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GENERALIZED FOURIER TRANSFORM FOR THE GENERATION OF COMPLEX FRACTIONAL MOMENTS

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ABSTRACT

Fourier transform of fractional order using the Mittag-Leffler-type function $E_q(x^q)$ and its complex type, was introduced together with its inversion formula. The obtained transform provided a suitable generalization of the characteristic function of random variables. It was shown that complex fractional moments which are complex moments of order nq$^{\text{th}}$ *of a certain distribution, are equivalent to Caputa fractional derivation of generalized characteristic function* (GCF) *in origin, n being a positive integer and* $0 < q \le 1$. *The case* q=1 *was reduced to the complex moments. Finally, after introducing fractional factorial moments of a positive random variable, we presented the relationship between integer moments, fractional moments* (FMs) *and fractional factorial moments* (FFMs) *of a positive random variable.*

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1. INTRODUCTION

It is well know that the Fourier transform of probability density function is characteristic function, that is

$$\varphi_X(t) = \left\langle e^{itX} \right\rangle = \int_{-\infty}^{\infty} e^{itx} f(x)dx, \tag{1}$$

Where the notation $\langle . \rangle$ *means expectation and on the other hand, we have:*

$$\phi_X(t) = \sum_{k=0}^{\infty} \left\langle (iX)^k \right\rangle \frac{t^k}{k!}, \tag{2}$$

This function generates complex moments of integer order, as we have:

$$\left\langle (iX)^k \right\rangle = \frac{d^k \phi_X(t)}{du^k} \Big|_{t=0}. \tag{3}$$

But in this work, we generalized $\phi_X(t)$ *in order to obtain complex non-integer moments.*

Recently, fractional moments of the type E[X^{kq}] *have been introduced* [2], *showing that such quantities have important features:* (i) *they are exact natural generalization of integer moments as like as fractional differential operators generalize the classical differential calculus;* (ii) *the interesting point is the relationship between fractional moments and the fractional special functions.*

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In this work, at first, we defined a Mittag-leffler-type function $\mathrm{E}_q\left(x^q\right)$ *and its complex type, hereafter called the generalized exponential function. This function is a product of Mittag-leffler function and a power function. Using complex type of this function, we defined generalized Fourier transform. The obtained transform provided a suitable generalization of the characteristic function of random variables; that is using the expectation of complex generalized exponential function, we could directly obtain the generalized characteristic function* GCF *of a certain random variable. It was shown that complex fractional moments which are complex moments of order* nq^{th} *of a certain distribution, are equivalent to Caputa fractional derivation of the* GCF *in origin, n being a positive integer and* $0 < q \leq 1$. *The case* q=1 *was reduced to the complex moments. In continue, after introducing fractional factorial moments of a positive random variable, we presented the relationship between integer moments, fractional moments* (FMs) *and fractional factorial moments* (FFMs) *of a positive random variable.*

Our main means of Fractional Calculus for this generalization were Reimann-Liouville and Caputo operators, fractional Taylor series.

2. PRELIMINARIES

In this section, we briefly review the definitions of fractional integrals and fractional derivatives, and the formal fractional right Riemann- Liouville Taylor series.

Definition: 1 *Let* $f(x)$ *is a function defined on the interval* [a,b] *and* q *is a positive real number. The right Riemann-Liouville fractional integral is defined by:*

$$ {}_aI_x^q f(x) = \frac{1}{\Gamma(q)}\int_a^x (x-t)^{q-1} f(t)dt, \qquad -\infty \leq a < x < \infty \tag{4}$$

and also the right "Riemann –Liouville fractional derivative" is defined by:

$$ {}_aD_x^q f(x) = \left(\frac{d}{dx}\right)^n \left({}_aI_x^{n-q} f(x)\right). \tag{5}$$

Definition: 2 *Let* $n = [q]+1$, *the right Caputo fractional derivative* $\left({}_a^CD_x^q f\right)(x)$ *is defined by:*

$$ {}_aI_x^{n-q} \frac{d^n}{dx^n} f(x) = \frac{1}{\Gamma(n-q)}\int_a^x (x-t)^{n-q-1} \frac{d^n}{dt^n} f(t)dt, \tag{6}$$

and the sequential fractional derivatives is given by:

$$ {}_a^CD_x^{kq} = \underbrace{{}_a^CD_x^q\, {}_a^CD_x^q \ldots {}_a^CD_x^q}_{Ktimes}$$

Definition: 3*Let* $f(x)$ *be a function defined on the right neighborhood of a, and be an infinitely fractionally-differentiable function at a, that is to say, all* $\left({}_a^CD_x^q\right)^k f(x), (k = 0,1,2,\ldots)$ *exist. The formal fractional right Riemann- Liouville Taylor series of a function is*

$$ f(x) = \sum_{k=0}^{\infty} \left({}_a^CD_x^q\right)^k f(x)\ \Big|_{x=a} . \left[\left({}_aI_x^q\right)^k (1)\right], \tag{7}$$

expilicity

$$ \left({}_0I_x^q\right)^k (1) = \frac{1}{\Gamma(kq+1)}(x-a)^{kq} \ .$$

where, ${}_a^CD_x^q$ *is the right Caputa fractional derivative and* ${}_aI_x^q$ *is the right Riemann- Liouville fractional integral .*

The fractional Taylor series of an infinitely fractionally differentiable function is based on fundamental theorem of Fractional Calculus (see [6]*). By fundamental theorem of fractional calculus, one can say that the right Caputa fractional derivative operation and the right Riemann- Liouville fractional integral operation are in inverse to each other.*

3. GENERALIZED FOURIER TRANSFORM

The explicit solutions to the equation

$$\left({}_0^C D_x^q y\right) - \lambda y(x) = 0 \quad ,\left(x\rangle 0,\ n-1\langle q \le n;\ n \in N,\ \lambda \in R\right) \tag{8}$$

in terms of this function, that is

$$y(x) = \mathrm{E}_q\left(\lambda x^q\right) .$$

Sequential fractional derivative of the function gives

$${}_0^C D_x^{kq} y = \lambda^k y. \tag{9}$$

and in general case

$${}_a^C D_x^q \ \mathrm{E}_q\left(\left(x-a\right)^q\right) = \mathrm{E}_q\left(\left(x-a\right)^q\right) \tag{10}$$

In addition, the generalized exponential function satisfied

$$E_q\left(\lambda\left(x+y\right)^q\right) = E_q\left(\lambda x^q\right) E_q\left(\lambda y^q\right) \ , \tag{11}$$

and

$$E_q\left(\lambda\left(x-x\right)^q\right) = E_q\left(\lambda x^q\right) E_q\left(\lambda\left(-x\right)^q\right) = E_q\left(0\right) = 1 \ ,$$

Therefore

$$E_q\left(\lambda\left(-x\right)^q\right) = E_q\left(\left(-1\right)^q \lambda x^q\right) = E_q^{-1}\left(\lambda x^q\right) \ ,$$

that is, $\mathrm{E}_q\left(x^q\right)$ *is the fractional analogue of* Exp (x).

The fractional Taylor series of this function is as following:

$$\mathrm{E}_q\left(\left(x-a\right)^q\right) = \sum_{k=0}^{\infty}\left[\left({}_a I_x^q\right)^k (1)\right] = \sum_{k=0}^{\infty} \frac{1}{\Gamma\left(kq+1\right)} \left(x-a\right)^{kq}, \tag{12}$$

because,

$$\left({}_a^C D_x^q\right)^k E_q\left(\left(x-a\right)^q\right) \Big|_{x=a} = 1. \tag{13}$$

It can be seen that,

$$L\left\{\mathrm{E}_q\left(x^q\right)\right\} = \frac{s^{q-1}}{s^q - 1}, \tag{14}$$

where L *is Laplace transform. With substitutions* q=1 *and a=0 the results* (8) *t0* (14) *have valid for the elementary exponential function.*

We define the generalized exponential function, $\mathrm{E}_q\left(x^q\right)$ *by the series below*

$$\sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma\left(kq+1\right)} \ , \tag{15}$$

and we have the complex generalized exponential function as following:

$$\mathrm{E}_q\left((ix)^q\right) = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma\left(kq+1\right)} . (i)^{kq} = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma\left(kq+1\right)} . e^{\frac{i\pi kq}{2}} , \tag{16}$$

and also we have:

$$\mathrm{E}_q((-ix)^q) = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)}.(-i)^{kq} = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)}.e^{\frac{-i\pi kq}{2}}. \tag{17}$$

Now that we have a generalization of the complex exponential function, it should; of course, be possible to construct a generalization of the Euler relation, that being

$$\mathrm{E}_q((ix)^q) = \cos_q(x^q) + i\sin_q(x^q). \tag{18}$$

From the real part of (16) *we obtain the equation for the generalized cosine function*

$$\cos_q(x^q) = \frac{1}{2}(\mathrm{E}_q((ix)^q) + \mathrm{E}_q((-ix)^q))$$

where by using (16) *and* (17) *in recent equation, we can rewrite:*

$$\cos_q(x^q) = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)}.\cos\frac{kq\pi}{2},$$

So that in the case q=1, *we have:*

$$\cos_1(x^1) = \sum_{k=0}^{\infty} \frac{x^{k}}{\Gamma(k+1)}.\cos\frac{k\pi}{2} = \cos(x),$$

and from the imaginary part of (16) *we obtain the equation for the generalized sine function*

$$\sin_q(x^q) = \frac{1}{2i}(\mathrm{E}_q((ix)^q) - \mathrm{E}_q((-ix)^q)),$$

where by using (16) *and* (17) *in recent equation, we can rewrite:*

$$\sin_q(x^q) = \sum_{k=0}^{\infty} \frac{x^{kq}}{\Gamma(kq+1)}.\sin\frac{kq\pi}{2},$$

So that in the special case q=1, *we have:*

$$\sin_1(x^1) = \sum_{k=0}^{\infty} \frac{x^{k}}{\Gamma(k+1)}.\sin\frac{k\pi}{2} = \sin(x).$$

Also we have

$$E_q\left((i(x+y))^q\right) = E_q\left((ix)^q\right)\ E_q\left((iy)^q\right),$$

Therefore we conclude that the function $\mathrm{E}_q\left((ix)^q\right)$ *is periodic with period* T_q *defined as the solution of the equation* $E_q(i^q(T_q)^q) = 1.$

Definition: 4 *Let* $f(x): R \to C,\quad x \to f(x)$*.The generalized Fourier transform of the function* f *is defined by integral*

$$\hat{f}_q(s) = \int_{-\infty}^{\infty} E_q((isx)^q)\, f(x)dx, \qquad s \in C \tag{19}$$

and for q=1, *we have the classical Fourier transform*

$$\hat{f}(s) = \int_{-\infty}^{\infty} e^{isx} f(x)dx,$$

and inverse Fourier transform is as following:

$$f(x)=\frac{1}{T_q}\int_{-\infty}^{\infty}E_q((-isx)^q)\,\hat{f}_q(s)ds.$$

4. THE GENERALIZED CHARACTERISTIC FUNCTION (GCF) OF A RANDOM VARIABLE

Definition: 5 *The generalized characteristic function of any random variable* X, $\tilde{\phi}_X(t)$ *is defined by:*

$$\tilde{\varphi}_X(t)=\left\langle E_q((iXt)^q)\right\rangle, \tag{20}$$

where, $\mathrm{E}_q\left((ixt)^q\right)$ *is the generalized exponential function. In the special case we* q=1, *obtain the ordinary characteristic function*

$$\phi_X(t)=\left\langle Exp(iXt)\right\rangle.$$

Theorem: 1 *Suppose that the fractional generalized characteristic function of a random variable* X *is finite in some open interval containing zero. Then, all the complex fractional moments exist and*

$$\tilde{\varphi}_X(t)=\sum_{k=o}^{\infty}\left\langle (iX)^{kq}\right\rangle\frac{t^{kq}}{\Gamma(kq+1)}, \tag{21}$$

that is, the complex fractional moments are the coefficients of the fractional Mac-Lourin series of $\tilde{\phi}_X(t)$ *and the generalized characteristic function is infinitely fractionally differentiable in that open interval, and for 0*<q ≤1 *and* $k=1,2,\ldots$

$$\left\langle (iX)^{kq}\right\rangle=\left({}_0^C D_x^{kq}\right)\ (\tilde{\varphi}_X(t))\ \Big|_{t=0}=\ ({}_0^C D_x^{q})^k(\tilde{\varphi}_X(t))\ \Big|_{t=0}, \tag{22}$$

also in the special case q=1 *we obtain:*

$$\left\langle (iX)^{k}\right\rangle=\left({}_0^C D_x^{q}\right)^k\phi_X(t)\ \Big|_{t=0}=\phi_X^{(k)}(t)\ \Big|_{t=0}\ .$$

Proof: *Since the fractional Mac-Lourin series of* $\mathrm{E}_q\left((ix)^q\right)$ *is*

$$\mathrm{E}_q\left((ix)^q\right)==\sum_{k=0}^{\infty}\frac{1}{\Gamma(kq+1)}(ix)^{kq}$$

it can be written:

$$\tilde{\varphi}_X(t)=\left\langle E_q((iXt)^q)\right\rangle=\left\langle\sum_{k=0}^{\infty}\frac{(iXt)^{qk}}{\Gamma(kq+1)}\right\rangle=\sum_{k=0}^{\infty}\left\langle (iX)^{kq}\right\rangle\frac{t^{kq}}{\Gamma(kq+1)}$$

in the other hand, by using (9), *we have:*

$$\begin{aligned}\left({}_0^C D_x^{q}\right)^k\tilde{\phi}_X(t)=\left({}_0^C D_x^{q}\right)^k\left(\left\langle E_q((iXt)^q)\right\rangle\right)&=\left\langle ({}_0^C D_x^{kq})(E((iXt)^q))\right\rangle\\&=\left\langle (iX)^{kq}E_q((iXt)^q)\right\rangle\ .\end{aligned}$$

5. THE FRACTIONAL FACTORIAL MOMENTS (FFMS)

Stirling functions of the first kind, S(n,k), *can be defined via their generating function*

$$(x)_n=\sum_{k=0}^{n}S(n,k)\ x^n,\qquad \left(x\in C,\ n\in N_0\right) \tag{23}$$

where

$$(x)_n=x(x-1)(x-2)\ldots(x-n+1)=\frac{\Gamma(1+x)}{\Gamma(1+x-n)},$$

and with the convention S(n,0) = $\delta_{n,0}$ (Kronecker' s delta)*. The latter gives a natural possibility to define "Stirling number of fractional order "* S(q,k) *with* $q \in C$ *and* $k \in N_0$*. In fact, these "Stirling functions", as one may call them, which were introduced by Butzer, Hauss and Schmidt* [1]*, may be defined via the generating function*

$$(x)_q = \frac{\Gamma(x+1)}{\Gamma(x-q+1)} = \sum_{k=0}^{\infty} S(q,k) x^k, \qquad (|x|<1,\ q \in C). \tag{24}$$

Theorem: 2 *Suppose* X *be a random variable with support* $[0,\infty)$ *and* $G_X(z)$ *is the probability generating function and finite in some open interval containing the origin. Then* $G_X(z)$ *is infinitely fractional differentiable in that open interval, and if* $\lim_{z\to 1} G_X(z)$ *is finite, then* $\langle X(X-1)\ldots(X-qk+1)\rangle$ *exists and is infinite,*

$$\lim_{Z\to 1} G_X^{(qK)}(z) = \langle X(X-1)\ldots(X-qk+1)\rangle,$$

where the notation (qk) *means the right Caputo fractional derivative* $({}_a^C D_x^{kq} f)(x)$ *and in the case* q=1*, we have:*

$$\lim_{Z\to 1} G_X^{(K)}(z) = \langle X(X-1)\ldots(X-k+1)\rangle,$$

where

$$G_X(z) = \sum_{k=o}^{\infty} P(x=k)\ z^k. \tag{25}$$

Proof: *It flows from the convergence of series (25) for* $|z|=1$ *and from Weierstrass theorem on uniformly convergent series of analytic functions that* $G_X(z)$ *is a analytic function of* z *in* $|z|<1$*, with (25) as its power series expansion. Since* $G_X(z)$ is *analytic, it is infinitely fractionally differentiable, and one can write meaningfully the fractional Taylor expansion*

$$G_X(z) = \sum_{k=o}^{\infty} \frac{z^{kq}}{\Gamma(kq+1)} G_X^{(qk)}(0), \qquad 0 \langle q \le 1$$

also,

$$\frac{d^{qk}}{dz^{qk}}\left(G_X(z) = \langle z^X \rangle\right),$$

$$G_X^{(qk)}(z) = \left\langle \frac{d^{qk}}{dz^{qk}}(z^X) \right\rangle,$$

we have,

$$G_X^{(qk)}(z) = \left\langle \frac{\Gamma(1+X)}{\Gamma(1+X-qk)} z^{X-qk} \right\rangle,$$

$$\lim_{z\to 1} G_X^{(qk)}(z) = \left\langle \frac{\Gamma(1+X)}{\Gamma(1+X-qk)} \right\rangle = \langle (X)_{qk} \rangle,$$

and if q=1*, we have,*

$$\lim_{z\to 1} G_X^{(K)}(z) = \left\langle \frac{\Gamma(1+X)}{\Gamma(1+X-k)} \right\rangle = \langle (X)_k \rangle.$$

5. THE RELATIONSHIP BETWEEN INTEGER MOMENTS WITH FMS AND FFMS

In this section, another theorem which brings the relationship between integer moments, FMs and FFMs is presented for a positive random variable.

Theorem: 3 *Suppose* X *be a random variable with support [*0, ∞).

(i) fractional moments of X *exist if and only if integer moments exist.*

(ii) fractional factorial moments of X *exist if and only if integer moments exist.*

Proof: (i) *let* q *be a non-integer number, it can be written:*

$$x^q = \sum_{n=0}^{\infty} \frac{1}{n!}(x^q)^{(n)}(a)(x-a)^n, \quad a>0$$

$$= \sum_{n=0}^{\infty} \frac{(a)_n}{n! a^{n-q}} \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} x^k a^{n-k},$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k} a^{n-k} x^k (a)_n}{k!(n-k)! a^{n-q}},$$

then, we have :

$$x^q = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{n-k} a^{q-k} (a)_n}{k!(n-k)!} \; x^k,$$

by taking expectation of recent équation, we have :

$$\left\langle X^q \right\rangle = \sum_{n=0}^{\infty} \sum_{k=0}^{n} c(n,k,q,a) \; \left\langle X^k \right\rangle.$$

(ii) *By taking expectation of expression (24), we have:*

$$\left\langle (X)_q \right\rangle = \sum_{k=0}^{n} S(q,k) \; \left\langle X^k \right\rangle.$$

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