

**GENERALIZED FOURIER TRANSFORM  
FOR THE GENERATION OF COMPLEX FRACTIONAL MOMENTS**

**M. Ganji<sup>1</sup> and F. Gharari<sup>\*2</sup>**

**<sup>1</sup>Department of Statistics, University of Mohaghegh Ardabili, Iran  
Daneshgah Avenue, Ardabil.**

**<sup>2</sup>Department of Mathematics, University of Mohaghegh Ardabili, Iran  
Daneshgah Avenue, Ardabil.**

**(Received on: 02-11-13; Revised & Accepted on: 21-01-14)**

**ABSTRACT**

Fourier transform of fractional order using the Mittag-Leffler-type function  $E_q(x^q)$  and its complex type, was introduced together with its inversion formula. The obtained transform provided a suitable generalization of the characteristic function of random variables. It was shown that complex fractional moments which are complex moments of order  $nq^{\text{th}}$  of a certain distribution, are equivalent to Caputa fractional derivation of generalized characteristic function (GCF) in origin,  $n$  being a positive integer and  $0 < q \leq 1$ . The case  $q=1$  was reduced to the complex moments. Finally, after introducing fractional factorial moments of a positive random variable, we presented the relationship between integer moments, fractional moments (FMs) and fractional factorial moments (FFMs) of a positive random variable.

**Mathematics Subject Classification:** MSC 60, MSC 58.

**Keywords:** Fractional calculus, Mittag-Leffler-type function, Stirling function.

**1. INTRODUCTION**

It is well known that the Fourier transform of probability density function is characteristic function, that is

$$\varphi_X(t) = \langle e^{itX} \rangle = \int_{-\infty}^{\infty} e^{itx} f(x) dx, \tag{1}$$

Where the notation  $\langle \cdot \rangle$  means expectation and on the other hand, we have:

$$\phi_X(t) = \sum_{k=0}^{\infty} \langle (iX)^k \rangle \frac{t^k}{k!}, \tag{2}$$

This function generates complex moments of integer order, as we have:

$$\langle (iX)^k \rangle = \left. \frac{d^k \phi_X(t)}{dt^k} \right|_{t=0}. \tag{3}$$

But in this work, we generalized  $\phi_X(t)$  in order to obtain complex non-integer moments.

Recently, fractional moments of the type  $E[X^{kq}]$  have been introduced [2], showing that such quantities have important features: (i) they are exact natural generalization of integer moments as like as fractional differential operators generalize the classical differential calculus; (ii) the interesting point is the relationship between fractional moments and the fractional special functions.

**Corresponding author: F. Gharari<sup>\*2</sup>**

**<sup>2</sup>Department of Mathematics, University of Mohaghegh Ardabili, Iran  
Daneshgah Avenue, Ardabil. E-mail: [fatemeh.gharari@yahoo.com](mailto:fatemeh.gharari@yahoo.com)**

In this work, at first, we defined a Mittag-leffler-type function  $E_q(x^q)$  and its complex type, hereafter called the generalized exponential function. This function is a product of Mittag-leffler function and a power function. Using complex type of this function, we defined generalized Fourier transform. The obtained transform provided a suitable generalization of the characteristic function of random variables; that is using the expectation of complex generalized exponential function, we could directly obtain the generalized characteristic function GCF of a certain random variable. It was shown that complex fractional moments which are complex moments of order  $nq^{\text{th}}$  of a certain distribution, are equivalent to Caputa fractional derivation of the GCF in origin,  $n$  being a positive integer and  $0 < q \leq 1$ . The case  $q=1$  was reduced to the complex moments. In continue, after introducing fractional factorial moments of a positive random variable, we presented the relationship between integer moments, fractional moments (FM) and fractional factorial moments (FFMs) of a positive random variable.

Our main means of Fractional Calculus for this generalization were Reimann-Liouville and Caputo operators, fractional Taylor series.

## 2. PRELIMINARIES

In this section, we briefly review the definitions of fractional integrals and fractional derivatives, and the formal fractional right Riemann- Liouville Taylor series.

**Definition: 1** Let  $f(x)$  is a function defined on the interval  $[a,b]$  and  $q$  is a positive real number. The right Riemann-Liouville fractional integral is defined by:

$${}_a I_x^q f(x) = \frac{1}{\Gamma(q)} \int_a^x (x-t)^{q-1} f(t) dt, \quad -\infty \leq a < x < \infty \quad (4)$$

and also the right "Riemann -Liouville fractional derivative" is defined by:

$${}_a D_x^q f(x) = \left( \frac{d}{dx} \right)^n \left( {}_a I_x^{n-q} f(x) \right). \quad (5)$$

**Definition: 2** Let  $n = [q] + 1$ , the right Caputo fractional derivative  $({}_a^C D_x^q f)(x)$  is defined by:

$${}_a I_x^{n-q} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-q)} \int_a^x (x-t)^{n-q-1} \frac{d^n}{dt^n} f(t) dt, \quad (6)$$

and the sequential fractional derivatives is given by:

$${}_a^C D_x^{kq} = \underbrace{{}_a^C D_x^q \quad {}_a^C D_x^q \quad \dots \quad {}_a^C D_x^q}_{K\text{times}}$$

**Definition: 3** Let  $f(x)$  be a function defined on the right neighborhood of  $a$ , and be an infinitely fractionally-differentiable function at  $a$ , that is to say, all  $({}_a^C D_x^q)^k f(x)$ , ( $k = 0,1,2,\dots$ ) exist. The formal fractional right Riemann- Liouville Taylor series of a function is

$$f(x) = \sum_{k=0}^{\infty} ({}_a^C D_x^q)^k f(x) \Big|_{x=a} \cdot \left[ ({}_a I_x^q)^k (1) \right], \quad (7)$$

explicitly

$$\left( {}_a I_x^q \right)^k (1) = \frac{1}{\Gamma(kq+1)} (x-a)^{kq} .$$

where,  ${}_a^C D_x^q$  is the right Caputa fractional derivative and  ${}_a I_x^q$  is the right Riemann- Liouville fractional integral .

The fractional Taylor series of an infinitely fractionally differentiable function is based on fundamental theorem of Fractional Calculus (see [6]). By fundamental theorem of fractional calculus, one can say that the right Caputa fractional derivative operation and the right Riemann- Liouville fractional integral operation are in inverse to each other.

### 3. GENERALIZED FOURIER TRANSFORM

The explicit solutions to the equation

$$\left({}^C_0 D_x^q y\right) - \lambda y(x) = 0, \quad (x > 0), \quad n-1 < q \leq n; \quad n \in N, \quad \lambda \in R \quad (8)$$

in terms of this function, that is

$$y(x) = E_q(\lambda x^q).$$

Sequential fractional derivative of the function gives

$${}^C_0 D_x^{kq} y = \lambda^k y. \quad (9)$$

and in general case

$${}^C_a D_x^q E_q((x-a)^q) = E_q((x-a)^q) \quad (10)$$

In addition, the generalized exponential function satisfied

$$E_q(\lambda(x+y)^q) = E_q(\lambda x^q) E_q(\lambda y^q), \quad (11)$$

and

$$E_q(\lambda(x-x)^q) = E_q(\lambda x^q) E_q(\lambda(-x)^q) = E_q(0) = 1,$$

Therefore

$$E_q(\lambda(-x)^q) = E_q((-1)^q \lambda x^q) = E_q^{-1}(\lambda x^q),$$

that is,  $E_q(x^q)$  is the fractional analogue of  $\text{Exp}(x)$ .

The fractional Taylor series of this function is as following:

$$E_q((x-a)^q) = \sum_{k=0}^{\infty} \left[ ({}^C_a I_x^q)^k (1) \right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} (x-a)^{kq}, \quad (12)$$

because,

$$\left({}^C_a D_x^q\right)^k E_q((x-a)^q) \Big|_{x=a} = 1. \quad (13)$$

It can be seen that,

$$L\{E_q(x^q)\} = \frac{s^{q-1}}{s^q - 1}, \quad (14)$$

where  $L$  is Laplace transform. With substitutions  $q=1$  and  $a=0$  the results (8) to (14) have valid for the elementary exponential function.

We define the generalized exponential function,  $E_q(x^q)$  by the series below

$$\sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)}, \quad (15)$$

and we have the complex generalized exponential function as following:

$$E_q((ix)^q) = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)} (i)^{kq} = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)} e^{\frac{i\pi kq}{2}}, \quad (16)$$

and also we have:

$$E_q((-ix)^q) = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)} (-i)^{kq} = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)} e^{\frac{-i\pi kq}{2}}. \quad (17)$$

Now that we have a generalization of the complex exponential function, it should; of course, be possible to construct a generalization of the Euler relation, that being

$$E_q((ix)^q) = \cos_q(x^q) + i \sin_q(x^q). \quad (18)$$

From the real part of (16) we obtain the equation for the generalized cosine function

$$\cos_q(x^q) = \frac{1}{2} (E_q((ix)^q) + E_q((-ix)^q))$$

where by using (16) and (17) in recent equation, we can rewrite:

$$\cos_q(x^q) = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)} \cdot \cos \frac{kq\pi}{2},$$

So that in the case  $q=1$ , we have:

$$\cos_1(x^1) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} \cdot \cos \frac{k\pi}{2} = \cos(x),$$

and from the imaginary part of (16) we obtain the equation for the generalized sine function

$$\sin_q(x^q) = \frac{1}{2i} (E_q((ix)^q) - E_q((-ix)^q)),$$

where by using (16) and (17) in recent equation, we can rewrite:

$$\sin_q(x^q) = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)} \cdot \sin \frac{kq\pi}{2},$$

So that in the special case  $q=1$ , we have:

$$\sin_1(x^1) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(k+1)} \cdot \sin \frac{k\pi}{2} = \sin(x).$$

Also we have

$$E_q((i(x+y))^q) = E_q((ix)^q) E_q((iy)^q),$$

Therefore we conclude that the function  $E_q((ix)^q)$  is periodic with period  $T_q$  defined as the solution of the equation  $E_q(i^q(T_q)^q) = 1$ .

**Definition: 4** Let  $f(x): R \rightarrow C$ ,  $x \rightarrow f(x)$ . The generalized Fourier transform of the function  $f$  is defined by integral

$$\hat{f}_q(s) = \int_{-\infty}^{\infty} E_q((isx)^q) f(x) dx, \quad s \in C \quad (19)$$

and for  $q=1$ , we have the classical Fourier transform

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x) dx,$$

and inverse Fourier transform is as following:

$$f(x) = \frac{1}{T_q} \int_{-\infty}^{\infty} E_q((-isx)^q) \hat{f}_q(s) ds.$$

#### 4. THE GENERALIZED CHARACTERISTIC FUNCTION (GCF) OF A RANDOM VARIABLE

**Definition: 5** The generalized characteristic function of any random variable  $X$ ,  $\tilde{\phi}_X(t)$  is defined by:

$$\tilde{\phi}_X(t) = \langle E_q((iXt)^q) \rangle, \quad (20)$$

where,  $E_q((ixt)^q)$  is the generalized exponential function. In the special case we  $q=1$ , obtain the ordinary characteristic function

$$\phi_X(t) = \langle \text{Exp}(iXt) \rangle.$$

**Theorem: 1** Suppose that the fractional generalized characteristic function of a random variable  $X$  is finite in some open interval containing zero. Then, all the complex fractional moments exist and

$$\tilde{\phi}_X(t) = \sum_{k=0}^{\infty} \langle (iX)^{kq} \rangle \frac{t^{kq}}{\Gamma(kq+1)}, \quad (21)$$

that is, the complex fractional moments are the coefficients of the fractional Mac-Lourin series of  $\tilde{\phi}_X(t)$  and the generalized characteristic function is infinitely fractionally differentiable in that open interval, and for  $0 < q \leq 1$  and  $k = 1, 2, \dots$

$$\langle (iX)^{kq} \rangle = ({}^C D_x^{kq}) (\tilde{\phi}_X(t)) \Big|_{t=0} = ({}^C D_x^q)^k (\tilde{\phi}_X(t)) \Big|_{t=0}, \quad (22)$$

also in the special case  $q=1$  we obtain:

$$\langle (iX)^k \rangle = ({}^C D_x^q)^k \phi_X(t) \Big|_{t=0} = \phi_X^{(k)}(t) \Big|_{t=0}.$$

**Proof:** Since the fractional Mac-Lourin series of  $E_q((ix)^q)$  is

$$E_q((ix)^q) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(kq+1)} (ix)^{kq}$$

it can be written:

$$\tilde{\phi}_X(t) = \langle E_q((iXt)^q) \rangle = \left\langle \sum_{k=0}^{\infty} \frac{(iXt)^{kq}}{\Gamma(kq+1)} \right\rangle = \sum_{k=0}^{\infty} \langle (iX)^{kq} \rangle \frac{t^{kq}}{\Gamma(kq+1)}$$

in the other hand, by using (9), we have:

$$\begin{aligned} ({}^C D_x^q)^k \tilde{\phi}_X(t) &= ({}^C D_x^q)^k \langle E_q((iXt)^q) \rangle = \langle ({}^C D_x^{kq})(E_q((iXt)^q)) \rangle \\ &= \langle (iX)^{kq} E_q((iXt)^q) \rangle. \end{aligned}$$

#### 5. THE FRACTIONAL FACTORIAL MOMENTS (FFMS)

Stirling functions of the first kind,  $S(n,k)$ , can be defined via their generating function

$$(x)_n = \sum_{k=0}^n S(n,k) x^k, \quad (x \in C, n \in N_0) \quad (23)$$

where

$$(x)_n = x(x-1)(x-2)\dots(x-n+1) = \frac{\Gamma(1+x)}{\Gamma(1+x-n)},$$

and with the convention  $S(n,0) = \delta_{n,0}$  (Kronecker's delta). The latter gives a natural possibility to define "Stirling number of fractional order"  $S(q,k)$  with  $q \in \mathbb{C}$  and  $k \in \mathbb{N}_0$ . In fact, these "Stirling functions", as one may call them, which were introduced by Butzer, Hauss and Schmidt [1], may be defined via the generating function

$$(x)_q = \frac{\Gamma(x+1)}{\Gamma(x-q+1)} = \sum_{k=0}^{\infty} S(q,k)x^k, \quad (|x| < 1, q \in \mathbb{C}). \quad (24)$$

**Theorem: 2** Suppose  $X$  be a random variable with support  $[0, \infty)$  and  $G_X(z)$  is the probability generating function and finite in some open interval containing the origin. Then  $G_X(z)$  is infinitely fractional differentiable in that open interval, and if  $\lim_{z \rightarrow 1} G_X(z)$  is finite, then  $\langle X(X-1)\dots(X-qk+1) \rangle$  exists and is finite,

$$\lim_{z \rightarrow 1} G_X^{(qk)}(z) = \langle X(X-1)\dots(X-qk+1) \rangle,$$

where the notation  $(qk)$  means the right Caputo fractional derivative  $({}^C D_x^{kq} f)(x)$  and in the case  $q=1$ , we have:

$$\lim_{z \rightarrow 1} G_X^{(k)}(z) = \langle X(X-1)\dots(X-k+1) \rangle,$$

where

$$G_X(z) = \sum_{k=0}^{\infty} P(X=k) z^k. \quad (25)$$

**Proof:** It flows from the convergence of series (25) for  $|z|=1$  and from Weierstrass theorem on uniformly convergent series of analytic functions that  $G_X(z)$  is a analytic function of  $z$  in  $|z| < 1$ , with (25) as its power series expansion. Since  $G_X(z)$  is analytic, it is infinitely fractionally differentiable, and one can write meaningfully the fractional Taylor expansion

$$G_X(z) = \sum_{k=0}^{\infty} \frac{z^{kq}}{\Gamma(kq+1)} G_X^{(kq)}(0), \quad 0 < q \leq 1$$

also,

$$\frac{d^{qk}}{dz^{qk}} (G_X(z) = \langle z^X \rangle),$$

$$G_X^{(qk)}(z) = \left\langle \frac{d^{qk}}{dz^{qk}} (z^X) \right\rangle,$$

we have,

$$G_X^{(qk)}(z) = \left\langle \frac{\Gamma(1+X)}{\Gamma(1+X-qk)} z^{X-qk} \right\rangle,$$

$$\lim_{z \rightarrow 1} G_X^{(qk)}(z) = \left\langle \frac{\Gamma(1+X)}{\Gamma(1+X-qk)} \right\rangle = \langle (X)_{qk} \rangle,$$

and if  $q=1$ , we have,

$$\lim_{z \rightarrow 1} G_X^{(k)}(z) = \left\langle \frac{\Gamma(1+X)}{\Gamma(1+X-k)} \right\rangle = \langle (X)_k \rangle.$$

## 5. THE RELATIONSHIP BETWEEN INTEGER MOMENTS WITH FMS AND FFMS

In this section, another theorem which brings the relationship between integer moments, FMs and FFMs is presented for a positive random variable.

**Theorem: 3** Suppose  $X$  be a random variable with support  $[0, \infty)$ .

(i) fractional moments of  $X$  exist if and only if integer moments exist.

(ii) fractional factorial moments of X exist if and only if integer moments exist.

**Proof:** (i) let  $q$  be a non-integer number, it can be written:

$$\begin{aligned} x^q &= \sum_{n=0}^{\infty} \frac{1}{n!} (x^q)^{(n)}(a) (x-a)^n, \quad a > 0 \\ &= \sum_{n=0}^{\infty} \frac{(a)_n}{n! a^{n-q}} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} x^k a^{n-k}, \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^{n-k} a^{n-k} x^k (a)_n}{k!(n-k)! a^{n-q}}, \end{aligned}$$

then, we have :

$$x^q = \sum_{n=0}^{\infty} \sum_{k=0}^n \frac{(-1)^{n-k} a^{q-k} (a)_n}{k!(n-k)!} x^k,$$

by taking expectation of recent equation, we have :

$$\langle X^q \rangle = \sum_{n=0}^{\infty} \sum_{k=0}^n c(n, k, q, a) \langle X^k \rangle.$$

(ii) By taking expectation of expression (24), we have:

$$\langle (X)_q \rangle = \sum_{k=0}^n S(q, k) \langle X^k \rangle.$$

## 6. REFERENCES

- [1] Butzer P. L., Hauss M., Schmid M., Fractional functions and Stirling numbers of fractional order, *Resoltate math*, 16 (1989) 16-48.
- [2] Ganji M., Eghbali N., Gharari F., Some fractional special functions and fractional moments, *Gen. Math. Notes*, (2013), 120-127.
- [3] Hilfer R., *Applications of fractional calculus in physics*, World Scientific, Singapore, (2000), 51-56.
- [4] Kilbas A. A., Srivastava H. M., Trujillo J. J., *Theory and Application of fractional Differential Equations*, El-sevier, Amsterdam, (2006).
- [5] Li M., Rongren J., Zhu T., Fractional vector calculus and fractional special function, arxiv: 1001.2889v1 [math-ph] 17 Jan 2010.
- [6] Li M., Rongren J., Zhu T., Series expansion in fractional calculus and fractional differential equations, arxiv: 0910.4819v2 [math-ph] 30 Oct 2009.
- [7] Oldham K., Spanier J., *The fractional calculus*, Academic Press, New York, (1974).
- [8] Podlubny I., *fractional differential equations*, Academic Press, San Diego, (1999).
- [9] Vasily E., Tarasov, *Annals of physics* 323, (2008), 2756-2778.

**Source of support: Nil, Conflict of interest: None Declared**