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LIE SYMMETRIES OF NONLINEAR DIFFUSION EQUATIONS WITH CONVECTION TERM

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1. INTRODUCTION

In the present paper, we consider nonlinear diffusion equations with convection term of the form

$$u_t = (k(u)u_x)_x + q(u),\ q(u) \neq 0 \tag{1}$$

where u = u(x, t) is the unknown function and k(u), q(u) are arbitrary smooth functions.

Equation (1) generalizes a great number of the well-known nonlinear second-order evolution equations, describing various processes in physics, chemistry, biology. Apart from their intrinsic theoretical interest, the equations of the type (1) are used to model a wide variety of phenomena in physics, engineering, chemistry, biology etc.

In the case k(u) = 1 and q(u) = 0, the classical heat equation

$$u_t = u_{xx} \tag{2}$$

follows from equation (1).

In the case q(u) = 0, the standard nonlinear heat equation

$$u_t = (k(u)u_x)_x \tag{3}$$

follows from equation (1). Lie symmetries of equation (3) were completely described by

Ovsyannikov[2]. Lie symmetries of equation (1) were described by different technique by Dorodnitsyn [3]. The conditional symmetry of nonlinear heat equation (1) was studied in [8, 9].

The paper is organized as follows. After this introductory section, in Section 2 the system of determining equations for Lie symmetry operators corresponding to (1) is obtained using a specific technique for addressing second -order partial differential equations with two independent variables and a dependent one. In the third section, four cases in which Eq. (1) admits symmetries will be studied. For these cases the Lie symmetry generators, the invariant quantities which could be attached to them and some special similarity solutions will be illuminated.

2. GENERAL APPROACHES

The Lie symmetries for a PDE equation are the classical ones which keep the equation invariant under a Lie group of local infinitesimal transformations. In this section we investigate the Lie symmetries for (1). We start from its equivalent form

$$u_t = k(u)\, u_{xx} + k_u\, u_x^2 + q(u) \tag{4}$$

Let us consider a one-parameter group of point-like transformations acting on the space of the independent variables (x, t) and on that of the dependent variable of Eq. (4), with the associated infinitesimal generator given by

$$\mathbf{v} = \xi\,\partial_x + \tau\partial_t + \phi\partial_u \tag{5}$$

The invariance condition **[5]** for the second-order PDE equation (4) is

$$pr^2\,(u_t - k(u)\, u_{xx} - k_u\, u_x^2 - q(u)) = 0 \tag{6}$$

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where pr^2 represents the second extension of operator U and has the general form

$$pr^2(v) = v + \varphi^x \frac{\partial}{\partial u_x} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{tt} \frac{\partial}{\partial u_{tt}} \tag{7}$$

The condition (6) is equivalent to the relation

$$\varphi^t = k(u)\varphi^{xx} + \varphi k_u u_{xx} + 2k_u u_x \varphi^x + \varphi u_x^2 k_{uu} + \varphi q_u \tag{8}$$

Using the formula **[5]** for coefficient functions ϕ^t, ϕ^x and ϕ^{zz} in equation (8) and equating the coefficients of the terms in the first and higher order partial derivatives of *u*, and and substituting u_t from (4) the determining equations for symmetry group of the non linear equation are found as follows

$$\varphi_t + (\varphi_u - \tau_t)q(u) = k(u)\varphi_{xx} + \varphi q_u \tag{9}$$

$$-\xi_t = 2k(u)\varphi_{xu} - k(u)\xi_{xx} + 2k_u\varphi_x \tag{10}$$

$$k_u\varphi_u - k_u\tau_t = k(u)\varphi_{uu} + \varphi k_{uu} + 2k_u(\varphi_u - \xi_x) \tag{11}$$

$$k(u)\varphi_u - k(u)\tau_t = k(u)(\varphi_u - 2\xi_x) + \varphi k_u \tag{12}$$

$$\xi_u k(u) = 0 \tag{13}$$

$$\tau_x k(u) = 0 \tag{14}$$

$$\tau_u k(u) = 0 \tag{15}$$

3. CONCRETE SYMMETRIES AND INVARIANTS

3.1 The case $k(u) = e^u$, $q(u) = e^{bu}$

For this chosen form of k(u), and q(u) the determining equations ((9)–(15)) of the classical symmetries associated with (4) are the following

$$\varphi_t + (\varphi_u - \tau_t)e^{bu} = e^u\varphi_{xx} + \varphi b e^{bu}$$

$$-\xi_t = 2e^u\varphi_{xu} - e^u\xi_{xx} + 2e^u\varphi_x$$

$$e^u\varphi_u - e^u\tau_t = e^u\varphi_{uu} + \varphi e^u + 2e^u(\varphi_u - \xi_x)$$

$$e^u\varphi_u - e^u\tau_t = e^u(\varphi_u - 2\xi_x) + \varphi e^u$$

$$-\xi_u e^u = 0$$

$$\tau_x e^u = 0$$

$$\tau_u e^u = 0$$

Thus solving these equations simultaneously we get $\phi = -c_3,\ \xi = c_2 + c_3(b-1)x/2,\ \tau = c_1 + c_3 bt$

Thus the Lie algebra of infinitesimal symmetries of non linear equation is spanned by the three vector fields

$$v_1 = \partial_t, \quad v_2 = \partial_x, \quad v_3 = bt\partial_t + \frac{(b-1)}{2}x\partial_x - \partial_u$$

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The commutation relation between the vector fields is given by the following table

	v_1	v_2	v_3
v_1	0	0	bv_1
v_2	0	0	$(b-1)v_2/2$
v_3	bv_1	$-(b-1)v_2/2$	0

The one-parameter groups G_i (i = 1, 2, 3) generated by the $\boldsymbol{v}_i$ are given by using exp (εv_i) (x, t, u) as follows

G_1 : $(x,t+\varepsilon,u)$, G_2 : $(x+\varepsilon, t, u)$ G_3 : $(xe^{\varepsilon(b-1)/2}, te^{\varepsilon b}, u-\varepsilon)$

Since each group G_i is a symmetry group.

The solution of equation corresponding to its different symmetry groups G_i (i =1, 2, 3) are obtained by using

$\tilde{u} = g \cdot u = g \cdot f(x, t)$ as follows

$u^{(1)} = f(x, t-\varepsilon)$, $u^{(2)} = f(x-\varepsilon, t)$, $u^{(3)} = f(xe^{-\varepsilon(b-1)/2}, e^{-\varepsilon b}t) - \varepsilon$,

3.2. The case k(u) =u^a, q(u) = u^n, a, n $\neq 0$

For this chosen form of k(u), and q(u) the determining equations ((9)–(15)) of the classical symmetries associated with (4) are the following

$$\varphi_t + (\varphi_u - \tau_t)u^n = u^a\varphi_{xx} + \varphi n u^{n-1}$$

$$-\xi_t = 2u^a\varphi_{xu} - u^a\xi_{xx} + 2au^{a-1}\varphi_x$$

$$au^{a-1}\varphi_u - au^{a-1}\tau_t = u^a\varphi_{uu} + \varphi a(a-1)u^{a-2} + 2au^{a-1}(\varphi_u - \xi_x)$$

$$u^a\varphi_u - u^a\tau_t = u^a(\varphi_u - 2\xi_x) + \varphi a u^{a-1}$$

$$u^a e^u = 0$$

$$\tau_x u^a = 0$$

$$\tau_u u^a = 0$$

Thus solving these equations simultaneously we get $\phi = -2c_3 u,\ \xi = c_2 + c_3(n-a-1)x,\ \tau = c_1 + 2c_3(n-1)t$

Thus the Lie algebra of infinitesimal symmetries of non linear equation is spanned by the three vector fields

$$v_1 = \partial_t,\quad v_2 = \partial_x,\quad v_3 = 2(n-1)t\partial_t + (n-a-1)x\partial_x - 2u\partial_u$$

The commutation relation between the vector fields is given by the following table

	v_1	v_2	v_3
v_1	0	0	$2(n-1)v_1$
v_2	0	0	$(n-a-1)v_2$
v_3	$2(n-1)v_1$	$-(n-a-1)v_2$	0

The one-parameter groups G_i (i = 1, 2, 3) generated by the $\boldsymbol{v}_i$ are given by using exp (εv_i) (x, t, u) as follows

G_1 : $(x, t+\varepsilon,u)$, G_2 : $(x+\varepsilon,t, u)$ G_3 : $(xe^{\varepsilon(n-a-1)}, te^{\varepsilon 2(n-1)}, ue^{-2\varepsilon})$

Since each group G_i is a symmetry group.

The solution of equation corresponding to its different symmetry groups G_i (i =1, 2, 3) are obtained by using

$\tilde{u} = g \cdot u = g \cdot f(x, t)$ as follows

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$u^{(1)} = f(x, t - \varepsilon)$, $u^{(2)} = f(x - \varepsilon, t)$, $u^{(3)} = e^{-2\varepsilon} f(xe^{-\varepsilon(n-a-1)}, e^{-\varepsilon 2(n-1)}t)$,

3.3 The case k(u) =1, q(u) = e^u

For this chosen form of k(u), and q(u) the determining equations ((9)–(15)) of the classical symmetries associated with (4) are the following

$$\varphi_t + (\varphi_u - \tau_t)e^u = \varphi_{xx} + \varphi e^u$$

$$-\xi_t = 2\varphi_{xu} - \xi_{xx}$$

$$\varphi_{uu} = 0$$

$$\tau_t = 2\xi_x$$

$$-\xi_u e^u = 0$$

$$\tau_x e^u = 0$$

$$\tau_u e^u = 0$$

Thus solving these equations simultaneously we get $\phi = -2c_3,\ \xi = c_2 + c_3 x,\ \tau = c_1 + c_3 2t$

Thus the Lie algebra of infinitesimal symmetries of non linear equation is spanned by the three vector fields

$$v_1 = \partial_t, \quad v_2 = \partial_x, \quad v_3 = 2t\partial_t + x\partial_x - 2\partial_u$$

The commutation relation between the vector fields is given by the following table

	v_1	v_2	v_3
v_1	0	0	$2v_1$
v_2	0	0	v_2
v_3	$2v_1$	$-v_2$	0

The one-parameter groups G_i (i = 1, 2, 3) generated by the $\boldsymbol{v}_i$ are given by using exp (εv_i) (x, t, u) as follows
G_1 : (x,t+ε,u), G_2 : (x+ε, t, u) G_3 : (xe^{ε}, $te^{\varepsilon 2}$, u-2ε)

Since each group G_i is a symmetry group.

The solution of equation corresponding to its different symmetry groups G_i (i =1, 2, 3) are obtained by using

$\tilde{u}$ = g · u = g · f (x, t) as follows

$u^{(1)} = f(x, t - \varepsilon)$, $u^{(2)} = f(x - \varepsilon, t)$, $u^{(3)} = f(xe^{-\varepsilon}, e^{-\varepsilon 2}t) - 2\varepsilon$,

3 4. The case k(u) =1, q(u) = u^n (n ≠0)

For this chosen form of k(u), and q(u) the determining equations ((9)–(15)) of the classical symmetries associated with (4) are the following

$$\varphi_t + (\varphi_u - \tau_t)u^n = \varphi_{xx} + \varphi n u^{n-1} e^u$$

$$-\xi_t = 2\varphi_{xu} - \xi_{xx}$$

$$\varphi_{uu} = 0$$

$$\tau_t = 2\xi_x$$

$$-\xi_u e^u = 0$$

$$\tau_x e^u = 0$$

$$\tau_u e^u = 0$$

Thus solving these equations simultaneously we get $\phi = -2c_3 u,\ \xi = c_2 + c_3(n-1)x,\ \tau = c_1 + 2c_3(n-1)t$

Thus the Lie algebra of infinitesimal symmetries of non linear equation is spanned by the three vector fields

$$v_1 = \partial_t, \quad v_2 = \partial_x, \quad v_3 = 2(n-1)t\partial_t + (n-1)x\partial_x - 2u\partial_u$$

The commutation relation between the vector fields is given by the following table

	v_1	v_2	v_3
v_1	0	0	$2(n-1)v_1$
v_2	0	0	$(n-a-1)v_2$
v_3	$-2(n-1)v_1$	$-(n-a-1)v_2$	0

The one-parameter groups G_i (i = 1, 2, 3) generated by the $\boldsymbol{v}_i$ are given by using exp (εv_i) (x, t, u) as follows

G_1 : $(x, t+\varepsilon, u)$, G_2 : $(x+\varepsilon, t, u)$ G_3 : $(xe^{\varepsilon(n-1)}, te^{\varepsilon 2(n-1)}, ue^{-2\varepsilon})$

Since each group G_i is a symmetry group.

The solution of equation corresponding to its different symmetry groups G_i (i =1, 2, 3) are obtained by using $\tilde{u}$ = g · u = g · $f(x, t)$ as follows

$u^{(1)} = f(x, t-\varepsilon)$, $u^{(2)} = f(x-\varepsilon, t)$, $u^{(3)} = e^{-2\varepsilon} f(xe^{-\varepsilon(n-1)}, e^{-\varepsilon 2(n-1)}t)$,

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