

ON g^* - \mathcal{G} -CLOSED SETS

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ABSTRACT

In this paper, we introduced and study the notions of g^* - \mathcal{G} -closed sets and study some of their properties.

1. INTRODUCTION

In 1970, Levine [8] first introduced the concept of generalized closed (briefly, g -closed) sets were defined and investigated. The idea of grill on a topological space was first introduced by Choquet [4] in 1947. It is observed from literature that the concept of grills is a powerful supporting tool, like nets and filters, in dealing with many topological concept quite effectively. In [15], Roy and Mukherjee defined and studied a typical topology associated rather naturally to the existing topology and a grill on a given topological space. The aim of this paper is to introduce g^* - \mathcal{G} -closed sets and investigate the relations of g^* - \mathcal{G} -closed sets between such sets.

2. PRELIMINARIES

Throughout this paper, (X, τ) (or X) represent a topological space on which no separation axioms are assumed unless otherwise mentioned. For a subset A of a space X , $cl(A)$ and $int(A)$ denote the closure of A and the interior of A , respectively. The power set of X will be denoted by $\wp(X)$. A collection \mathcal{G} of a nonempty subsets of a space X is called a grill [1] on X if

- (1) $A \in \mathcal{G}$ and $A \subseteq B \Rightarrow B \in \mathcal{G}$.
- (2) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

For any point x of a topological space (X, τ) , $\tau(x)$ denote the collection of all open neighbourhoods of x . We recall the following results which are useful in the sequel.

Definition: 2.1 [15] Let (X, τ) be a topological space and \mathcal{G} be a grill on X . The mapping $\phi: \wp(X) \rightarrow \wp(X)$, denoted by $\phi_G(A, \tau)$ for $A \in \wp(X)$ or simply $\phi(A)$ called the operator associated with the grill \mathcal{G} and the topology τ and is defined by $\phi_G(A) = \{x \in X \mid A \cap U \in \mathcal{G}, \forall U \in \tau(x)\}$. Let \mathcal{G} be a grill on a space X . Then a map $\Psi: \wp(X) \rightarrow \wp(X)$ is defined by $\Psi(A) = A \cup \phi(A)$, for all $A \in \wp(X)$. The map Ψ satisfies Kuratowski closure axioms. Corresponding to a grill \mathcal{G} on a topological space (X, τ) , there exists a unique topology τ_G on X given by $\tau_G = \{U \subseteq X \mid \Psi(X-U) = X-U\}$, where for any $A \subseteq X$, $\Psi(A) = A \cup \phi(A) = \tau_G - cl(A)$. For any grill \mathcal{G} on a topological space by (X, τ, \mathcal{G}) .

Definition: 2.2 A subset A of a topological space (X, τ) is called

- 1) a pre-open set [12] if $A \subseteq int(cl(A))$ and a pre-closed set if $cl(int(A)) \subseteq A$.
- 2) a semi-open set [7] if $A \subseteq cl(int(A))$ and a semi-closed set if $int(cl(A)) \subseteq A$.
- 3) a semi-preopen set [1] if $A \subseteq cl(int(cl(A)))$ and a semi-pre-closed set [2] if $(int(cl(A))) \subseteq A$.

Definition: 2.3 A subset A of a topological space (X, τ) is called

- 1) a generalized closed set (briefly g -closed) [8] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- 2) a generalized semi-closed set (briefly gs -closed) [2] if $scl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- 3) an α -generalized closed set (briefly αg -closed) [9] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- 4) a generalized semi-preclosed set (briefly gsp -closed) [5] if $spcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- 5) a generalized preclosed set (briefly gp -closed) [11] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- 6) a generalized grill closed set (briefly $g\mathcal{G}$ -closed) [6] if $\Psi(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ) .
- 7) a generalized closed set (briefly g^* -closed) [17] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open in (X, τ) .

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Theorem: 2.4 [15]

- 1) If \mathcal{G}_1 and \mathcal{G}_2 are two grills on a space X with $\mathcal{G}_1 \subset \mathcal{G}_2$, then $\tau_{\mathcal{G}_1} \subset \tau_{\mathcal{G}_2}$.
- 2) If \mathcal{G} is a grill on a space X and $B \notin \mathcal{G}$, then B is closed in (X, τ, \mathcal{G}) .
- 3) For any subset A of a space X and any grill \mathcal{G} on X , $\phi(A)$ is $\tau_{\mathcal{G}}$ -closed.

Theorem: 2.5 [15] Let (X, τ) be a topological space and \mathcal{G} be any grill on X . Then

- 1) $A \subseteq B (\subseteq X) \Rightarrow \phi(A) \subseteq \phi(B)$;
- 2) $A \subseteq X$ and $A \notin \mathcal{G} \Rightarrow \phi(A) = \phi$;
- 3) $\phi(\phi(A)) \subseteq \phi(A) = \text{cl}(\phi(A)) \subseteq \text{cl}(A)$, for any $A \subseteq X$;
- 4) $\phi(A \cup B) = \phi(A) \cup \phi(B)$ for any $A, B \subseteq X$;
- 5) $A \subseteq \phi(A) \Rightarrow \text{cl}(A) = \tau_{\mathcal{G}}\text{-cl}(A) = \text{cl}(\phi(A)) = \phi(A)$;
- 6) $U \in \tau$ and $\tau \setminus \{ \phi \} \subseteq \mathcal{G} \Rightarrow U \subseteq \phi(U)$;
- 7) If $U \in \tau$ then $U \cap \phi(A) = U \cap \phi(U \cap A)$, for any $A \subseteq X$.

Theorem: 2.6 Let (X, τ) be a topological space and \mathcal{G} be any grill on X . Then, for any $A, B \subseteq X$.

- 1) $A \subseteq \Psi(A)$ [15];
- 2) $\Psi(\phi) = \phi$ [15];
- 3) $\Psi(A \cup B) = \Psi(A) \cup \Psi(B)$ [15];
- 4) $\Psi(\Psi(A)) = \Psi(A)$ [15];
- 5) $\text{int}(A) \subset \text{int}(\Psi(A))$;
- 6) $\text{int}(\Psi(A \cap B)) \subset \text{int}(\Psi(A))$;
- 7) $\text{int}(\Psi(A \cap B)) \subset \text{int}(\Psi(B))$;
- 8) $\text{int}(\Psi(A)) \subset \Psi(A)$;
- 9) $A \subseteq B \Rightarrow \Psi(A) \subseteq \Psi(B)$.

Theorem: 2.7 [16] Let (X, τ) be a topological space and \mathcal{G} be any grill on X . Then, for any $A, B \subseteq X$,

- (1) $\phi(A) \subseteq \Psi(A) = \tau_{\mathcal{G}}\text{-cl}(A) \subseteq \text{cl}(A)$;
- (2) $A \cup \Psi(\text{int}(A)) \subseteq \text{cl}(A)$;
- (3) $A \subseteq \phi(A)$ and $B \subseteq \phi(B) \Rightarrow \Psi(A \cap B) \subseteq \Psi(A) \cap \Psi(B)$.

3. g^* - \mathcal{G} -CLOSED SETS

Definition: 3.1 A subset A of (X, τ) is called a g^* - \mathcal{G} -closed set if $\Psi(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open in (X, τ) .

Theorem: 3.2 Every closed set is a g^* - \mathcal{G} -closed. But not conversely.

Proof: Let A be a closed set. Then $\text{cl}(A) = A$. Let U be any g -open set such that $A \subseteq U$. Then $\text{cl}(A) \subseteq U$. We know that $\Psi(A) = A \cup \phi(A) = \tau_{\mathcal{G}}\text{-cl}(A) \subseteq \text{cl}(A)$. Therefore $\Psi(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open.

Example: 3.3 Let $X = \{a, b, c\}$, $\tau = \{ \{ \phi \}, \{b\}, \{X\} \}$ and $\mathcal{G} = \{ \{a\}, \{a, b\}, \{X\} \}$. Let $A = \{c\}$. A is a g^* - \mathcal{G} -closed set but not closed set.

Theorem: 3.4 Every g -closed set is a g^* - \mathcal{G} -closed. But not conversely.

Proof: Let A be a g -closed set. Then $\text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open. We know that $\Psi(A) = A \cup \Psi(A) = \tau_{\mathcal{G}}\text{-cl}(A) \subseteq \text{cl}(A)$. Therefore $\Psi(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open.

Example: 3.5 Let $X = \{a, b, c\}$, $\tau = \{ \{ \phi \}, \{a\}, \{b\}, \{a, b\}, \{X\} \}$ and $\mathcal{G} = \{ \{a\}, \{a, b\}, \{X\} \}$. Let $A = \{c\}$. A is a g^* - \mathcal{G} -closed set but not g -closed set of (X, τ) .

Theorem: 3.6 Every g^* - \mathcal{G} -closed set is a gsp closed. But not conversely.

Proof: Let A be a g^* - \mathcal{G} -closed set in (X, τ) . Then $\Psi(A) \subseteq U$. Whenever $A \subseteq U$ and U is g -open. From the above theorem $\text{cl}(A) \subseteq U$. But every closed set is a semi-pre-closed set, we have $\text{spcl}(A) \subseteq U$ and also every open set is g -open. Therefore $\text{spcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Example: 3.7 Let $X = \{a, b, c\}$, $\tau = \{ \{ \phi \}, \{b\}, \{X\} \}$ and $\mathcal{G} = \{ \{a\}, \{a, b\}, \{X\} \}$. Let $A = \{b\}$. A is a g^* - \mathcal{G} -closed set but not gsp -closed set of (X, τ) .

Theorem: 3.8 Every g^* - \mathcal{G} -closed set is a gp -closed. But not conversely.

Proof: Let A be a g^* - g -closed set in (X, τ) . Then $\Psi(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open. From the above $\text{cl}(A) \subseteq U$. But every closed set is a pre-closed set, we have $\text{pcl}(A) \subseteq U$ and also every open set is g -open. Therefore $\text{pcl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Example: 3.9 Let $X = \{a, b, c\}$, $\tau = \{\{\phi\}, \{b\}, \{X\}\}$ and $\mathcal{g} = \{\{a\}, \{a, b\}, \{X\}\}$. Let $A = \{b\}$. Then A is g -closed set but not g^* - g -closed set of (X, τ) .

Remark: 3.10 If A and B are g^* - g -closed set, then $A \cup B$ is also a g^* - g -closed set.

Theorem: 3.11 Every g^* - g -closed set is a αg -closed set. But not conversely.

Proof: Let A be a g^* - g -closed set in (X, τ) . Then $\Psi(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open. We know $\text{cl}(A) \subseteq U$. From the above and every closed set is α -closed set. We have $\alpha \text{cl}(A) \subseteq U$ and also every open set is g -open. Therefore $\alpha \text{cl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Example: 3.12 Let $X = \{a, b, c\}$, $\tau = \{\{\phi\}, \{b\}, \{X\}\}$ and $\mathcal{g} = \{\{a\}, \{a, b\}, \{X\}\}$. Let $A = \{b\}$. Then A is a αg -closed set but not g^* - g -closed set of (X, τ) .

Theorem: 3.13 Every g^* - g -closed set is a g s-closed set. But not conversely.

Proof: Let A be g^* - g -closed set in (X, τ) . Then $\Psi(A) \subseteq U$ whenever $A \subseteq U$ and U is g -open. We know $\text{cl}(A) \subseteq U$ from the above and every closed set is semi-closed set. We have $\text{scl}(A) \subseteq U$ and also every open set is g -open. Therefore $\text{scl}(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

Example: 3.14 Let $X = \{a, b, c\}$, $\tau = \{\{\phi\}, \{b\}, \{X\}\}$ and $\mathcal{g} = \{\{a\}, \{a, b\}, \{X\}\}$. Let $A = \{b, c\}$. Then A is a g s-closed set but not a g^* - g -closed set of (X, τ) .

Theorem: 3.15 A subset A of (X, τ) is a g^* - g -closed set if and only if $\Psi(A) - A$ does not contain any non-empty g - g -closed set.

Proof: Necessity: Let F be a g - g -closed set of (X, τ) such that $F \subseteq \Psi(A) - A$. Then $A \subseteq X - F$.

Since A is g^* - g -closed and $X - F$ is g - g -open, $\Psi(A) \subseteq X - F$. This implies $F \subseteq X - \Psi(A)$.

So $F \subseteq (X - \Psi(A)) \cap (\Psi(A) - A) \subseteq (X - \Psi(A)) \cap \Psi(A) = \phi$. Therefore $F = \phi$.

Sufficiency: Suppose A is a subset of (X, τ) such that $\Psi(A) - A$ does not contain any non-empty g - g -closed set. Let U be a g - g -closed set of (X, τ) such that $A \subseteq U$. If $\Psi(A) \not\subseteq U$, then $\Psi(A) \cap \Psi(U) \neq \phi$. Since $\Psi(A)$ is a closed set. Then we have $\phi \neq \Psi(A) \cap \Psi(U)$ is a g - g -closed set of (X, τ) . Then $\phi \neq \Psi(A) \cap \Psi(U) \subseteq \Psi(A) - A$. So $\Psi(A) - A$ contains a non-empty g - g -closed set. This contradicts the hypothesis. Therefore A is a g^* - g -closed set.

Theorem: 3.16 If A is a g^* - g -closed set of (X, τ) such that $A \subseteq B \subseteq \Psi(A)$, then B is also a g^* - g -closed set of (X, τ) .

Proof: Let U be a g - g -closed set of (X, τ) such that $B \subseteq U$. Then $A \subseteq U$. Since A is g^* - g -closed, $\Psi(A) \subseteq U$. Now $\Psi(B) \subseteq \Psi(\Psi(A)) = \Psi(A) \subseteq U$. Therefore B is also a g^* - g -closed set of (X, τ) .

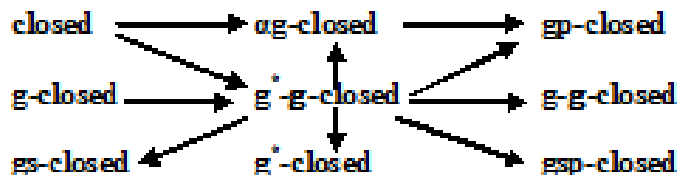
Theorem: 3.17 Every g^* - g -closed set is a g^* -closed. But not conversely.

Example: 3.18 Let $X = \{a, b, c\}$, $\tau = \{\{\phi\}, \{a\}, \{c\}, \{a, c\}, \{X\}\}$ and $\mathcal{g} = \{\{a\}, \{a, b\}, \{X\}\}$. Let $A = \{a, c\}$. Then A is a g^* -closed but not g^* - g -closed set of (X, τ) .

Theorem: 3.19 Every g^* - g -closed set is a g - g -closed set. But not conversely.

Example: 3.20 Let $X = \{a, b, c\}$, $\tau = \{\{\phi\}, \{a\}, \{c\}, \{a, c\}, \{X\}\}$ and $\mathcal{g} = \{\{a\}, \{a, b\}, \{X\}\}$. Let $A = \{c\}$. Then A is a g - g -closed set but not g^* - g -closed set of (X, τ) .

Remark: 3.21



4. g^* - \mathcal{G} -CONTINUOUS

Definition: 4.1 A function $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ is called g^* - \mathcal{G} -continuous if $f^{-1}(V)$ is a g^* - \mathcal{G} -closed set of (X, τ, \mathcal{G}) for every closed set V of (Y, σ) .

Theorem: 4.2 Every continuous map is g^* - \mathcal{G} -continuous. But not conversely.

Proof: Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ be a continuous map. Let V be a closed set in Y . Then $f^{-1}(V)$ is closed. Since every closed set is g^* - \mathcal{G} -closed set. We have $f^{-1}(V)$ is a g^* - \mathcal{G} -closed set. Therefore f is g^* - \mathcal{G} -continuous.

Example: 4.3 Let $X = Y = \{a, b, c\}$, $\tau = \{\{\phi\}, \{b\}, \{X\}\}$ and $\sigma = \{\{\phi\}, \{a\}, \{a, b\}, \{Y\}\}$ and $\mathcal{G} = \{\{a\}, \{a, b\}, \{X\}\}$. Define $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. f is not continuous since $\{c\}$ is a closed set of (Y, σ) . But $f^{-1}(c) = \{c\}$ is not closed set of (X, τ, \mathcal{G}) . However f is g^* - \mathcal{G} -continuous.

Theorem: 4.4 Every g -continuous map is g^* - \mathcal{G} -continuous map. But not conversely.

Proof: Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ be a g -continuous map. Let V be a g -closed set in Y . Then $f^{-1}(V)$ is g -closed. Since every g -closed is g^* - \mathcal{G} -closed set. We have $f^{-1}(V)$ is a g^* - \mathcal{G} -closed set. Therefore f is g^* - \mathcal{G} -continuous.

Example: 4.5 Let $X = Y = \{a, b, c\}$, $\tau = \sigma = \{\{\phi\}, \{a\}, \{a, b\}, \{x\}\}$ and $\mathcal{G} = \{\{a\}, \{a, c\}, \{x\}\}$. Define $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ by $f(a) = a$, $f(b) = b$ and $f(c) = c$. f is not g -continuous. Since $\{c\}$ is a closed set in (Y, σ) . But $f^{-1}(c) = \{c\}$ is not a g -closed set of (X, τ, \mathcal{G}) . However f is g^* - \mathcal{G} -continuous.

Theorem: 4.6 Every g^* - \mathcal{G} -continuous map is gsp continuous and hence an αg -continuous, gp -continuous and gs -continuous. But not conversely.

Proof: Let $f : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ be a g^* - \mathcal{G} -continuous map. Let V be a closed set of (Y, σ) . Since f is g^* - \mathcal{G} -continuous, $f^{-1}(V)$ is a g^* - \mathcal{G} -closed set of (X, τ, \mathcal{G}) . By the theorems 3.6, 3.8, 3.11 and 3.13, $f^{-1}(V)$ is gsp -closed, gp -closed, αg -closed and gs -closed set of (X, τ) .

Example: 4.7 Let $X = Y = \{a, b, c\}$, $\tau = \{\{\phi\}, \{a\}, \{a, b\}, \{x\}\}$, $\sigma = \{\{\phi\}, \{c\}, \{x\}\}$ and $\mathcal{G} = \{\{a\}, \{a, c\}, \{x\}\}$. Define $g : (X, \tau, \mathcal{G}) \rightarrow (Y, \sigma)$ by $g(a) = a$, $g(b) = b$ and $g(c) = c$. f is not g^* -continuous. Since $\{a, b\}$ is a closed set in (Y, σ) , but $f^{-1}(a, b) = \{a, b\}$ is not g^* - \mathcal{G} -closed. However f is gsp continuous, gp -continuous, αg -continuous and gs -continuous.

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