

α -COMPATIBLE MAPPINGS OF TYPE (P) AND COMMON α -FIXED POINT THEOREM

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(Received on: 04-03-13; Revised & Accepted on: 20-01-14)

ABSTRACT

In this paper we introduce the concept of α -compatible mappings of type (P), which is equivalent to the notion of α -compatible mappings as well as compatible mappings of type (P) under certain conditions. We prove a common α -fixed point theorem for four α -compatible mappings of type (P).

1. INTRODUCTION

The theory of fixed points is a quite popular and attractive area of researches in Mathematics. It has equally drawn attention of people working both in Pure as well as Applied Mathematics. Fixed points have long been used in Analysis to solve various kinds of differential and integral equations. It has wider applications to the theory of positive matrices.

The theory of fixed points took its proper shape with the landmark result of Polish mathematician S. Banach popularly known as Banach Contraction Principle. Till then many workers including S. Brouwer, J. Schauder, G. D. Birkoff, O. D. Kellog, M. Balanzat, Y. J. Cho, G. Jungck, B. Fisher, S. M. Kang, R. Kannan, R. P. Pant, etc. have contributed and given the present shape to the theory.

G. Jungck [1] has given a generalization of the Banach's contraction theorem by using the concept of commuting mappings. S. Sessa [9] generalized the concept of commuting mapping by using the concept of weakly commuting mappings.

Further G. Jungck [2] generalized weak commutativity by introducing the concept of compatible mappings. Jungck and others proved common fixed point theorems using this concept ([2]-[4], [7], [8]).

In [5] Jungck introduced the notion of weakly compatible maps. In [6] Jungck, introduced the concept of compatible mappings of type (A) and proved common fixed point theorems for compatible mappings of type (A) on a complete metric space.

Recently Pathak-Chang-Cho-Kang [11] introduced the concept of compatible mappings of type (P) in metric space (X, d) and compare it with the compatible and compatible mappings of type (A).

In a paper [14,15] author introduced the concept of α -fixed point, α -commuting mappings, weakly α -commuting mappings, α -compatible mappings, weakly α -compatible mappings and α -compatible mappings of type (A) and proved some common α -fixed point theorems.

In this paper, we introduce the concept of α -compatible mappings of type (P) in metric space (X, d) , which is equivalent to the concept of α -compatible mappings and as well as α -compatible mappings of type (A) under certain conditions. We prove a common α -fixed point theorem for four α -compatible mappings of type (P) on a complete metric spaces.

2. α -COMPATIBLE MAPPINGS OF TYPE (P)

In this section, we introduce the concept of α -compatible mappings of type (P) in metric space (X, d) and show that the concept of α -compatible mappings, α -compatible mappings of type (A) and α -compatible mappings of type (P) are equivalent under some conditions and give some properties of α -compatible mappings of type (P) for our main result. Throughout this paper (X, d) denotes a metric space.

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We also recall the following definitions and properties of α -compatible mappings and α -compatible mappings of type (A). [14,15]

Definition: 2.1 Let α, S and T be self maps of a metric space (X, d) . Then S and T are called α -compatible if $(\alpha \circ S)$ and $(\alpha \circ T)$ are compatible if

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} (\alpha \circ S)(x_n) = \lim_{n \rightarrow \infty} (\alpha \circ T)(x_n) = t \quad \text{for some } t \text{ in } X.$$

Definition: 2.2 Let α, S and T be self maps of a metric space (X, d) . Then S and T are said to be α -compatible mappings of type (A) if $(\alpha \circ S)$ and $(\alpha \circ T)$ are compatible mappings of type (A) if

$$\lim_{n \rightarrow \infty} d((\alpha \circ T)(\alpha \circ S)(x_n), (\alpha \circ S)(\alpha \circ S)(x_n)) = 0$$

and
$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} (\alpha \circ S)(x_n) = \lim_{n \rightarrow \infty} (\alpha \circ T)(x_n) = t \quad \text{for some } t \text{ in } X.$$

Definition: 2.3 Let α, S and T be self maps of a metric space (X, d) . Then S and T are said to be α -compatible mappings of type (P) if $(\alpha \circ S)$ and $(\alpha \circ T)$ are compatible mappings of type (P) if

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \rightarrow \infty} (\alpha \circ S)(x_n) = \lim_{n \rightarrow \infty} (\alpha \circ T)(x_n) = t \quad \text{for some } t \text{ in } X.$$

Definition: 2.4 Let $\alpha, S: (X, d) \rightarrow (X, d)$ be mappings. Then $(\alpha \circ S)$ is said to be sequentially continuous at a point $t \in X$ if for every sequence $\{x_n\}$ in X , such that

$$\lim_{n \rightarrow \infty} d(x_n, t) = 0, \quad \text{we have } \lim_{n \rightarrow \infty} d((\alpha \circ S)x_n, (\alpha \circ S)t) = 0.$$

The following propositions show that Definitions 2.1 and 2.2 are equivalent under some conditions:

Proposition: 2.1 Let $\alpha, S, T: (X, d) \rightarrow (X, d)$ be mappings such that $(\alpha \circ S)$ and $(\alpha \circ T)$ are continuous. If S and T are α -compatible, then they are α -compatible of type (A).

Proposition: 2.2 Let $\alpha, S, T: (X, d) \rightarrow (X, d)$ be mappings such that one of $(\alpha \circ S)$ and $(\alpha \circ T)$ is continuous. If S and T are α -compatible mappings of type (A), then they are α -compatible.

Remark: 2.1 In [15] we can find two examples that Proposition 2.1 and 2.2 are not true if S and T are not continuous on X .

We can also show that $(\alpha \circ S)$ and $(\alpha \circ T)$ are continuous, then S and T are α -compatible if and only if they are α -compatible mappings of type (P), as follows:

Proposition: 2.3 Let $\alpha, S, T: (X, d) \rightarrow (X, d)$ be mappings such that $(\alpha \circ S)$ and $(\alpha \circ T)$ are continuous. Then S and T are α -compatible mappings if and only if they are α -compatible mappings of type (P).

Proof: Let $\{x_n\}$ be a sequence in X such that $(\alpha \circ S)(x_n), (\alpha \circ T)(x_n) \rightarrow t$ for some $t \in X$. Since $(\alpha \circ S)$ and $(\alpha \circ T)$ are continuous, then we have

$$\lim_{n \rightarrow \infty} (\alpha \circ S)(\alpha \circ S)(x_n) = \lim_{n \rightarrow \infty} (\alpha \circ S)(\alpha \circ T)(x_n) = (\alpha \circ S)t$$

$$\lim_{n \rightarrow \infty} (\alpha \circ T)(\alpha \circ S)(x_n) = \lim_{n \rightarrow \infty} (\alpha \circ T)(\alpha \circ T)(x_n) = (\alpha \circ T)t$$

Suppose that S and T are α -compatible mappings. Then

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) = 0$$

By the triangle inequality of the metric d , we have

$$\begin{aligned} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) &\leq d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ S)(\alpha \circ T)(x_n)) + d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) \\ &\leq d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ S)(\alpha \circ T)(x_n)) + d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) \\ &\quad + d((\alpha \circ T)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) \end{aligned}$$

Taking $n \rightarrow \infty$, since S and T are α -compatible and $(\alpha \circ S)$ and $(\alpha \circ T)$ are continuous, then

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = 0$$

Therefore, S and T are α -compatible mappings of type (P).

Conversely, Suppose that S and T are α -compatible mappings of type (P). That is,

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = 0$$

By the triangle inequality of the metric d, we have

$$\begin{aligned} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) &\leq d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ S)(\alpha \circ S)(x_n)) + d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) \\ &\leq d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ S)(\alpha \circ S)(x_n)) + d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) \\ &\quad + d((\alpha \circ T)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) \end{aligned}$$

Taking $n \rightarrow \infty$, since S and T are α -compatible of type (P) and $(\alpha \circ S)$ and $(\alpha \circ T)$ are continuous, then we have

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) = 0$$

Therefore, S and T are α -compatible mappings. This completes the proof.

Proposition: 2.4 Let $\alpha, S, T: (X, d) \rightarrow (X, d)$ be mappings such that one of $(\alpha \circ S)$ and $(\alpha \circ T)$ is continuous. Then S and T are α -compatible mappings of type (A) if and only if they are α -compatible mappings of type (P).

Proof: Let $\{x_n\}$ be a sequence in X such that $(\alpha \circ S)(x_n), (\alpha \circ T)(x_n) \rightarrow t$ for some $t \in X$. Assume without loss of generality, that $(\alpha \circ T)$ is continuous, then we have

$$\lim_{n \rightarrow \infty} (\alpha \circ T)(\alpha \circ S)(x_n) = \lim_{n \rightarrow \infty} (\alpha \circ T)(\alpha \circ T)(x_n) = (\alpha \circ T)t$$

Suppose that S and T are α -compatible mappings of type (A), that is,

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = 0$$

and

$$\lim_{n \rightarrow \infty} d((\alpha \circ T)(\alpha \circ S)(x_n), (\alpha \circ S)(\alpha \circ S)(x_n)) = 0$$

By the triangle inequality of the metric d, we have

$$d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) \leq d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) + d((\alpha \circ T)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n))$$

Taking $n \rightarrow \infty$, since S and T are α -compatible of type (A) and $(\alpha \circ T)$ is continuous, then

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = 0$$

Therefore, S and T are α -compatible mappings of type (P).

Conversely, Suppose that S and T are α -compatible mappings of type (P), that is,

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = 0$$

By the triangle inequality of the metric d, we have

$$d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) \leq d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) + d((\alpha \circ T)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n))$$

Taking $n \rightarrow \infty$, since S and T are α -compatible of type (P) and $(\alpha \circ T)$ is continuous, then

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) = 0$$

Similarly, we have

$$\lim_{n \rightarrow \infty} d((\alpha \circ T)(\alpha \circ T)(x_n), (\alpha \circ S)(\alpha \circ T)(x_n)) = 0$$

Therefore, S and T are α -compatible mappings of type (A). This completes the proof.

We give following examples to show that the results of Proposition 2.3 and 2.4 are not true if $(\alpha \circ S)$ and $(\alpha \circ T)$ are not continuous.

Example: 2.1 Let $X = \mathbb{R}$, the set of reals with usual metric $d(x, y) = |x - y|$. Define $\alpha, S, T: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\alpha(x) = \begin{cases} x^2 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases} \quad S(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 2 & \text{if } x = 0, \end{cases} \quad \text{and } T(x) = \begin{cases} 1/x^2 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0, \end{cases}$$

Then $(\alpha \circ S)$ and $(\alpha \circ T)$ are not continuous at $x = 0$. Consider a sequence $\{x_n\}$ in X defined by $x_n = n^3$, $n = 1, 2, \dots$. Then we have, as $n \rightarrow \infty$

$$(\alpha \circ S)x_n = 1/n^6 \rightarrow 0, \quad (\alpha \circ T)x_n = 1/n^{12} \rightarrow 0$$

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) = \lim_{n \rightarrow \infty} d(n^{24}, n^{24}) = 0$$

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = \lim_{n \rightarrow \infty} d(n^{12}, n^{48}) \\ = |n^{12} - n^{48}| = \infty$$

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = \lim_{n \rightarrow \infty} d(n^{24}, n^{48}) \\ = |n^{24} - n^{48}| = \infty$$

$$\lim_{n \rightarrow \infty} d((\alpha \circ T)(\alpha \circ S)(x_n), (\alpha \circ S)(\alpha \circ S)(x_n)) = \lim_{n \rightarrow \infty} d(n^{24}, n^{12}) \\ = |n^{24} - n^{12}| = \infty$$

Thus S and T are α -compatible mappings but neither α -compatible mappings of type (A) nor α -compatible mappings of type (P).

Example: 2.2 Let $X = [0, 2]$ with the usual metric $d(x, y) = |x - y|$. Define $\alpha, S, T: X \rightarrow X$ such that

$$\alpha(x) = \begin{cases} \sqrt{x} & \text{if } x \in [0, 1) \\ 2\sqrt{x} & \text{if } x \in [1, 2], \end{cases} \quad S(x) = \begin{cases} x^2 & \text{if } x \in [0, 1) \\ 2 & \text{if } x \in [1, 2], \end{cases} \quad \text{and } T(x) = \begin{cases} (2-x)^2 & \text{if } x \in [0, 1) \\ 2 & \text{if } x \in [1, 2] \end{cases}$$

Then $(\alpha \circ S)$ and $(\alpha \circ T)$ are not continuous at $x = 1$. Let $\{x_n\} \subseteq [0, 2]$ be a sequence such that $x_n \rightarrow 1$ and assume that $x_n < 1$ for all n , then

$$(\alpha \circ T)x_n = 2 - x_n \rightarrow 1 \text{ from right hand side and } (\alpha \circ S)x_n = x_n \rightarrow 1 \text{ from right hand side.}$$

Since $2 - x_n > 1$, for all n , thus

$$(\alpha \circ S)(\alpha \circ T)x_n = (\alpha \circ S)(2 - x_n) = 2 \text{ and } (\alpha \circ T)(\alpha \circ S)x_n = (\alpha \circ T)(x_n) = 2 - x_n$$

Consequently,

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) = \lim_{n \rightarrow \infty} |2 - (2 - x_n)| \rightarrow 1$$

Also,

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = \lim_{n \rightarrow \infty} d((\alpha \circ S)(x_n), (\alpha \circ T)(2 - x_n)) \\ = \lim_{n \rightarrow \infty} d(x_n, 2) = \lim_{n \rightarrow \infty} |x_n - 2| \rightarrow 1$$

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = \lim_{n \rightarrow \infty} |2 - (\alpha \circ T)(2 - x_n)| \\ = |2 - 2| \rightarrow 0 \text{ as } x_n \rightarrow 1 \text{ and}$$

$$\lim_{n \rightarrow \infty} d((\alpha \circ T)(\alpha \circ S)(x_n), (\alpha \circ S)(\alpha \circ S)(x_n)) = \lim_{n \rightarrow \infty} |(2 - x_n) - x_n| \\ = \lim_{n \rightarrow \infty} |1 - 2x_n| \rightarrow 0 \text{ as } x_n \rightarrow 1$$

Thus S and T are α -compatible mappings of type (A) but they are neither α -compatible nor α -compatible mappings of type (P).

Example: 2.3 Let $X = \mathbb{R}$, the set of reals with usual metric $d(x, y) = |x - y|$. Define $\alpha, S, T: \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\alpha(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases} \quad S(x) = \begin{cases} x/2 & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases} \quad \text{and } T(x) = \begin{cases} x/3 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0, \end{cases}$$

Then $(\alpha \circ S)$ and $(\alpha \circ T)$ are not sequentially continuous at $x = 0$. Consider a sequence $\{x_n\}$ in X defined by $x_n = n^2$, $n = 1, 2, \dots$. Then we have, as $n \rightarrow \infty$

$$(\alpha \circ S)x_n = 2/n^2 \rightarrow 0, \quad (\alpha \circ T)x_n = 3/n^2 \rightarrow 0$$

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ S)(x_n)) = \lim_{n \rightarrow \infty} d(2n^2/3, 3n^2/2) \\ = |2n^2/3 - 3n^2/2| = \infty$$

$$\lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = \lim_{n \rightarrow \infty} d(n^2, n^2) = 0$$

$$\begin{aligned} \lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ T)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) &= \lim_{n \rightarrow \infty} d(2n^2 / 3, n^2) \\ &= |2n^2 / 3 - n^2| = \infty \end{aligned}$$

$$\begin{aligned} \lim_{n \rightarrow \infty} d((\alpha \circ T)(\alpha \circ S)(x_n), (\alpha \circ S)(\alpha \circ S)(x_n)) &= \lim_{n \rightarrow \infty} d(3n^2 / 2, n^2) \\ &= |3n^2 / 2 - n^2| = \infty \end{aligned}$$

Thus S and T are α -compatible mappings of type (P) but neither α -compatible of type (A) nor α -compatible.

Next we give several properties of α -compatible mappings of type (P) for our main theorems.

Proposition: 2.5 Let $\alpha, S, T: (X, d) \rightarrow (X, d)$ be mappings. If S and T are α -compatible mappings of type (P) and $(\alpha \circ S)(t) = (\alpha \circ T)(t)$ for some $t \in X$, then

$$(\alpha \circ S)(\alpha \circ T)(t) = (\alpha \circ S)(\alpha \circ S)(t) = (\alpha \circ T)(\alpha \circ T)(t) = (\alpha \circ T)(\alpha \circ S)(t).$$

Proof: Suppose that $\{x_n\}$ is a sequence in X defined by $x_n = t$, for $n = 1, 2, 3, \dots$, and $(\alpha \circ S)(t) = (\alpha \circ T)(t)$. Then we have,

$$(\alpha \circ S)(x_n), (\alpha \circ T)(x_n) \rightarrow (\alpha \circ S)(t) \text{ as } n \rightarrow \infty.$$

Since S and T are α -compatible of type (P), then we have

$$\begin{aligned} d((\alpha \circ S)(\alpha \circ S)(t), (\alpha \circ T)(\alpha \circ T)(t)) &= \lim_{n \rightarrow \infty} d((\alpha \circ S)(\alpha \circ S)(x_n), (\alpha \circ T)(\alpha \circ T)(x_n)) = 0 \\ \text{and so } (\alpha \circ S)(\alpha \circ S)(t) &= (\alpha \circ T)(\alpha \circ T)(t). \end{aligned}$$

Since $(\alpha \circ T)(t) = (\alpha \circ S)(t)$

therefore $(\alpha \circ T)(\alpha \circ T)(t) = (\alpha \circ T)(\alpha \circ S)(t)$

and $(\alpha \circ S)(\alpha \circ T)(t) = (\alpha \circ S)(\alpha \circ S)(t)$

So, $(\alpha \circ S)(\alpha \circ T)(t) = (\alpha \circ S)(\alpha \circ S)(t) = (\alpha \circ T)(\alpha \circ T)(t) = (\alpha \circ T)(\alpha \circ S)(t).$

Proposition: 2.6 Let S and T be α -compatible mappings of type (P) from a metric space (X, d) into itself. Suppose $\lim_{n \rightarrow \infty} (\alpha \circ S)x_n = \lim_{n \rightarrow \infty} (\alpha \circ T)x_n = t$ for some t in X. Then

- (1) $\lim_{n \rightarrow \infty} (\alpha \circ T)(\alpha \circ T)x_n = (\alpha \circ S)t$ if $(\alpha \circ S)$ is sequentially continuous.
- (2) $\lim_{n \rightarrow \infty} (\alpha \circ S)(\alpha \circ S)x_n = (\alpha \circ T)t$ if $(\alpha \circ T)$ is sequentially continuous.
- (3) $(\alpha \circ S)(\alpha \circ T)(t) = (\alpha \circ S)(\alpha \circ S)(t)$ and $(\alpha \circ S)t = (\alpha \circ T)t$ if $(\alpha \circ S)$ and $(\alpha \circ T)$ are sequentially continuous at t.

Proof: Suppose that $\lim_{n \rightarrow \infty} (\alpha \circ S)x_n = \lim_{n \rightarrow \infty} (\alpha \circ T)x_n = t$ for some $t \in X$.

- (1) Since, $(\alpha \circ S)$ is sequentially continuous, then we have

$$\lim_{n \rightarrow \infty} (\alpha \circ S)(\alpha \circ S)x_n = (\alpha \circ S)t.$$

By triangle inequality, we have

$$d((\alpha \circ T)(\alpha \circ T)x_n, (\alpha \circ S)t) \leq d((\alpha \circ T)(\alpha \circ T)x_n, (\alpha \circ S)(\alpha \circ S)x_n) + d((\alpha \circ S)(\alpha \circ S)x_n, (\alpha \circ S)t)$$

Letting $n \rightarrow \infty$, since S and T are α -compatible mappings of type (P), then we have

$$\lim_{n \rightarrow \infty} d((\alpha \circ T)(\alpha \circ T)x_n, (\alpha \circ S)t) = 0$$

so $\lim_{n \rightarrow \infty} (\alpha \circ T)(\alpha \circ T)x_n = (\alpha \circ S)t.$

- (2) The proof of $\lim_{n \rightarrow \infty} (\alpha \circ S)(\alpha \circ S)x_n = (\alpha \circ T)t$ follows on the similar lines as argued in(1).

- (3) Since, $(\alpha \circ T)$ is sequentially continuous at t, we have

$$\lim_{n \rightarrow \infty} (\alpha \circ T)(\alpha \circ T)x_n = (\alpha \circ T)t$$

Since, $(\alpha \circ S)$ is sequentially continuous at t, by (1) also we have

$$\lim_{n \rightarrow \infty} (\alpha \circ T)(\alpha \circ T)x_n = (\alpha \circ S)t$$

Hence, by the uniqueness of the limit, we have $(\alpha \circ S)t = (\alpha \circ T)t$

By Proposition 2.5, we have

$$(\alpha\circ S)(\alpha\circ T)t = (\alpha\circ T)(\alpha\circ S)t.$$

This completes the proof.

Throughout this section, suppose that a function $\phi : [0, \infty)^{14} \rightarrow [0, \infty)$ satisfies the followings:

- (i) ϕ is a upper semi-continuous and non-decreasing in each co-ordinate variable.
- (ii) $\phi(t) = \max \{ \phi(t, 0, 2t, 0, t, 0, 2t, t, 0, t, 2t, t, t, t), \phi(t, 0, 0, 2t, t, 2t, 0, t, 2t, t, 0, t, t, t), \phi(0, t, 0, 0, t, 0, 0, 0, t, 0, t, 0, 0, 0) \} < t$, for some $t > 0$.

Lemma: 2.1 Let α, A, B, S and T be self maps of complete metric space (X, d) into itself such that

$$(2.1) \quad (\alpha\circ A)(X) \subseteq (\alpha\circ T)(X) \text{ and } (\alpha\circ B)(X) \subseteq (\alpha\circ S)(X)$$

$$(2.2) \quad [d((\alpha\circ A)x, (\alpha\circ B)y)]^2 \leq \phi(d((\alpha\circ S)x, (\alpha\circ A)x) d((\alpha\circ T)y, (\alpha\circ B)y), d((\alpha\circ S)x, (\alpha\circ B)y) d((\alpha\circ T)y, (\alpha\circ A)x), d((\alpha\circ S)x, (\alpha\circ A)x) d((\alpha\circ S)x, (\alpha\circ B)y), d((\alpha\circ T)y, (\alpha\circ A)x) d((\alpha\circ T)y, (\alpha\circ B)y), [d((\alpha\circ S)x, (\alpha\circ T)y)]^2, d((\alpha\circ S)x, (\alpha\circ A)x) d((\alpha\circ T)y, (\alpha\circ A)x), d((\alpha\circ T)y, (\alpha\circ B)y) d((\alpha\circ S)x, (\alpha\circ B)y), d((\alpha\circ S)x, (\alpha\circ T)y) d((\alpha\circ S)x, (\alpha\circ A)x), d((\alpha\circ S)x, (\alpha\circ T)y) d((\alpha\circ T)y, (\alpha\circ A)x), d((\alpha\circ S)x, (\alpha\circ T)y) d((\alpha\circ T)y, (\alpha\circ B)y), d((\alpha\circ S)x, (\alpha\circ T)y) d((\alpha\circ S)x, (\alpha\circ B)y), d((\alpha\circ S)x, (\alpha\circ T)y) d((\alpha\circ A)x, (\alpha\circ B)y), d((\alpha\circ S)x, (\alpha\circ A)x) d((\alpha\circ A)x, (\alpha\circ B)y), d((\alpha\circ T)y, (\alpha\circ B)y) d((\alpha\circ A)x, (\alpha\circ B)y))$$

for all $x, y \in X$, where ϕ satisfies (i) and (ii), then there is a Cauchy sequence $\{y_n\}$ in X , defined by,

$$y_{2n-1} = (\alpha\circ T)x_{2n-1} = (\alpha\circ A)x_{2n-2} \text{ and } y_{2n} = (\alpha\circ S)x_{2n} = (\alpha\circ B)x_{2n-1} \text{ for } n = 1, 2, 3, \dots$$

Proof: Let $x_0 \in X$ be arbitrary since $(\alpha\circ A)(X) \subseteq (\alpha\circ T)(X)$ and $(\alpha\circ B)(X) \subseteq (\alpha\circ S)(X)$, we can choose x_1, x_2 in X , such that

$$y_1 = (\alpha\circ T)x_1 = (\alpha\circ A)x_0 \text{ and } y_2 = (\alpha\circ S)x_2 = (\alpha\circ B)x_1$$

In general we can choose x_{2n-1} and x_{2n} in X , such that

$$y_{2n-1} = (\alpha\circ T)x_{2n-1} = (\alpha\circ A)x_{2n-2} \text{ and } y_{2n} = (\alpha\circ S)x_{2n} = (\alpha\circ B)x_{2n-1} \tag{2.3}$$

Thus the indicated sequence $\{y_n\}$ exists. To show that $\{y_n\}$ is Cauchy, by (2.2) and (2.3) imply that

$$[d(y_{2n+1}, y_{2n+2})]^2 = [d((\alpha\circ A)x_{2n}, (\alpha\circ B)x_{2n+1})]^2 \leq \phi(d((\alpha\circ S)x_{2n}, (\alpha\circ A)x_{2n}) d((\alpha\circ T)x_{2n+1}, (\alpha\circ B)x_{2n+1}), d((\alpha\circ S)x_{2n}, (\alpha\circ B)x_{2n+1}) d((\alpha\circ T)x_{2n+1}, (\alpha\circ A)x_{2n}), d((\alpha\circ S)x_{2n}, (\alpha\circ A)x_{2n}) d((\alpha\circ S)x_{2n}, (\alpha\circ B)x_{2n+1}), d((\alpha\circ T)x_{2n+1}, (\alpha\circ A)x_{2n}) d((\alpha\circ T)x_{2n+1}, (\alpha\circ B)x_{2n+1}), [d((\alpha\circ S)x_{2n}, (\alpha\circ T)x_{2n+1})]^2, d((\alpha\circ S)x_{2n}, (\alpha\circ A)x_{2n}) d((\alpha\circ T)x_{2n+1}, (\alpha\circ A)x_{2n}), d((\alpha\circ T)x_{2n+1}, (\alpha\circ B)x_{2n+1}) d((\alpha\circ S)x_{2n}, (\alpha\circ B)x_{2n+1}), d((\alpha\circ S)x_{2n}, (\alpha\circ T)x_{2n+1}) d((\alpha\circ S)x_{2n}, (\alpha\circ A)x_{2n}), d((\alpha\circ S)x_{2n}, (\alpha\circ T)x_{2n+1}) d((\alpha\circ T)x_{2n+1}, (\alpha\circ A)x_{2n}), d((\alpha\circ S)x_{2n}, (\alpha\circ T)x_{2n+1}) d((\alpha\circ T)x_{2n+1}, (\alpha\circ B)x_{2n+1}), d((\alpha\circ S)x_{2n}, (\alpha\circ T)x_{2n+1}) d((\alpha\circ S)x_{2n}, (\alpha\circ B)x_{2n+1}), d((\alpha\circ S)x_{2n}, (\alpha\circ T)x_{2n+1}) d((\alpha\circ A)x_{2n}, (\alpha\circ B)x_{2n+1}), d((\alpha\circ S)x_{2n}, (\alpha\circ A)x_{2n}) d((\alpha\circ A)x_{2n}, (\alpha\circ B)x_{2n+1}), d((\alpha\circ T)x_{2n+1}, (\alpha\circ B)x_{2n+1}) d((\alpha\circ A)x_{2n}, (\alpha\circ B)x_{2n+1}))$$

or

$$[d(y_{2n+1}, y_{2n+2})]^2 \leq \phi(d(y_{2n}, y_{2n+1}) d(y_{2n+1}, y_{2n+2}), 0, d(y_{2n}, y_{2n+1}) d(y_{2n}, y_{2n+2}), 0, [d(y_{2n}, y_{2n+1})]^2, 0, d(y_{2n+1}, y_{2n+2}) d(y_{2n}, y_{2n+2}), d(y_{2n}, y_{2n+1}) d(y_{2n}, y_{2n+1}), 0, d(y_{2n}, y_{2n+1}) d(y_{2n+1}, y_{2n+2}), d(y_{2n}, y_{2n+1}) d(y_{2n}, y_{2n+2}), d(y_{2n}, y_{2n+1}) d(y_{2n+1}, y_{2n+2}), d(y_{2n+1}, y_{2n+2}) d(y_{2n+1}, y_{2n+2}))$$

Let $d(y_n, y_{n+1}) = d_n$ then we get

$$d_{2n+1}^2 \leq \phi(d_{2n} d_{2n+1}, 0, d_{2n}(d_{2n} + d_{2n+1}), 0, d_{2n}^2, 0, d_{2n+1}(d_{2n} + d_{2n+1}), d_{2n}^2, 0, d_{2n} d_{2n+1}, d_{2n}(d_{2n} + d_{2n+1}), d_{2n} d_{2n+1}, d_{2n} d_{2n+1}, d_{2n+1}^2) \tag{2.4}$$

We want to show that $d_{2n+1} \leq d_{2n}$. For this assume that for some n , $d_{2n+1} > d_{2n}$. Now by (2.4), we have

$$\begin{aligned} \mathcal{C}_{2n \square 1}^2 &\leq \phi(d_{2n+1}^2, 0, 2d_{2n+1}^2, 0, d_{2n+1}^2, 0, 2d_{2n+1}^2, d_{2n+1}^2, 0, d_{2n+1}^2, \\ &2d_{2n+1}^2, d_{2n+1}^2, d_{2n+1}^2, d_{2n+1}^2) < d_{2n+1}^2 \end{aligned}$$

Which is a contradiction. Therefore $d_{2n+1} \leq d_{2n}$, similarly again using (2.2) we obtain

$$\begin{aligned} d_{2n}^2 &\leq \phi(d_{2n} d_{2n-1}, 0, 0, d_{2n-1}(d_{2n-1} + d_{2n}), d_{2n-1}^2, d_{2n}(d_{2n-1} + d_{2n}), 0, d_{2n-1}d_{2n}, \\ &d_{2n-1}(d_{2n-1} + d_{2n}), d_{2n-1}^2, 0, d_{2n-1} d_{2n}, d_{2n}^2, d_{2n-1} d_{2n}) \end{aligned} \quad (2.5)$$

If possible let $d_{2n} > d_{2n-1}$ for some n , then (2.5) gives

$$\begin{aligned} d_{2n}^2 &\leq \phi(d_{2n}^2, 0, 0, 2d_{2n}^2, d_{2n}^2, 2d_{2n}^2, 0, d_{2n}^2, 2d_{2n}^2, d_{2n}^2, 0, d_{2n}^2, d_{2n}^2, d_{2n}^2) \\ &< d_{2n}^2 \end{aligned}$$

Which is a contradiction. Therefore $d_{2n} \leq d_{2n-1}$. Thus $\{d_n\}$ is non-increasing sequence in \mathbb{R}^+ let $\{d_n\} \rightarrow r \in \mathbb{R}^+$. If $r \neq 0$, then since ϕ is upper semi-continuous, we have from (2.4) and (2.5) as $n \rightarrow \infty$

$$r^2 \leq \phi(r^2, 0, 2r^2, 0, r^2, 0, 2r^2, r^2, 0, r^2, 2r^2, r^2, r^2, r^2) < r^2$$

and $r^2 \leq \phi(r^2, 0, 0, 2r^2, r^2, 2r^2, 0, r^2, r^2, 2r^2, 0, r^2, r^2, r^2) < r^2$

Which is a contradiction, therefore $d_n \rightarrow 0$. Now we shall show that sequence $\{y_n\}$ is Cauchy. If not so then there exist an $\varepsilon > 0$ and sequence of positive integer $\{m(k)\}$ and $\{n(k)\}$ with $k \leq n(k) < m(k)$ such that

$$c_k = d(y_{m(k)}, y_{n(k)}) \geq \varepsilon; \quad k = 1, 2, \dots \quad (2.6)$$

Let $m(k)$ be the least integer exceeding $n(k)$ for which (2.6) is true then by the well ordering principle,

$d(y_{m(k)-1}, y_{n(k)}) > \varepsilon$. Now

$$\varepsilon \leq c_k \leq d(y_{m(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{n(k)}) < d_{m(k)-1} + \varepsilon \rightarrow \varepsilon \text{ as } k \rightarrow \infty$$

and thus $c_k \rightarrow \varepsilon$. Further c_k can have different values under the following four cases, viz.,

- (a) m is even and n is odd; (b) m and n are odd;
- (c) m is odd and n is even; (d) m and n are even.

Now, in case (a), we have

$$\begin{aligned} c_k &= d(y_{2m}, y_{2n-1}) \\ &\leq d(y_{2m}, y_{2m+1}) + d(y_{2m+1}, y_{2n}) + d(y_{2n}, y_{2n-1}) \end{aligned}$$

letting $n \rightarrow \infty$ we get

$$\varepsilon \leq 0 + 0 + \lim_{n \rightarrow \infty} d(y_{2m+1}, y_{2n}) \quad (2.7)$$

Now using (2.2) and (2.3), we obtain

$$\begin{aligned} [d(y_{2m+1}, y_{2n})]^2 &= [d((\alpha \circ A)x_{2m}, (\alpha \circ B)x_{2n-1})]^2 \\ &\leq \phi(d_{2m} d_{2n-1}, (c_k + d_{2n-1})(c_k + d_{2m}), d_{2m}(c_k + d_{2n-1}), \\ &d_{2n-1}(c_k + d_{2m}), \mathcal{C}_k^2, d_{2m}(c_k + d_{2m}), d_{2n-1}(c_k + d_{2n-1}), \\ &c_k d_{2m}, c_k(c_k + d_{2m}), c_k d_{2n-1}, c_k(c_k + d_{2n-1}), \\ &c_k(d_{2m} + c_k + d_{2n-1}) d_{2m}(c_k + d_{2n-1}), d_{2n-1}(d_{2m} + c_k + d_{2n-1})). \end{aligned}$$

Since ϕ is upper semi-continuous so letting $n \rightarrow \infty$ and using (ii), we get

$$\lim_{n \rightarrow \infty} [d(y_{2m+1}, y_{2n})]^2 \leq \phi(0, \varepsilon^2, 0, 0, \varepsilon^2, 0, 0, 0, \varepsilon^2, 0, \varepsilon^2, \varepsilon^2, 0, 0) < \varepsilon^2$$

So we get a contradiction and hence $\varepsilon = 0$. Similarly, other cases also give $\varepsilon = 0$. Therefore, the sequence $\{y_n\}$ is Cauchy sequence.

Now we prove the our main result.

Theorem: 2.1 Let α, A, B, S, T be self maps of complete metric space (X, d) satisfying (2.1), (2.2) and (2.8). The pairs A, S and B, T are α -compatible of type (P).

One of $(\alpha\circ A), (\alpha\circ B), (\alpha\circ S)$ and $(\alpha\circ T)$ is sequentially continuous at their coincidence point.

Then A, B, S and T have a unique common α -fixed point in X .

Proof: By Lemma 2.1, the sequence $\{y_n\}$ as defined by (2.2) is Cauchy, so it converges to a point z in X . Consequently the subsequence $\{(\alpha\circ A)_{x_{2n}}\}, \{(\alpha\circ S)_{x_{2n}}\}, \{(\alpha\circ B)_{x_{2n-1}}\}$ and $\{(\alpha\circ T)_{x_{2n-1}}\}$ converges to z .

Now suppose $(\alpha\circ A)$ is sequentially continuous, since A and S are α -compatible of type (P), then by Proposition 2.6 we have

$$\begin{aligned}
 &(\alpha\circ A)(\alpha\circ S)_{x_{2n}} \rightarrow (\alpha\circ A)z \quad \text{and} \quad (\alpha\circ S)(\alpha\circ S)_{x_{2n}} \rightarrow (\alpha\circ A)z \quad \text{as } n \rightarrow \infty. \quad \text{By (2.2),} \\
 &[d((\alpha\circ A)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)_{x_{2n-1}})]^2 \leq \phi (d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ T)_{x_{2n-1}}, (\alpha\circ B)_{x_{2n-1}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)_{x_{2n-1}}) d((\alpha\circ T)_{x_{2n-1}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)_{x_{2n-1}}), \\
 &\quad d((\alpha\circ T)_{x_{2n-1}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ T)_{x_{2n-1}}, (\alpha\circ B)_{x_{2n-1}}), \\
 &\quad [d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)_{x_{2n-1}})]^2, \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ T)_{x_{2n-1}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}), \\
 &\quad d((\alpha\circ T)_{x_{2n-1}}, (\alpha\circ B)_{x_{2n-1}}) d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)_{x_{2n-1}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)_{x_{2n-1}}) d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)_{x_{2n-1}}) d((\alpha\circ T)_{x_{2n-1}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)_{x_{2n-1}}) d((\alpha\circ T)_{x_{2n-1}}, (\alpha\circ B)_{x_{2n-1}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)_{x_{2n-1}}) d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)_{x_{2n-1}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)_{x_{2n-1}}) d((\alpha\circ A)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)_{x_{2n-1}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ A)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)_{x_{2n-1}}), \\
 &\quad d((\alpha\circ T)_{x_{2n-1}}, (\alpha\circ B)_{x_{2n-1}}) d((\alpha\circ A)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)_{x_{2n-1}})
 \end{aligned}$$

Since ϕ is upper semi-continuous, so letting $n \rightarrow \infty$, we obtain

$$\begin{aligned}
 &d((\alpha\circ A)z, z)]^2 \leq \phi (0, d((\alpha\circ A)z, z)]^2, 0, 0, d((\alpha\circ A)z, z)]^2, 0, 0, 0, d((\alpha\circ A)z, z)]^2, 0, \\
 &\quad d((\alpha\circ A)z, z)]^2, d((\alpha\circ A)z, z)]^2, 0, 0)
 \end{aligned}$$

So that $(\alpha\circ A)z = z$, since $(\alpha\circ A)(X) \subseteq (\alpha\circ T)(X)$, there exists a point v in X such that $z = (\alpha\circ A)z = (\alpha\circ T)v$. Again using (2.2), we get

$$\begin{aligned}
 &[d((\alpha\circ A)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)v)]^2 \leq \phi (d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ T)v, (\alpha\circ B)v), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)v) d((\alpha\circ T)v, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)v), \\
 &\quad d((\alpha\circ T)v, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ T)v, (\alpha\circ B)v), [d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)v)]^2, \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ T)v, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}), \\
 &\quad d((\alpha\circ T)v, (\alpha\circ B)v) d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)v), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)v) d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)v) d((\alpha\circ T)v, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)v) d((\alpha\circ T)v, (\alpha\circ B)v), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)v) d((\alpha\circ T)v, (\alpha\circ B)v), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)v) d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)v), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ T)v) d((\alpha\circ A)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)v), \\
 &\quad d((\alpha\circ S)(\alpha\circ S)_{x_{2n}}, (\alpha\circ A)(\alpha\circ S)_{x_{2n}}) d((\alpha\circ A)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)v), \\
 &\quad d((\alpha\circ T)v, (\alpha\circ B)v) d((\alpha\circ A)(\alpha\circ S)_{x_{2n}}, (\alpha\circ B)v)
 \end{aligned}$$

letting $n \rightarrow \infty$, we have

$$\begin{aligned}
 &[d(z, (\alpha\circ B)v)]^2 \leq \phi (0, 0, 0, 0, 0, 0, [d(z, (\alpha\circ B)v)]^2, 0, 0, 0, 0, 0, 0, [d(z, (\alpha\circ B)v)]^2) \\
 &\quad < [d(z, (\alpha\circ B)v)]^2
 \end{aligned}$$

Which is a contradiction, therefore $z = (\alpha\circ B)v$.

Now B and T are α -compatible mappings of type (P) and $(\alpha\circ T)v = (\alpha\circ B)v = z$, therefore by Proposition 2.5, we have $(\alpha\circ T)(\alpha\circ B)v = (\alpha\circ B)(\alpha\circ T)v$

Hence $(\alpha \circ T)z = (\alpha \circ B)z$.

Now using (2.2) again, we get

$$[d((\alpha \circ A)_{x_{2n}}, (\alpha \circ B)z)]^2 \leq \phi(d((\alpha \circ S)_{x_{2n}}, (\alpha \circ A)_{x_{2n}}) d((\alpha \circ T)z, (\alpha \circ B)z), d((\alpha \circ S)_{x_{2n}}, (\alpha \circ B)z) d((\alpha \circ T)z, (\alpha \circ A)_{x_{2n}}), \\ d((\alpha \circ S)_{x_{2n}}, (\alpha \circ A)_{x_{2n}}) d((\alpha \circ S)_{x_{2n}}, (\alpha \circ B)z), d((\alpha \circ T)z, (\alpha \circ A)_{x_{2n}}) d((\alpha \circ T)z, (\alpha \circ B)z), \\ [d((\alpha \circ S)_{x_{2n}}, (\alpha \circ T)z)]^2, d((\alpha \circ S)_{x_{2n}}, (\alpha \circ A)_{x_{2n}}) d((\alpha \circ T)z, (\alpha \circ A)_{x_{2n}}), \\ d((\alpha \circ T)z, (\alpha \circ B)z) d((\alpha \circ S)_{x_{2n}}, (\alpha \circ B)z), d((\alpha \circ S)_{x_{2n}}, (\alpha \circ T)z) d((\alpha \circ S)_{x_{2n}}, (\alpha \circ A)_{x_{2n}}), \\ d((\alpha \circ S)_{x_{2n}}, (\alpha \circ T)z) d((\alpha \circ T)z, (\alpha \circ A)_{x_{2n}}), d((\alpha \circ S)_{x_{2n}}, (\alpha \circ T)z) d((\alpha \circ T)z, (\alpha \circ B)z), \\ d((\alpha \circ S)_{x_{2n}}, (\alpha \circ T)z) d((\alpha \circ S)_{x_{2n}}, (\alpha \circ B)z), d((\alpha \circ S)_{x_{2n}}, (\alpha \circ T)z) d((\alpha \circ A)_{x_{2n}}, (\alpha \circ B)z), \\ d((\alpha \circ S)_{x_{2n}}, (\alpha \circ A)_{x_{2n}}) d((\alpha \circ A)_{x_{2n}}, (\alpha \circ B)z), d((\alpha \circ T)z, (\alpha \circ B)z) d((\alpha \circ A)_{x_{2n}}, (\alpha \circ B)z))$$

letting $n \rightarrow \infty$, above inequality reduces to

$$[d(z, (\alpha \circ B)z)]^2 \leq \phi(0, [d(z, (\alpha \circ B)z)]^2, 0, 0, [d(z, (\alpha \circ B)z)]^2, 0, 0, 0, [d(z, (\alpha \circ B)z)]^2, 0, \\ [d(z, (\alpha \circ B)z)]^2, [d(z, (\alpha \circ B)z)]^2, 0, 0) \\ < [d(z, (\alpha \circ B)z)]^2$$

so that $z = (\alpha \circ B)z = (\alpha \circ T)z = (\alpha \circ A)z$.

Since $(\alpha \circ B)(X) \subseteq (\alpha \circ S)(X)$ there exists a point w in X such that $z = (\alpha \circ B)z = (\alpha \circ S)w$.

Again condition (2.2) imply that

$$[d((\alpha \circ A)w, z)]^2 = [d((\alpha \circ A)w, (\alpha \circ B)z)]^2 \\ \leq \phi(d((\alpha \circ S)w, (\alpha \circ A)w) d((\alpha \circ T)z, (\alpha \circ B)z), d((\alpha \circ S)w, (\alpha \circ B)z) d((\alpha \circ T)z, (\alpha \circ A)w), \\ d((\alpha \circ S)w, (\alpha \circ A)w) d((\alpha \circ S)w, (\alpha \circ B)z), d((\alpha \circ T)z, (\alpha \circ A)w) d((\alpha \circ T)z, (\alpha \circ B)z), \\ [d((\alpha \circ S)w, (\alpha \circ T)z)]^2, d((\alpha \circ S)w, (\alpha \circ A)w) d((\alpha \circ T)z, (\alpha \circ A)w), \\ d((\alpha \circ T)z, (\alpha \circ B)z) d((\alpha \circ S)w, (\alpha \circ B)z), d((\alpha \circ S)w, (\alpha \circ T)z) d((\alpha \circ S)w, (\alpha \circ A)w), \\ d((\alpha \circ S)w, (\alpha \circ T)z) d((\alpha \circ T)z, (\alpha \circ A)w), d((\alpha \circ S)w, (\alpha \circ T)z) d((\alpha \circ T)z, (\alpha \circ B)z), \\ d((\alpha \circ S)w, (\alpha \circ T)z) d((\alpha \circ S)w, (\alpha \circ B)z), d((\alpha \circ S)w, (\alpha \circ T)z) d((\alpha \circ A)w, (\alpha \circ B)z) \\ d((\alpha \circ S)w, (\alpha \circ A)w) d((\alpha \circ A)w, (\alpha \circ B)z), d((\alpha \circ T)z, (\alpha \circ B)z) d((\alpha \circ A)w, (\alpha \circ B)z))$$

or

$$[d((\alpha \circ A)w, z)]^2 \leq \phi(0, 0, 0, 0, 0, [d((\alpha \circ A)w, z)]^2, 0, 0, 0, 0, 0, 0, [d((\alpha \circ A)w, z)]^2, 0) \\ < [d((\alpha \circ A)w, z)]^2$$

so that $(\alpha \circ A)w = z$, since A and S are α -compatible mappings of type (P) and $(\alpha \circ A)w = (\alpha \circ S)w = z$ we obtain using Proposition 2.5

$$(\alpha \circ S)(\alpha \circ A)w = (\alpha \circ A)(\alpha \circ S)w \text{ and hence } (\alpha \circ S)z = (\alpha \circ A)z = z.$$

Therefore z is a common α -fixed point of A, B, S and T .

Similarly, we can complete the result by taking $(\alpha \circ B)$ or $(\alpha \circ S)$ or $(\alpha \circ T)$ as sequentially continuous. Uniqueness follows easily from (2.2).

Now we present the following example to prove the validity of Theorem 2.1.

Example: 2.4 Let $X = [0, 1]$ with usual metric in real line. Define A, B, S, T and α by $A(x) = x/8, B(x) = x/6, S(x) = x/2, T(x) = 2x/3$ and $\alpha(x) = x/3$ for all $x \in [0, 1]$. Then clearly the function $(\alpha \circ A), (\alpha \circ B), (\alpha \circ S)$ and $(\alpha \circ T)$ are sequentially continuous and satisfy

$$(\alpha \circ A)(x) = [0, 1/24] \subseteq [0, 2/9] = (\alpha \circ T)(x)$$

$$(\alpha \circ B)(x) = [0, 1/18] \subseteq [0, 1/6] = (\alpha \circ S)(x)$$

Moreover

$$|(\alpha \circ A)(x) - (\alpha \circ S)(x)| = |x/24 - x/6| = x/8 \rightarrow 0 \quad \text{if and only if } x \rightarrow 0$$

$$|(\alpha \circ A)(\alpha \circ A)(x) - (\alpha \circ S)(\alpha \circ S)(x)| = |x/576 - x/36| = 5x/192 \rightarrow 0 \quad \text{if and only if } x \rightarrow 0$$

Therefore A and S are α -compatible mappings of type (P). Similarly,

$$|(\alpha \circ B)(x) - (\alpha \circ T)(x)| = |x/18 - 2x/9| = x/6 \rightarrow 0 \quad \text{if and only if } x \rightarrow 0$$

$$|(\alpha \circ B)(\alpha \circ B)(x) - (\alpha \circ T)(\alpha \circ T)(x)| = |x/324 - 4x/81| = 5x/108 \rightarrow 0 \quad \text{if and only if } x \rightarrow 0$$

Thus B and T are also α -compatible mappings of type (P). Let us define the function ϕ as

$\phi(t_1, t_2, \dots, t_{14}) = h \max \{ t_1, t_2, \dots, t_{14} \}$ for all $t_i \in \mathbb{R}^+$; $i = 1, 2, \dots, 14$; $1/16 \leq h < 1/2$, then ϕ satisfies condition (i) and (ii).

Also we obtain

$$|(\alpha \circ A)(x) - (\alpha \circ B)(y)| = |x/24 - y/18| = (1/72) |3x - 4y|$$

$$|(\alpha \circ S)(x) - (\alpha \circ T)(y)| = |x/6 - 2y/9| = (1/18) |3x - 4y|$$

$$|(\alpha \circ A)(x) - (\alpha \circ B)(y)|^2 = (1/16) |(\alpha \circ S)(x) - (\alpha \circ T)(y)|^2 \\ \leq \phi(d((\alpha \circ S)(x), (\alpha \circ A)(x)), d((\alpha \circ T)(y), (\alpha \circ B)(y)), \dots, [d((\alpha \circ S)(x), (\alpha \circ T)(y))]^2, \dots)$$

So that condition (2) is satisfied and thus the hypothesis of Theorem 2.1 is satisfied and clearly $x = 0$ is the unique common α -fixed point of A, B, S and T.

REFERENCES

1. G. Jungck; Commuting maps and fixed points, *Amer. Math. Monthly*, 83 (1979), 261-263.
2. G. Jungck ; Compatible mappings and common fixed points, *Internat. J. Math. And Math. Sci.*, 9 (1986), 771-779.
3. G. Jungck; Compatible mappings and common fixed points (2), *Internat. J. Math. and Math. Sci.*, 11 (1988), 285-288.
4. G. Jungck; Common fixed points of commuting and compatible maps on compacta, *Proc. Amer. Math. Soc.*, 103 (1988), 977-983.
5. G. Jungck; Common fixed points for non continuous nonself maps on non metric spaces, *Far East J. Math. Sci.*, 4 (1996), 199-215.
6. G. Jungck, P. P. Murthy and Y. J. Cho; Compatible mapping of type (A) and common fixed points, *Math. Japonica*, 38 (2) (1993), 381-390.
7. H. Kaneko and S. Sessa; Fixed point theorems for compatible multi-valued and single-valued mappings, *Internat. J. Math. and Math. Sci.*, 12 (1989), 257-262.
8. S. M. Kang, Y. J. Cho and G. Jungck; Common fixed points of compatible mapping, *Internat. J. Math. and Math. Sci.*, 13 (1990), 61-66.
9. Ramesh Kumar and Renu Chugh; Compatible mappings of type (P) and common fixed point theorem, *Vikram Mathematical Journal*, 15 (1995), 1-16.
10. H. K. Pathak, S. S. Chang and Y. J. Cho; Fixed point theorems for compatible mapping of type (P), *Indian J. Math.*, 36(2) (1994), 151-166.
11. H. K. Pathak, Y. J. Cho, S. S. Chang and S. M. Kang; Compatible mappings of type (P) and fixed point theorems in metric spaces and probabilistics metric spaces, *Novi Sad J. Math.*, 26(2) (1996), 87-109.
12. S. Sessa; On weak a commutativity conditions of mappings in fixed point considerations, *Publ. Inst. Math. (Beograd)*, 32 (1982), 149-153.
13. S. P. Singh and B. A. Meade; On common fixed point theorems, *Bull. Austral. Math.*, 16 (1977), 49-53.
14. Anjali Srivastava and Renuka Rathore; α -Reciprocal continuity and α -fixed point, *Bull. Cal. Math. Soc. Vol.* (2006)
15. Anjali Srivastava and Renuka Rathore; α -Compatible mappings of type (A) and common α -fixed point. (Communicated).

Source of support: Nil, Conflict of interest: None Declared