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DEGREE OF APPROXIMATION OF FUNCTION BELONGING TO THE LIPSCHITZ CLASS BY (E, q) (C,1) MEANS OF IT'S FOURIER SERIES

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Abstract

In this paper, a theorem on the degree of approximation of the function belonging to the Lipschitz class by almost (E,q) (C,1) product means of its Fourier series has been established.

1. Definitions and Notations:

Definition: 1. Let $\sum_{n=0}^{\infty} u_n$ be a given infinite series with the sequence of n^{th} partial sums $\{s_n\}$. If

$$C_k^1 = \frac{1}{k+1}\sum_{r=0}^{k} s_r \to s \qquad \text{as } n\to\infty \tag{1}$$

then an infinite series $\sum_{n=0}^{\infty} u_n$ with the partial sums s_n is said to be summable (C, 1) to the definite number s.

Definition 2. The (E, q) transform of the (C, 1) transform C_n^1 defines the (E, q) (C, 1) transform of the partial sum's s_n of the series $\sum_{n=0}^{\infty} u_n$. Thus if

$$(EC)_n^q = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k} C_k^1 = \frac{1}{(1+q)^n}\sum_{k=0}^{n}\binom{n}{k} q^{n-k}\frac{1}{k+1}\sum_{r=0}^{k} s_r \to s \tag{2}$$

as $n\to\infty$

where C_n^1 denoted the (C, 1) transform of s_n; then an infinite series $\sum_{n=0}^{\infty} u_n$ with the partial sums s_n is said to be summable (E,q) (C,1) to the definite number s and we write

$$(EC)_n^q \to s[\,(E,q)\ (C,1)] \text{ as } n \to \infty$$

Let f(t) be periodic with period 2π and integrable in the sense of Lebesgue. The Fourier series of f(t) is given by

$$f(t) \sim \frac{1}{2}a_0 + \sum_{n=0}^{\infty}(a_n \cos nt + b_n \sin nt) \tag{3}$$

A function $f \in \text{Lip}\ \alpha$ if

$$f(x + t) - f(x) = O(|t|^{\alpha}) \text{ for } 0 < \alpha \le 1. \tag{4}$$

The degree of approximation of a function $f : R \to R$ by a trigonometric polynomial t_n of order n is defined by Zygmund (1968; 1; p.114)

$$\|t_n - f\|_\infty = \sup\{|t_n(x) - f(x)| : x \in R\} \tag{5}$$

We shall use following notation :

$\phi(t) = f(x + t) + f(x - t) - 2f(x)$

2. Main Theorem: The degree. of approximation of functions belonging to Lip α by Cesàro means and by Nörlund means has been discussed by a number of researcher's like Alexits (1961), Quereshi (1981,

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1982), Quereshi and Neha (1990) Chandra (1975), Sahney and Goel(1973), Khan(1974), Leinder (2005) and Rhoades(2003). But till now no work seems to have been done to obtain the degree of approximation of the function belonging to Lip α by (E,q) (C,1) product means of its Fourier series. In an attempt to make study in this direction, one theorem on the degree of approximation of function of Lip α class by product summability means of the form (E,q) (C,1) has been obtained as following :

Theorem: If $f : R \to R$ is 2π-periodic, Lebesgue integrable on $[-\pi, \pi]$ and belonging to the Lipschitz class then the degree of approximation of f by the (E,q) (C,1) product means of its Fourier series satisfies for n = 0, 1, 2, ...

$$\left\|(EC)_n^q - f\right\|_\infty = O\left[\frac{1}{(n+1)^\alpha}\right] \quad ;0<\alpha<1.$$

3. Proof of the theorem:

The n^{th} partial sum $S_n(x)$ of the series (3) at t = x is written as

$$S_n(x)-f(x)=\frac{1}{2\pi}\int_0^\pi \phi(t).\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin t/2}dt$$

$$C_k^1(x)-f(x)=\frac{1}{2\pi(k+1)}\int_0^\pi \phi(t)\sum_{r=0}^{k}\frac{\sin(2r+1)t/2}{\sin\ t/2}dt$$

$$(EC)_n^q(x)-f(x)=\frac{1}{2\pi(1+q)^n}\int_0^\pi \phi(t)\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\sin(2r+1)t/2}{\sin t/2}dt$$

Therefore

$$\left|(EC)_n^q(x)-f(x)\right|\le\frac{1}{2\pi(1+q)^n}\int_0^\pi|\phi(t)|\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\sin(2r+1)t/2}{\sin t/2}\right|dt$$

$$\le\frac{1}{2\pi(1+q)^n}\left[\int_0^{1/(n+1)}+\int_{1/(n+1)}^{\pi}\right]|\phi(t)|\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\sin(2r+1)t/2}{\sin t/2}\right|dt$$

$\le I_1 + I_2$ say (6)

Now,

$$I_1=\frac{1}{2\pi(1+q)^n}\int_0^{1/(n+1)}|\phi(t)|\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\sin(2r+1)t/2}{\sin\ t/2}\right|dt$$

$$\le\frac{1}{2\pi(1+q)^n}\int_0^{1/(n+1)}|\phi(t)|\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{(2r+1)\sin t/2}{\sin\ t/2}\right|dt$$

(For $0\le t\le\frac{1}{n+1}$, $\sin nt\le n\sin t$)

$$\le\frac{1}{2\pi(1+q)^n}\int_0^{1/(n+1)}|\phi(t)|\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}(k+1)\right|dt$$

$$=O\left[\frac{(n+1)}{(1+q)^n}\int_0^{1/(n+1)}|\phi(t)|\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|dt\right]$$

$$=O\left[(n+1)\int_0^{1/(n+1)}|\phi(t)|dt\right]\quad\left(\text{since}\sum_{k=0}^{n}\binom{n}{k}q^{n-k}=(1+q)^n\right)$$

$$=O\left[(n+1)\int_0^{1/(n+1)}t^\alpha dt\right]$$

$$=O\left[\frac{1}{(n+1)^\alpha}\right] \tag{7}$$

Let us consider I_2

$$I_2=\frac{1}{2\pi(1+q)^n}\int_{1/(n+1)}^{\pi}|\phi(t)|\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\frac{1}{k+1}\sum_{r=0}^{k}\frac{\sin(2r+1)t/2}{\sin t/2}\right|dt$$

$$\le\frac{1}{2(1+q)^n}\int_{1/(n+1)}^{\pi}\frac{|\phi(t)|}{t}\left|\sum_{k=0}^{n}\binom{n}{k}q^{n-k}\right|dt$$

$$\left(\begin{gathered}\sin t/2\ge t/\pi\\ \&\sin nt\le1\end{gathered}\right)$$

$$=O\left[\int_{1/(n+1)}^{\pi}\frac{|\phi(t)|}{t}dt\right]$$

$$=O\left[\int_{1/(n+1)}^{\pi}t^{\alpha-1}dt\right]$$

$$= O\left[\frac{1}{(n+1)^{\alpha}}\right] \tag{8}$$

Combining (6), (7) and (8), we get

$$\left|(EC)_n^q(x) - f(x)\right| = O\left[\frac{1}{(n+1)^{\alpha}}\right] \;;\; 0<\alpha<1$$

Thus,

$$\left\|(EC)_n^q - f\right\|_{\infty} = \sup_{-\pi\le x\le\pi}\left|(EC)_n^q(x) - f(x)\right|$$

$$= O\left[\frac{1}{(n+1)^{\alpha}}\right] \;;0<\alpha<1.$$

This completes the proof of the theorem.

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