varphi(d(z, z)), \varphi(d(z, Az))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Az))\},$$

that is $d^2(Az, z) \le 0$, so $d(Az, z) \le 0$. But $d(Az, z) \ge 0$.

Therefore $d(Az, z) = 0$ and hence $Az = z$. So $Tz = Az = z$.

Again from condition (b), we have

$$d^2(Ax_{2n+1}, Bz) \le \max\{\varphi(d(Tx_{2n+1}, Tz))\varphi(d(Tx_{2n+1}, Ax_{2n+1})), \varphi(d(Tx_{2n+1}, Tz))\varphi(d(Tz, Bz)), \varphi(d(Tx_{2n+1}, Ax_{2n+1}))\varphi(d(Tz, Bz)), \varphi(d(Tx_{2n+1}, Ax_{2n+1}))\varphi(d(Tz, Bz)), \varphi(d(Tx_{2n+1}, Bz))\varphi(d(Tz, Ax_{2n+1}))\}.$$

Letting as $n \rightarrow \infty$, we have

$$d^2(z, Bz) \le \max\{\varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, z))\varphi(d(z, Bz)), \varphi(d(z, Bz))\varphi(d(z, z))\},$$

implies that $d(z, Bz) = 0$. Hence $z = Bz$.

Thus $z = Fz = Ez = Tz$, showing that z is a common fixed point of A, B and T. Similarly we can prove that z is a common fixed point of A, B and T when the pair (B, T) is compatible of type (R).

UNIQUENESS

Let z and w be two common fixed points of A, B and T, so $z = Az = Bz = Tz$ and $w = Aw = Bw = Tw$. From condition (ii), we have

$$d^2(z, w) = d^2(Az, Fw)$$

$$\le \max\{\varphi(d(Tz, Tw))\varphi(d(Tz, Az)), \varphi(d(Tz, Tw))\varphi(d(Tw, Az)), \varphi(d(Tz, Tw))\varphi(d(Tw, Bw)), \varphi(d(Tz, Az))\varphi(d(Tw, Bw)), \varphi(d(Tz, Bw))\varphi(d(Tw, Az))\}$$

$= \max\{\varphi(d(z, w))\varphi(d(z, z)), \varphi(d(z, w))\varphi(d(w, z)), \varphi(d(z, w))\varphi(d(w, w)), \varphi(d(z, z))\varphi(d(w, w)), \varphi(d(z, w))\varphi(d(w, z))\}$

$= \varphi(d(z, w))\varphi(d(w, z)) < d^2(z, w)$,

implies $d(z, w) < d(w, z)$, a contradiction, hence the proof.

Theorem: 3.2 Suppose A, B, T and C be four self mappings of a complete metric space (X, d) into itself satisfying the conditions
(a) $A(X) \subset C(X)$, $T(X) \subset B(X)$.

(b) $d^2(Ax, Ty) \le \max\{\varphi(d(Bx, Cy))\varphi(d(Bx, Ax)), \varphi(d(Bx, Cy))\varphi(d(Cy, Ty)), \varphi(d(Bx, Ax))\varphi(d(Cy, Ty)), \varphi(d(Bx, Ty))\varphi(d(Cy, Ax))\}$,
for all $x, y \in X$.

(c) one *of* A, B, T and C is continuous. And if

(d) the pairs (A, B) and (T, C) are compatible of type (R). Then A, B, T and C have a unique common fixed point.

Proof: Let x_0 in X be arbitrary. Take a point x_1 in X such that $Ax_0 = Cx_1$, since $A(X) \subset C(X)$. Let x_2 be a point in X such that $Tx_1 = Bx_2$, since $T(X) \subset B(X)$. In general we can choose $x_{2n}, x_{2n+1}, x_{2n+2}, \dots$ such that $Ax_{2n} = Cx_{2n+1}$ and $Tx_{2n+1} = Bx_{2n+2}$, so that we obtain a sequence

$$Ax_0, Tx_1, Ax_2, Tx_3, \dots \qquad (3.2.1)$$

From condition (b), we have

$$d^2(Ax_{2n}, Tx_{2n+1}) \le \max\{\varphi(d(Bx_{2n}, Cx_{2n+1}))\varphi(d(Bx_{2n}, Ax_{2n})), \varphi(d(Bx_{2n}, Cx_{2n+1}))\varphi(d(Cx_{2n+1}, Tx_{2n+1})), \varphi(d(Bx_{2n}, Ax_{2n}))\varphi(d(Cx_{2n+1}, Tx_{2n+1})), \varphi(d(Bx_{2n}, Tx_{2n+1}))\varphi(d(Cx_{2n+1}, Ax_{2n}))\}$$

$$= \max\{\varphi(d(Tx_{2n-1}, Ax_{2n}))\varphi(d(Tx_{2n-1}, Ax_{2n})), \varphi(d(Tx_{2n-1}, Ax_{2n}))\varphi(d(Ax_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n-1}, Ax_{2n}))\varphi(d(Ax_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2n-1}, Tx_{2n+1}))\varphi(d(Ax_{2n}, Ax_{2n}))\}$$

$$= \max\{\varphi(d(Tx_{2n-1}, Ax_{2n}))\varphi(d(Tx_{2n-1}, Ax_{2n})), \varphi(d(Tx_{2n-1}, Ax_{2n}))\varphi(d(Ax_{2n}, Tx_{2n+1}))\},$$

i.e. $$d^2(Ax_{2n}, Tx_{2n+1}) \le \varphi(d(Ax_{2n}, Tx_{2n-1})) \max\{\varphi(d(Ax_{2n}, Tx_{2n-1})), \varphi(d(Ax_{2n}, Tx_{2n+1}))\}$$

$$\le d(Ax_{2n}, Tx_{2n-1}) \max\{d(Ax_{2n}, Tx_{2n-1}), d(Ax_{2n}, Tx_{2n+1})\}$$

$$\Rightarrow \quad d(Ax_{2n}, Tx_{2n+1}) \le \varphi(d(Ax_{2n}, Tx_{2n-1}))$$

$$\le d(Ax_{2n}, Tx_{2n-1}). \qquad (3.2.2)$$

By similar procedure we can prove that

$d(Ax_{2n}, Tx_{2n-1}) < d(Ax_{2n-2}, Tx_{2n-1})$.

Therefore the sequence $\{d(Ax_{2n}, Tx_{2n+1})\}$ is non decreasing and hence convergent, say converges to some real number c. Since φ is contractive modulus and letting $n\to\infty$, we get $c \le \varphi(c) \le c$ and so $c = 0$. To show that the sequence (3.2.1) is Cauchy, it is sufficient to show that $\{Ax_{2n}\}$ is Cauchy sequence. Suppose it is not so, hence there exist $\in > 0$ and a sequence of integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) \ge n(k) \ge k$ such that $d(Ax_{2m(k)}, Ax_{2n(k)}) > \in$. Let m be the smallest integer greater than $n(k)$ such that $d(Ax_{2m}, Ax_{2n}) > \in$ and $d(Ax_{2m-2}, Ax_{2n}) \le \in$. Therefore

$\in < d(Ax_{2m}, Ax_{2n})$

$\le d(Ax_{2m}, Ax_{2m-1}) + d(Ax_{2m-1}, Ax_{2m-2}) + d(Ax_{2m-2}, Ax_{2n})$

$\le d(Ax_{2m}, Ax_{2m-1}) + d(Ax_{2m-1}, Ax_{2m-2}) + \in$,

Letting $k\to\infty$, we get

$$\lim_{k} d(Ax_{2m}, Ax_{2n}) = \in. \qquad (3.2.3)$$

Also we have

$$d(Ax_{2m}, Ax_{2n}) \le d(Ax_{2m}, Tx_{2n+1}) + d(Ax_{2n}, Tx_{2n+1}) \tag{3.2.4}$$

From condition (b), we have

$$\begin{aligned} d^2(Ax_{2m}, Tx_{2n+1}) \le \max\{&\varphi(d(Bx_{2m}, Cx_{2n+1}))\varphi(d(Bx_{2m}, Ax_{2m})), \varphi(d(Bx_{2m}, Cx_{2n+1}))\varphi(d(Cx_{2n+1}, Tx_{2n+1})), \\ &\varphi(d(Bx_{2m}, Ax_{2m}))\varphi(d(Cx_{2n+1}, Tx_{2n+1})), \varphi(d(Bx_{2m}, Tx_{2n+1}))\varphi(d(Cx_{2n+1}, Ax_{2m}))\} \end{aligned}$$

$$\begin{aligned} = \max\{&\varphi(d(Tx_{2m-1}, Ax_{2n}))\varphi(d(Tx_{2m-1}, Ax_{2m})), \varphi(d(Tx_{2m-1}, Ax_{2n}))\varphi(d(Ax_{2n}, Tx_{2n+1})), \\ &\varphi(d(Tx_{2m-1}, Ax_{2m}))\varphi(d(Ax_{2n}, Tx_{2n+1})), \varphi(d(Tx_{2m-1}, Tx_{2n+1}))\varphi(d(Ax_{2n}, Ax_{2m}))\} \end{aligned} \tag{3.2.5}$$

Also we have

$$d(Tx_{2m-1}, Tx_{2n+1}) \le d(Tx_{2m-1}, Ax_{2m-2}) + d(Ax_{2m-2}, Ax_{2n}) + d(Ax_{2n}, Tx_{2n+1}) \tag{3.2.6}$$

Using (3.2.2), (3.2.5) and letting $k\to\infty$, inequality (3.2.4) gives

$$\lim_k d(Ax_{2m}, Tx_{2n+1}) < \in .$$

Consequently (3.2.2) gives that

$$\in = \lim_k d(Ax_{2m}, Ax_{2n}) < \in,$$

which is a contradiction. Hence $\{Ax_{2n}\}$ is a Cauchy sequence and consequently the sequence (3.2.1) is a Cauchy sequence. Since X is complete, the sequence (3.2.1) converges to a limit z in X. Hence the subsequences $\{Ax_{2n}\} = \{Cx_{2n+1}\}$ and $\{Tx_{2n-1}\} = \{Bx_{2n}\}$ also converge to the limit point z.

Suppose that the mapping B is continuous, then $BBx_{2n} \to Bz$ and $BAx_{2n} \to Bz$ as $n\to\infty$. Since the pair (A, B) is compatible of type (R), by proposition 2.7, we get

$AAx_{2n} \to Bz$ and $ABx_{2n} \to Bz$ as $n\to\infty$.

Now from (b)

$$\begin{aligned} d^2(AAx_{2n}, Tx_{2n+1}) \le \max\{&\varphi(d(BAx_{2n}, Cx_{2n+1}))\varphi(d(BAx_{2n}, AAx_{2n})), \varphi(d(BAx_{2n}, Cx_{2n+1}))\varphi(d(C x_{2n+1}, T x_{2n+1})), \\ &\varphi(d(BAx_{2n}, AAx_{2n}))\varphi(d(C x_{2n+1}, T x_{2n+1})), \varphi(d(BAx_{2n}, T x_{2n+1}))\varphi(d(C x_{2n+1}, AAx_{2n}))\}, \end{aligned}$$

Letting $n\to\infty$, we get,

$$d^2(Bz, z) \le \max\{\varphi(d(Bz, z))\varphi(d(Bz, Bz)), \varphi(d(Bz, z))\varphi(d(z, z)), \varphi(d(Bz, Bz))\varphi(d(z, z)), \varphi(d(Bz, z))\varphi(d(z, Bz))\}$$

$$= \varphi(d(Bz, z))\varphi(d(Bz, z))$$

i.e. $\quad d(Bz, z) \le \varphi(d(Bz, z)) \le d(Bz, z)$.

Hence $\varphi(d(Bz, z)) = 0$ i.e. $Bz = z$.

Further

$$\begin{aligned} d^2(Az, Tx_{2n+1}) \le \max\{&\varphi(d(Bz, Cx_{2n+1}))\varphi(d(Bz, Az), \varphi(d(Bz, Cx_{2n+1}))\varphi(d(Cx_{2n+1}, Tx_{2n+1})), \varphi(d(Bz, Az))\varphi(d(Cx_{2n+1}, Tx_{2n+1})), \\ &\varphi(d(Bz, Tx_{2n+1}))\varphi(d(Cx_{2n+1}, Az))\}. \end{aligned}$$

Since $Cx_{2n+1} \to z$, $Tx_{2n+1} \to z$ as $n\to\infty$ and $Bz = z$, so letting $n\to\infty$ we get

$$d^2(Az, z) \le \max\{\varphi(d(z, z))\varphi(d(z, Az)), \varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, Az))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Az))\} = 0,$$

which implies that $d(Az, z) \le 0$, so we have $d(Az, z) = 0$ and hence $Az = z$.

Thus $Az = Bz = z$. Since $A(X) \subset C(X)$, there is a point $u \in X$ such that $z = Az = Cu$.

Now we prove that $Cu = Tu$.

Now from (b)

$d^2(Az, Tu) = d^2(AAx_{2n}, T x_{2n+1})$

$$\le \max\{\varphi(d(BAx_{2n}, Cx_{2n+1}))\varphi(d(BAx_{2n}, AAx_{2n})), \varphi(d(BAx_{2n}, Cx_{2n+1}))\varphi(d(Cx_{2n+1}, Tx_{2n+1})), \varphi(d(BAx_{2n}, AAx_{2n}))\varphi(d(Cx_{2n+1}, T x_{2n+1})), \varphi(d(BAx_{2n}, T x_{2n+1}))\varphi(d(C x_{2n+1}, AAx_{2n}))\}$$

$$= \max\{\varphi(d(z, z))\varphi(d(z, z)), \varphi(d(z, z))\varphi(d(z, Tu)), \varphi(d(z, z))\varphi(d(z, Tu)), \varphi(d(z, Tu))\varphi(d(z, z))\}$$

so $d^2(Az, Tu) \le 0$ implies $d(z, Tu) = 0$ and $Tu = z$, hence $z = Cu = Tu$.

Let $y_n = u$ for $n \ge 1$, then $Ty_n \to Tu = z$ and $Cy_n \to Cu = z$ as $n\to\infty$.

Since the pair (T, C) is compatible of type (R) from proposition 2.6, we get $TTu=CCu$.

This gives $Tz = Cz$.

Further from (b)

$$d^2(Ax_{2n}, Tz) \le \max\{\varphi(d(Bx_{2n}, Cz))\varphi(d(Bx_{2n}, Ax_{2n})), \varphi(d(Bx_{2n}, Cz))\varphi(d(Cz,Tz)), \varphi(d(Bx_{2n}, Ax_{2n}))\varphi(d(Cz,Tz)), \varphi(d(Bx_{2n}, Tz))\varphi(d(Cz, Ax_{2n}))\}$$

letting $n\to\infty$, we get

$$d^2(z, Tz) \le \max\{\varphi(d(z, Tz))\varphi(d(z, z)), \varphi(d(z, Tz))\varphi(d(Tz, Tz))\ \varphi(d(z, z))\varphi(d(Tz, Tz)), \varphi(d(z, Tz))\varphi(d(Tz, z))\}$$

$$= \varphi(d(z, Tz))\varphi(d(z, Tz))$$

i.e. $d(Tz, z) \le \varphi(d(Tz, z)) \le d(Tz, z)$.

Hence $\varphi(d(Tz, z)) = d(Tz, z) = 0$.

i.e. $Tz = z$. Hence $z = Tz$ and $z = Cz = Tz$. Since $T(X) \subset B(X)$, there is a point $v \in X$ such that

$z = Tz = Bv$. Now

$d^2(Av, z) = d^2(Az, Tz)$

$$\le \max\{\varphi(d(Bv, Cz))\varphi(d(Bv, Av)), \varphi(d(Bv, Cz))\varphi(d(Cz, Tz)), \varphi(d(Bv, Av))\varphi(d(Cz, Tz)), \varphi(d(Bv, Tz))\varphi(d(Cz, Av))\},$$

$$\le \max\{\varphi(d(z, z)\varphi(d(z, Av), \varphi(d(z, z)\varphi(d(z, z), \varphi(d(z, Av)\varphi(d(z, z), \varphi(d(z, z)\varphi(d(z, Av)\}= 0,$$

Thus we have $d(Av, z) = 0$ and $Av = z$. Take $y_n = v$ then $Ay_n \to Av = z$, $By_n \to Bv = z$. Since (A, B) is compatible of type (R) by proposition 2.6, we get $AAu=BBu$ which implies that $Az = Bz$.

Hence z is a common fixed point of A, B, C and T when A is a continuous. The proof is similar that z is common fixed point of A, B, C and T, when T is continuous.

UNIQUENESS

Let z and w be two common fixed point of A, B, C and T i.e. $z = Az = Bz = Tz = Cz$ and $w = Aw = Bw = Tw = Cw$. From condition (b) we have

$d^2(z, w) = d^2(Az, Tw)$

$$\le \max\{\varphi(d(Bz, Cw))\varphi(d(Bz, Az)), \varphi(d(Bz, Cw))\varphi(d(Cw, Tw)), \varphi(d(Bz, Az))\varphi(d(Cw, Tw)), \varphi(d(Bz, Tw))\varphi(d(Cw, Az))\},$$

$$= \max\{\varphi(d(z, w)\varphi(d(z, z), \varphi(d(z, w)\varphi(d(w, w), \varphi(d(z, z)\varphi(d(w, w), \varphi(d(z, w)\varphi(d(w, z)\}$$

Therefore $d(z, w) \leq \varphi(d(z, w)) \leq d(z, w)$ i.e. $\varphi(d(z, w)) = d(z, w)$.

Thus $d(z, w) = 0$ i.e. $w = z$. Hence the common fixed point is unique.

REFERENCES

[1] Al–Thagafi, M.A., Shahzad, N., Generalised I–nonexpansive selfmaps and invariant approximations, *Acta Math. Sin*. (*Engl. Ser*.) **24**, 5 (2008), 867–876.

[2] H. Bouhadjera, On Common Fixed Point Theorems for Three and Four Self Mappings Satisfying Contractive Conditions, *Acta Univ. Palacki. Olomuc., Fac. rer. nat., Mathematica* **49**, 1 (2010), 25–31.

[3] G. Jungck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci*. **9,** (1986), 771-779.

[4] H.K. Pathak, S.S. Chang and Y.J. Cho, Fixed point theorems for compatibles of type (P), *Indian J. Math*., **36** (2) (1994), 151–156.

[5] Y. Rohen, M. Ranjit and L. Shambhu, Common fixed point of compatible mappings of type(C) in Banach spaces, *Proc. of Math. Soc.* BHU, vol. **20** (2004), 77-87.

[6] Y. Rohen and M. Ranjit, Common fixed point of compatible mappings of type(R) in complete metric spaces, *International Journal of Mathematical Sciences and Engineering Applications* (*IJMSEA*), *India*, Vol. **2**, No. IV (2008), 295-303.

[7] Y. Rohen and Bidya Devi, Relationship between various types of compatible mappings, *International Journal of Mathematical Sciences and Engineering Applications* (*IJMSEA*), *India*, Vol. **4**, No.I (March, 2010), 67-82.

[8] Y. Rohen, Fixed point theorems of compatible mappings of type(R) in metric spaces, *Int. J. Open Problems Compt. Math.*, Vol. 3, No. 5, (Dec 2010) 132-143.

[9] Y. Rohen, M. Krishnanando and L. Shambhu, Common fixed point theorems of compatible mappings of type(R), *International Journal of Mathematical Sciences and Engineering Applications* (*IJMSEA*), *India*, Vol. **6**, No. I (Jan., 2012), 355-361.

[10] Y. Rohen, M. Krishnanando, O. Budhichandra and Mairembam Bina, Weak compatible and compatible mappings of type (R) in cone metric space, *International Journal of Mathematical Sciences and Engineering Applications* (*IJMSEA*), *India*, Vol. **6**, No. II (March., 2012), 161-170.

[11] Y. Rohen, M. Koireng Meitei and Leenthoi N., Common fixed points of compatible mappings of type (R), *General Mathematics Notes*, **10**(1) (May 2012), 58-62.

Source of support: Nil, Conflict of interest: None Declared