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FIXED POINT THEOREM IN COMPLEX VALUED METRIC SPACES USING SIX MAPS

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ABSTRACT

Recently Rahul Tiwari [10] proved common fixed point theorem with six maps in complex valued metric spaces. In this paper we obtain a common fixed point theorem for six maps in complex valued metric spaces having commuting and weakly compatible and satisfying different type of inequality. Our theorem generalizes and extends the result of R. Tiwari [10].

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Keywords: *Weakly compatible maps, fixed points, common fixed points, complex valued metric spaces.*

1. INTRODUCTION

The study of metric space expressed the most common important role to many fields both in pure and applied science [3]. Many authors generalized and extended the notion of a metric space such as vector valued metric space of Perov [2], a cone metric spaces of Huang and Zhang [8], a modular metric spaces of Chistyakov [13], etc.

Azam, Fisher and Khan [1] first introduced the complex valued metric spaces which is more general than well known metric spaces and also gave common fixed point theorems for maps satisfying generalized contraction condition.

2. PRELIMINARIES

Let $\mathbb{C}$ be the set of all complex numbers. For $z_1, z_2 \in \mathbb{C}$, define partial order $\leq$ on $\mathbb{C}$ by $z_1 \leq z_2$ if and only if

$Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$.

That is $z_1 \leq z_2$ if one of the following conditions holds
(i) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(ii) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
(iii) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
(iv) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$;

In particular, we will write $z_1 < z_2$ if $z_1 \neq z_2$ and one of (ii), (iii) and (iv) is satisfied and we will write $z_1 < z_2$.

Definition: 2.1[1]Let X be a non-empty set and $d: X \times X \to \mathbb{C}$ be a map, then d is said to be complex valued metric if
(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and if only $x = y$;
(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Pair (X, d) is called a complex valued metric space.

Example: 2.2 Define a map $d: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ by $d(z_1, z_2) = e^{ip}|z_1 - z_2|$ where $p \in R$. Then $(\mathbb{C}, d)$ is a complex valued metric.

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Definition: 2.3 [1] Let (X, d) be a complex valued metric space then

(i) Any point $x \in X$ is said to be an interior point of $A \subseteq X$ if there exists $0 < r \in \mathbb{C}$ such that
$$B(x, r) = \{y \in X | d(x, y) < r\} \subseteq A.$$
(ii) Any point $x \in X$ is said to be a limit point of A if for every $0 < r \in \mathbb{C}$, we have $B(x, r) \cap (A - X) \neq \emptyset$.
(iii) Any subset A of X is said to be an open if each element of A is an interior point of A.
(iv) Any subset A of X is said to be a closed if each limit point of A belongs to A.
(v) A sub-basis of a Hausdorff topology τ on X is a family given by $F = \{B(x, r) | x \in X \ and \ 0 < r\}$.

Definition: 2.4 [1] Let $\{x_n\}$ be a sequence in complex valued metric space (X, d) and $x \in X$. Then

(i) It is said to be a convergent sequence, $\{x_n\}$ converges to x and x is the limit point of $\{x_n\}$, if for every $c \in \mathbb{C}$, with $0 < c$ there is a natural number N such that $d(x_n, x) < c$, for all $n > N$. We denote it by $\lim_{n\to\infty} x_n = x$
(ii) It is said to be a Cauchy sequence, if for every $c \in \mathbb{C}$, with $0 < c$ there is a natural number N such that $d(x_n,\ x_{n+m}) < c$, for all $n > N$ and $m \in \mathbb{N}$.
(iii) (X, d) is said to be complete complex valued metric space if every Cauchy sequence in X is convergent .

Lemma: 2.5 [1] Any sequence $\{x_n\}$ in complex valued metric space (X, d), converges to x if and only if

$|d(x_n,\ x)| \to 0$ as $n \to \infty$.

Lemma: 2.6 [1] Any sequence $\{x_n\}$ in complex valued metric space (X, d), Cauchy sequence if and only if

$|d(x_n,\ x_{n+m})| \to 0$ as $n \to \infty$ where $m \in \mathbb{N}$.

Definition: 2.7 Let S and T be self maps of a non-empty set X. Then

(i) Any point $x \in X$ is said to be a fixed point T if $Tx = x$.
(ii) Any point $x \in X$ is said to be a coincidence point of S and T if $Sx = Tx$ and we shall called $w = Sx = Tx$ that a point of coincidence of S and T.
(iii) Any point $x \in X$ is said to be a common fixed point of S and T if $Sx = Tx = x$.

Definition: 2.8 [5] Two self maps S and T of a non-empty set X are commuting if

$TSx = STx$, for all $x \in X$.

Definition: 2.9 [12] Let S, T be self maps of metric space (X, d) then S, T are said to be weakly commuting if

$d(STx, TSx) \leq d(Sx, Tx)$, for all $x \in X$.

Definition: 2.10 [6] Let S, T be self maps of metric space (X, d) then S, T are said to be compatible if

$$\lim_{n\to\infty} d(STx, TSx_n) = 0$$

Whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$, for some $z \in X$.

Remark: 2.11 In general, commuting maps are weakly commuting and weakly commuting maps are compatible, but the converse are not necessarily true and some examples can be found in [5-7, 9]

Definition: 2.12[7] Two self maps S, T of a non-empty set X are said to be weakly compatible if $STx = TSx$ whenever $Sx = Tx$.

Lemma: 2.13 [9] Let $T: X \to X$ be a map, then there exists a subset E of X such that $T(E) = T(X)$ and $T: E \to X$ is one to one.

3. MAIN RESULT

Theorem 3.1: Let (X, d) be a complex valued metric space and P, Q, R, S, T, U be self maps of X satisfying the following conditions

$$TU(X) \subseteq P(X) \text{ and } RS(X) \subseteq Q(X) \tag{3.1}$$

$$d(RSx, TUy) \leq ad(Px, Qy) + b\big(d(Px, RSx) + d(Qy, TUy) + d(Px, TUy) + d(Qy, RSx)\big) \tag{3.2}$$

For all $x, y \in X$ where $a, b \geq 0$ and $a + 4b < 1$.

Assume that pairs (TU, Q) and (RS, P) are weakly compatible. Pairs $(T, U), (T, Q), (U, Q), (R, S), (R, P)$ and (S, P) are commuting pairs of maps. Then T, U, R, S, Q and P have a unique common fixed point in X.

Proof: Let $x_0 \in X$. By (3.1) we can define inductively a sequence $\{y_n\}$ in X such that

$y_{2n} = RSx_{2n} = Qx_{2n}$ and $y_{2n+1} = TUx_{2n+1} = Px_{2n+2}$ for all $n = 1, 2, 3, ...$ (3.3)

By (3.2), we have

$$d(y_{2n}, y_{2n+1}) = d(RSx_{2n}, TUx_{2n+1})$$

$$\leq ad(Px_{2n}, Qx_{2n+1}) + b\begin{pmatrix} d(Px_{2n}, RSx_{2n}) + d(Qx_{2n+1}, TUx_{2n+1}) + \\ d(Px_{2n}, TUx_{2n+1}) + d(Qx_{2n+1}, RSx_{2n}) \end{pmatrix}$$

$$= ad(y_{2n-1}, y_{2n}) + b\big(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1}) + d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n})\big)$$

$$\leq (a + 2b)d(y_{2n-1}, y_{2n}) + 2bd(y_{2n}, y_{2n+1})$$

Which implies that

$d(y_{2n}, y_{2n+1}) \leq \frac{a+2b}{1-2b} d(y_{2n-1}, y_{2n}) = kd(y_{2n-1}, y_{2n})$ Where $k = \frac{a+2b}{1-2b} < 1$.

Similarly we obtain $d(y_{2n+1}, y_{2n+2}) \leq kd(y_{2n}, y_{2n+1})$

Therefore,

$d(y_{n+1}, y_{n+2}) \leq kd(y_n, y_{n-1}) \leq \cdots k^{n+1} d(y_0, y_1)$ for $n = 1, 2, 3, ...$

Now, for all $m > n$,

$$d(y_n, y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \cdots d(y_{m-1}, y_m)$$

$$\leq (k^n + k^{n+1} + \cdots + k^{m-1}) d(y_1, y_0)$$

$$\leq \frac{k^n}{k-1} d(y_1, y_0)$$

$$d|(y_n, y_m)| \leq \frac{k^n}{k-1} |d(y_1, y_0)| k^n |d(y_1, y_0)|$$

Which implies that $d|(y_n, y_m)| \to 0$ as $n, m \to \infty$. Hence $\{y_n\}$ is a Cauchy sequence.

Since X is complete, there exists a point z in X such that

$$\lim_{n\to\infty} RSx_{2n} = \lim_{n\to\infty} Qx_{2n+1} = \lim_{n\to\infty} TUx_{2n+1} = \lim_{n\to\infty} Px_{2n+2} = z$$

Since $TU(X) \subseteq P(X)$, there exists a point $u \in X$ such that $z = Pu$.

Then by (3.2), we have

$$d(RSu, z) \leq d(RSu, TUx_{2n-1}) + d(TUx_{2n-1}, z)$$

$$\leq ad(Pu,\ Qx_{2n-1}) + b\begin{pmatrix} d(Pu, RSu) + d(Qx_{2n-1}, TUx_{2n-1}) \\ +d(Pu, TUx_{2n-1}) + d(Qx_{2n-1}, RSu) + d(TUx_{2n-1}, z) \end{pmatrix}$$

Taking the limit as $\to \infty$, we obtain

$$d(RSu, z) \leq ad(z, z) + b\big(d(z, RSu) + d(z, z) + d(z, z) + d(z, RSu)\big) + d(z, z)$$

$$= 2bd(Rsu, z), \text{ a contradiction}$$

Since $a + 4b < 1$. Therefore $RSu = Pu = z$. Since $(X) \subseteq Q(X)$, there exists a point v in X such that $z = Qv$.

Then by (3.2), we have

$d(z, TUv) = d(RSu, TUv)$

$$\leq ad(Pu, Qv) + b\big(d(Pu, RSu) + d(Qv, TUv) + d(Pu, TUv) + d(Qv, RSu)\big)$$

$$= ad(z, z) + b\big(d(z, z) + d(z, TUv) + d(z, TUv) + d(z, z)\big)$$

$= 2bd(z, TUv)$, which is a contradiction.

Therefore $TUv = Qv = z$ and so $RSu = Pu = TUv = Qv = z$.

Similarly, Q and TU are weakly compatible maps, we have $TUz = Qz$

Now we claim that z is a fixed point of TU. If $\neq z$, then by (3.2), we have

$d(z, TUz) = d(RSz, TUz)$

$$\leq ad(Pz, Qz) + b\big(d(Pz, RSz) + d(Qz, TUz) + d(Pz, TUz) + d(Qz, RSz)\big)$$

$$= ad(z, TUz) + b\big(d(z, z) + d(TUz, TUz) + d(z, TUz) + d(TUz, z)\big)$$

$= (a + 2b)d(z, TUz)$, a contradiction.

Therefore $TUz = z$. Hence $TUz = Qz = z$. We have therefore proved that $RSz = TUz = Pz = Qz = z$. So z is common fixed point of P, Q, RS and U .

By commuting conditions of pairs we have

$Tz = T(TUz) = T(UTz) = TU(Tz)$.

$Tz = T(Pz) = P(Tz)$ and $Uz = U(TUz) = (UT)(Uz) = (TU)(Uz)$

$Uz = U(Pz) = P(Uz)$, which follows that Tz and Uz are common fixed points of (TU, P)

Then $Tz = z = Uz = Pz = TUz$

Similarly $Rz = z = Sz = Qz = RSz$

Therefore z is a common fixed point of T, U, R, S, P and Q .

For uniqueness of z, let w be another common fixed point of T, U, R, S, P and Q.

Then by (3.2), we have

$d(z, w) = d(RSz, TUw)$

$$\leq ad(Pz, Qw) + b\big(d(Pz, RSz) + d(Qw, TUw) + d(Pz, TUw) + d(Qw, RSz)\big)$$

$$= ad(z, w) + b\big(d(z, z) + d(w, w) + d(z, w) + d(w, z)\big)$$

$= (a + 2b)d(z, w)$, a contradiction.

So, $= w$.

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