International Journal of Mathematical Archive-4(12), 2013, 250-252
Available online through www.ijma.info ISSN 2229 – 5046

I_{gg}-Closed sets

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(Received on: 16-11-13; Revised & Accepted on: 11-12-13)

ABSTRACT

We define $\mathbf{I}_{gg}$ -closed sets in an ideal topological space and discuss their properties.

Keywords and Phrases: *$\mathbf{I}_g$-closed set, $\mathbf{I}_g$-open set, g-closed set, g-open set and g-local function.*

AMS 2010 Subject classification: *54 A 05, 54 A 10.*

1. INTRODUCTION AND PRELIMINARIES

A nonempty collection **I** of subsets of a given set X is said to be an ideal on X if (i). $A \in \mathbf{I}$ and $B \subset A$ implies $B \in \mathbf{I}$ (heredity) and (ii). $A \in \mathbf{I}$ and $B \in \mathbf{I}$ implies $A \cup B \in \mathbf{I}$ (finite additivity). If (X, τ) is a topological space and **I** is an ideal on X, then $(X, \tau, \mathbf{I})$ is called an ideal topological space [4]. For each subset A of X, $A^\star(\mathbf{I}, \tau) = \{x \in X: U_x \cap A \notin \mathbf{I}$, for every open set U_x containing x}, is called the local function of A [4] with respect to **I** and τ. We simply write $A^\star$ instead of $A^\star(\mathbf{I}, \tau)$ in case there is no chance for confusion. We often use the properties of the local function stated in Theorem 2.3 of [3] without mentioning it. Moreover, $cl^\star(A) = A \cup A^\star$ [6] defines a Kuratowski closure operator for a topology $\tau^\star$ on X which is finer than τ. A subset A of an ideal topological space (X, τ) is said to be g-closed [5], if $cl(A) \subset U$ whenever $A \subset U$ and U is open. The complement of a g-closed set is called a g-open set [5]. A subset A of an ideal space $(X, \tau, \mathbf{I})$ is said to be $\mathbf{I}_g$-closed [2], if $cl^\star(A) \subset U$ whenever $A \subset U$ and U is open. The complement of a $\mathbf{I}_g$-closed set is called a $\mathbf{I}_g$-open set [2]. The collection of all g -open sets in a topological space (X, τ) is denoted by τ_g. The g-closure of A denoted by $cl_g(A)$[1] is the intersection of all g-closed sets containing A and the the g-interior of A denoted by $int_g(A)$[1] is the union of all g-open sets contained in A.

For each subset A of X, $A_g^\star(\mathbf{I}, \tau) = \{x \in X: U_x \cap A \notin \mathbf{I}$, for every g-open set U_x containing x}, is called the g-local function of A [1] with respect to **I** and τ_g and is denoted by $A_g^\star$. Also, $cl_g^\star(A) = A \cup A_g^\star$ [1] is a Kurotowski closure operator for a topology $\tau_g^\star = \{X - A: cl_g^\star(A) = A\}$[1] on X which is finer than τ_g.

2. I_{gg} -CLOSED SETS

Definition: 2.1 A subset A of an ideal topological space $(X, \tau, \mathbf{I})$ is said to be an $\mathbf{I}_{gg}$ -closed set if $A_g^\star \subset U$ whenever $A \subset U$ where U is a g-open set in X, equivalently, $cl_g^\star(A) \subset U$ whenever $A \subset U$ where U is a g-open set in X. A is said to be an $\mathbf{I}_{gg}$-open set if X–A is an $\mathbf{I}_{gg}$-closed set.

Theorem: 2.2 Let $(X, \tau, \mathbf{I})$ be an ideal topological space and $A \subset X$. Then the following are equivalent.

(a) A is $\mathbf{I}_{gg}$ -closed.
(b) For all $x \in cl_g^\star(A)$, $cl_g(\{x\}) \cap A \neq \emptyset$
(c) $cl_g^\star(A) - A$ contains no nonempty g-closed set.
(d) $A_g^\star - A$ contains no nonempty g-closed set.

Proof:
(a) ⟹ (b). Suppose that $x \in cl_g^\star(A)$. If $cl_g(\{x\}) \cap A = \emptyset$, then $A \subset X - cl_g(\{x\})$. Since A is I_{gg}-closed, $cl_g^\star(A) \subset X - cl_g(\{x\})$ which is a contradiction to the fact that $x \in cl_g^\star(A)$.
This proves (b).

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(b) ⇒(c): Suppose $F \subset cl_g^\star(A)–A$, F is a g-closed set and $x \in F$. Since $F \subset X–A$ and F is g-closed, $cl_g(\{x\}) \cap A \subset cl_g(F) \cap A = F \cap A = \emptyset$. Since $x \in cl_g^\star(A)$, by (b), $cl_g(\{x\}) \cap A \neq \emptyset$, a contradiction which proves (c).

(c) ⇒(a): Let U be a g-open set containing A. Since $cl_g^\star(A)$ is closed, X-U is g-closed, $cl_g^*(A) \cap (X-U)$ is g-closed and $cl_g^\star(A) \cap (X–U) \subset cl_g^\star(A) \cap (X–A) = cl_g^\star(A)–A$. By hypothesis, $cl_g^\star(A) \cap (X–U) = \emptyset$, and so $cl_g^\star(A) \subset U$. Thus, A is $\mathbf{I}_{gg}$-closed.

The equivalence of (c) and (d) follows from the fact that $cl_g^\star(A)–A = A_g^\star – A$.

Theorem: 2.3 Let (X, τ, **I**) be a topological space and $A \subset X$. Then the following are equivalent.

(a) A is $\mathbf{I}_{gg}$-closed.

(b) $A \cup (X–cl_g^\star(A))$ is $\mathbf{I}_{gg}$-closed.

(c) $cl_g^\star(A) –A$ is $\mathbf{I}_{gg}$-open.

(d) $A_g^\star –A$ is $\mathbf{I}_{gg}$-open.

Proof:

(a) ⇒(b): Suppose that A is $\mathbf{I}_{gg}$-closed. Let $A \cup (X–cl_g^\star(A)) \subset U$ where U is g-open. Then $X–U \subset (X–A) \cap cl_g^\star(A) = cl_g^\star(A)–A$ where X–U is g-closed. By Theorem 2.2(c), $cl_g^\star(A) –A$ contains no nonempty g-closed set, $X–U = \emptyset$ implies that X=U. Since X is the only g-open set containing A, $A \cup (X–cl_g^\star(A))$ is $\mathbf{I}_{gg}$-closed.

(b) ⇒(a): Suppose that $A \cup (X–cl_g^\star(A))$ is $\mathbf{I}_{gg}$-closed. If F is any g-closed set contained in $cl_g^\star(A)–A$, then $A \cup (X–cl_g^\star(A)) \subset X- F$ where X–F is g-open. Therefore, $cl_g^\star(A) \cup cl(X–cl_g^\star(A)) \subset X–F$ and so $X \subset X–F$, it follows that $F = \emptyset$. Hence A is $\mathbf{I}_{gg}$-closed.

The equivalence of (b) and (c) follows from the fact that $X–(cl_g^\star(A)–A) = A \cup (X–cl_g^\star(A))$. The equivalence of (c) and (d) follows from the fact that $cl_g^\star(A) –A = A_g^\star–A$.

Theorem: 2.4 For a subset A of an ideal topological space (X, τ, **I**), A is $\mathbf{I}_{gg}$-open if and only if $F \subset int_g^\star(A)$ whenever $F \subset A$, where F is a g-closed set in (X, τ, **I**).

Proof: Suppose that A is $\mathbf{I}_{gg}$-open. If F is g-closed and $F \subset A$, then $X – A \subset X – F$, and so $cl_g^\star(X–A) \subset X–F$. Therefore, $F \subset int_g^\star(A)$.

Conversely, suppose the condition holds. Let U be a g-open set such that $X–A \subset U$. Then $X–U \subset A$ and so $X–U \subset int_g^\star(A)$ which implies that $cl_g^\star(X–A) \subset U$. Thus X–A is $\mathbf{I}_{gg}$-closed and so A is $\mathbf{I}_{gg}$-open.

Theorem: 2.5 Let (X, τ, **I**) be an ideal topological space. Then every g-open subset of (X, τ, **I**) is $\tau_g^\star$-closed if and only if every subset of (X, τ, **I**) is $\mathbf{I}_{gg}$-closed.

Proof: Suppose that every g-open subset of (X, τ, **I**) is $\tau_g^\star$-closed. Let $A \subset U$ where $A \subset X$ and U is g-open. Since U is g-open, $A_g^\star \subset U_g^\star \subset U$ and so A is $\mathbf{I}_{gg}$-closed. Conversely, suppose that every subset of (X, τ, **I**) is $\mathbf{I}_{gg}$-closed. Let $A \subset X$ be a g-open set. Since $A \subset A$, $A_g^\star \subset A$, A is a $\tau_g^\star$ -closed set. Hence, every g-open subset of (X, τ, **I**) is $\tau_g^\star$-closed if and only if every subset of (X, τ, **I**) is $\mathbf{I}_{gg}$-closed.

Theorem: 2.6 Let (X, τ, **I**) be an ideal topological space. If A and B are subsets of X such that $A \subset B \subset cl_g^\star(A)$ and A is an $\mathbf{I}_{gg}$-closed subset of X, then B is also an $\mathbf{I}_{gg}$-closed set.

Proof: Since $cl_g^\star(B)–B \subset cl_g^\star(A)–A$ and $cl_g^\star(A)–A$ contains no nonempty g-closed set, $cl_g^\star(B)–B$ contains no nonempty g-closed set. This implies that B is $\mathbf{I}_{gg}$-closed.

Theorem: 2.7 Let (X, τ, **I**) be an ideal topological space. If A and B are subsets of X such that $int_g^\star(A) \subset B \subset A$ and A is an $\mathbf{I}_{gg}$-open set, then B is also an $\mathbf{I}_{gg}$-open set.

Theorem: 2.8 Let (X, τ, **I**) be an ideal topological space. If A and B are subsets of X such that $A \subset B \subset A_g^\star$ and A is $\mathbf{I}_{gg}$-closed, then A and B are g-closed sets.

Proof: Since $A \subset B \subset A_g^\star$, $A_g^\star = B_g^\star = A^\star = B^\star$ by [1, Theorem 3.10]. Let $A \subset U$ and $U \in \tau$. Since A is $\mathbf{I}_{gg}$-closed, $A^\star = B^\star = B_g^\star = A_g^\star \subset U$ which implies that A and B are g-closed sets.

Theorem: 2.9 Let (X, τ, **I**) be an ideal topological space and A⊂X. Then A is $\mathbf{I}_{gg}$-closed if and only if A = F–N, where F is $\tau_g{}^\star$-closed and N contains no nonempty g-closed set.

Proof: If A is $\mathbf{I}_{gg}$-closed, then by Theorem 2.2(c), N= $A_g{}^\star$–A contains no nonempty g-closed set. If F=$cl_g{}^\star(A)$, then F is $\tau_g{}^\star$-closed such that F–N = (A ∪ $A_g{}^\star$) – ($A_g{}^\star$–A) = (A ∪ $A_g{}^\star$) ∩ ((X– $A_g{}^\star$)∪A)=A. Conversely, suppose that A = F–N where F is $\tau_g{}^\star$-closed and N contains no nonempty g-closed set. Let U be a g-open set such that A⊂U. Then F–N ⊂ U which implies that F∩(X–U) ⊂N. Now, A⊂F and $F_g{}^\star$ ⊂ F implies that $A_g{}^\star$∩(X–U) ⊂ $F_g{}^\star$∩(X–U) ⊂ F∩(X–U) ⊂ N. By hypothesis, since $A_g{}^\star$∩(X–U) is g-closed, $A_g{}^\star$∩(X–U) = **∅,** and so $A_g{}^\star$⊂U which implies that A is $\mathbf{I}_{gg}$-closed.

Theorem: 2.10 Let (X, τ, **I**) be an ideal topological space and A ⊂ X. Then the following hold.
(a) Every g-closed set is $\mathbf{I}_{gg}$-closed.
(b) The union of two $\mathbf{I}_{gg}$-closed sets is $\mathbf{I}_{gg}$-closed.
(c) The intersection of a closed set with an $\mathbf{I}_{gg}$-closed set is $\mathbf{I}_{gg}$-closed.
(d) The union of a g-closed set with an $\mathbf{I}_{gg}$-closed is $\mathbf{I}_{gg}$-closed.

Proof:
(a) If A is g-closed in (X, τ, **I**), then A is $\tau_g{}^\star$ -closed. Therefore, $A_g{}^\star$ ⊂ A⊂ U whenever A ⊂ U and U ∈ τ_g implies that A is $\mathbf{I}_{gg}$-closed.

(b) Let A and be B be $\mathbf{I}_{gg}$-closed sets and U be a g-open set such that A∪B⊂ U. Since $A_g{}^\star$ ⊂U, $B_g{}^\star$ ⊂ U and by [1, Theorem 3.7(h)], $(A∪B)_g{}^\star$ =$A_g{}^\star$∪ $B_g{}^\star$⊂U which implies that A∪B is an $\mathbf{I}_{gg}$-closed set.

(c) Let A be an $\mathbf{I}_{gg}$-closed set and B be a closed set in (X, τ, **I**). Suppose that A ∩ B⊂U and U is g-open in X. Then A ⊂ U∪(X- B). Now X-B is open and hence U ∪ (X-B) is g-open. Since A is $\mathbf{I}_{gg}$-closed, $cl_g{}^\star$ (A) ⊂U ∪ (X - B). Therefore, $cl_g{}^\star$(A)∩B⊂U which implies that $cl_g{}^\star$(A ∩ B) ⊂ U. So A∩ B is an $\mathbf{I}_{gg}$ -closed set.

(d) The proof follows from (a) and (b).

The following example shows that the converse of Theorem 2.10(a) is not true.

Example: 2.11 Let X = {a, b, c}, τ = {∅, X,{a},{a, b}} and **I**={∅,{a}}. If A = {a} and A ⊂U where U = {a} is open, then cl(A) = X⊄ U but $A_g{}^\star$ =∅ ⊂ U which implies that A is g-closed.

The following example shows that the concept of $\mathbf{I}_g$-closed sets and $\mathbf{I}_{gg}$ -closed sets are independent.

Example: 2.12 In Example 2.11, if A ={a} and U={a}∈τ, then cl⋆(A) = X ⊄ U but $A_g{}^\star$ = ∅ ⊂ U which implies that A is $\mathbf{I}_{gg}$-closed but not $\mathbf{I}_g$-closed.

Theorem: 2.13 Let (X, τ, **I**) be an ideal topological space and A⊂ X. If A ⊂ $A_g{}^\star$ and A is $\mathbf{I}_{gg}$-closed, then A is $\mathbf{I}_g$-closed.

Proof: Let A⊂ U where A⊂X and U∈τ. Since U is g-open, $A_g{}^\star$⊂U by hypothesis. Since A⊂$A_g{}^\star$ and by [1, Theorem 3.10], A⋆ = $A_g{}^\star$ ⊂ U which implies that A is $\mathbf{I}_g$-closed.

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Source of support: Nil, Conflict of interest: None Declared