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ON SUMS OF POLYNOMIAL CONJUGATE EP MATRICES

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ABSTRACT

Necessary and sufficient conditions are determined for a sum of polynomial con-EP matrices to be polynomial con-EP and it is shown that the sum and parallel sum of parallel summable polynomial con-EP matrices are polynomial con-EP.

Keywords: *EP matrix, polynomial matrix, Generalized inverse.*

Ams classification: *15A09, 15A15, 15A57.*

INTRODUCTION

In this paper we shall study the question of when of polynomial conjugate EP (polynomial con-EP) matrices is polynomial con-EP. We give necessary and sufficient conditions for sum of polynomial con-EP matrices to be polynomial con-EP. We also show that sum and parallel sum of parallel summable (p.s) [7], polynomial con-EP matrices are polynomial con-EP. The results of this paper for polynomial con-EP matrices are analogous to that of EP matrices, studied in [4].

Throughout we shall deal with $n \times n$ complex polynomial matrices. An n-square matrix $A(\lambda)$ which is a polynomial in the scalar variable λ from a field C represented by $A(\lambda) = A_m\lambda^m + A_{m-1}\lambda^{m-1} + ... + A_1\lambda + A_0$ where the leading coefficient $A_m \neq 0$, A_i's are square matrices in $V_{n\times n}$ is defined a polynomial matrix. Let $\overline{A}$, A^T, A^* and A^- denote the conjugate, transpose, conjugate transpose and generalized inverse (A A^- A=A) of A respectively. $A^\dagger$ denotes the Moore-penrose inverse satisfying the following four equations: AXA=A, XAX=X, $(AX)^*$=AX and $(XA)^*$ =XA of [7]. Any matrix A is called polynomial con-EP if R(A)=R(A^T) or N(A)=N(A^T) or $AA^\dagger = A^\dagger A$ and is called polynomial con-EP, if A is polynomial con-EP and rk(A)=r, where N(A), R(A) and rk(A) denote the null space, range space and rank of A respectively[5]. Any two matrices A and B are said to be p.s. if $N(A+B) \subseteq N(B)$ and $N(A+B)^* \subseteq N(B)^*$ or equivalently $N(A+B) \subseteq N(A)$ and $N(A+B)^* \subseteq N(A)^*$. If A and B are p.s. then parallel sum of A and B denoted by A:B and defined as A : B = $A(A+B)^- B$ of [7], if A and B are p.s. then the following hold [7]

(1) $A:B = B:A$
(2) A^* and B^* are p.s. and $(A:B)^* = A^* : B^*$
(3) If U is nonsingular them UA and UB are p.s. and UA:UB=U(A:B)
(4) R(A:B)=R(A) $\cap$ R(B)
(5) (A: B):E= A:(B:E) if all the parallel sum operations involved are defined.

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Let M= $\begin{bmatrix} A & B \\ C & D \end{bmatrix}$ be an $n \times n$ matrix. Then the schur complement of A in M, denoted by M/A is defined as D-CA $^{-}$B [3]. For further properties of schur complements one may refer [1] and [2].

Theorem: 1 Let A_j (i=1 to n) be polynomial con-EP matrices. Then $A = \sum_{i=1}^{n} A_j$ is polynomial con-EP if any one of the following equivalent conditions hold.

(i) N(A) $\subseteq$ N(A_i) for each i.

(ii) rk $\begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ \cdot \\ \cdot \\ A_n \end{bmatrix}$ = rk(A).

Proof: Equivalence of (i) and (ii) is already proved in [4] Since each A_i is polynomial con-EP $N(A_i) = N(A_i^T)$ for each i. $N(A) \subseteq N(A_i)$ for each i implies $N(A) \subseteq \cap N(A_i) = \cap N(A_i^T) = \cap N(A_i^T)$ and $rk(A) = rk(A^T)$. Hence $N(A) = N(A^T)$. Thus A is polynomial con-EP. Hence the Theorem.

Remark 1: In the above Theorem if A is nonsingular then the conditions hold automatically and A is polynomial con-EP. But, it fails if we relax the condition on the A_i’ s.

Example 1: A= $\begin{bmatrix} \lambda^2 & \lambda \\ \lambda & i \end{bmatrix}$ is polynomial con-EP, B= $\begin{bmatrix} \lambda^3 & \lambda^2 + i \\ \lambda^2 & i \end{bmatrix}$ is not polynomial con-EP then A+B is not polynomial con-EP. However, $N(A+B) \subseteq N(A)$ and $N(A+B) \subseteq N(B)$; rk $\begin{bmatrix} A \\ B \end{bmatrix}$ = rk(A+B).

Remark 2: If rank is additive, that is rk(A) = $\sum rk(A_i)$ then by Theorem 11 of [3], $R(A_i) \cap R(A_j) = \{0\}, i \neq j$, which implies $N(A) \subseteq N(A_i)$ for each i, hence A is polynomial con-EP. That the conditions given in Theorem 1 are weaker than the condition of rank additivity can be seen by the following example.

Example 2: Let A= $\begin{bmatrix} \lambda^2 & \lambda \\ \lambda & i \end{bmatrix}$ and B= $\begin{bmatrix} \lambda^3 & \lambda^2 \\ \lambda^2 & i \end{bmatrix}$ A, B and A+B are polynomial con-EP$_1$ matrices. Conditions (i) and (ii) of Theorem 1 hold. But rk(A+B) $\neq$ rk(A)+rk(B).

Theorem 2: Let A_i (i=1 to n) be polynomial con-EP$_1$ matrices such that $\sum_{i \neq j} (A_i)^* A_j = 0$. Then A= $\sum A_i$ is polynomial con-EP.

Proof: As in the proof of Theorem 2 in [6], Let $\sum_{i \neq j} (A_i)^* A_j = 0$ implies $N(A) \subseteq N(A_i)$ for each i. Since each A_i is polynomial con-EP, A is polynomial con-EP. By theorem 1 hence the theorem

Remark: 3

Theorem 2 fails if we relax the condition that A_i’s are polynomial con-EP. For instance

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & i\lambda^2 & 0 \\ i\lambda^2 & 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & i\lambda^2 & 0 \\ i\lambda^2 & 0 & 0 \\ 0 & i\lambda^2 & 0 \end{bmatrix}$ are not polynomial con-EP, then A+B is also not polynomial con-EP. However $B^* A + A^* B = 0$.

Remark: 4 The condition given in Theorem 2 implies those in Theorem 1, but not conversely. This can be seen by the following.

Example: 3 Let $A = \begin{bmatrix} \lambda^2 & i \\ i & \lambda \end{bmatrix}$ and $B = \begin{bmatrix} \lambda^2 & \lambda \\ \lambda & i \end{bmatrix}$. A and B are polynomial con-EP matrices. N(A+B) $\subseteq$ N(A) and N(B).

But $A^T B + B^T A = \begin{bmatrix} 2\lambda^4 + 2\lambda i & \lambda^3 + \lambda^2(i+1) - 1 \\ \lambda^3 + \lambda^2(i-1) - 1 & 4\lambda i \end{bmatrix} \neq 0$.

Remark: 5 We note that the conditions given in Theorem 1 and Theorem 2 are only sufficient for the sum of polynomial con-EP matrices to be polynomial con-EP. But not necessary and this is illustrated in the following.

Example: 4 Let A= $\begin{bmatrix} \lambda^2 & i \\ -i & 0 \end{bmatrix}$ and B= $\begin{bmatrix} \lambda^2 & \lambda \\ \lambda & i \end{bmatrix}$. A and B are con-EP $_2$. Neither the conditions in Theorem 1 nor in Theorem 2 hold. However A+B is polynomial con-EP.

If A and B are polynomial con-EP matrices by Result 2.1 of [5]. We get $A^* = K_1 \overline{A}$, and $B^* = K_2 \overline{B}$, where K_1 and K_2 are nonsingular $n \times n$ matrices. If $K_1 = K_2$, then A+B is polynomial con-EP. If $(K_1 - K_2)$ is nonsingular then the above conditions are also necessary for the sum of polynomial con-EP matrices to be polynomial con-EP. This is given in the following Theorem.

Theorem: 3 Let $A^* = K_1 \overline{A}$ and $B^* = K_2 \overline{B}$ such that $(K_1 - K_2)$ is a nonsingular matrix. Then A+B is polynomial con-EP if and any only if $N(A+B) \subseteq N(B)$.

Proof: $A^* = K_1 \overline{A}$ and $B^* = K_2 \overline{B}$ by Result 2.1 of [5] A and B are polynomial con-EP matrices. Since $N(A+B) \subseteq N(B)$ We can see that, $N(A+B) \subseteq N(A)$. Hence by Theorem 1, A+B is polynomial con-EP.

Conversely, let us assume that A+B is polynomial con-EP, then by Theorem1 of [5], $A^* + B^* = (A+B)^* = G(\overline{A+B})$ for some $n \times n$ matrix G. Hence $K_1 \overline{A} + K_2 \overline{B} = G\overline{(A+B)}$. This implies $K\overline{A} = H\overline{B}$, where $K = K_1 - G$ and $H = G - K_2$.

$(K+H)\overline{A} = H\overline{(A+B)}$ and $(K+H)\overline{B} = K\overline{(A+B)}$. By hypothesis, $K+H = K_1 - K_2$ is nonsingular. $N\overline{(A+B)} \subseteq N(H\overline{(A+B)}) = N(K+H)\overline{A} = N(\overline{A})$,which implies $N(A+B) \subseteq N(A)$.

Similarly, $N\overline{(A+B)} \subseteq N(K\overline{(A+B)}) = N(K+H)\overline{B} = N(\overline{B})$ implies $N(A+B) \subseteq N(B)$. Thus A+B is polynomial con-EP implies, $N(A+B) \subseteq N(A)$ and N(B). Hence the Theorem.

Remark 6: The condition ($K_1 - K_2$) to be nonsingular is essential in Theorem 3. This is illustrated in the following.

Example 5: $A=\begin{bmatrix}\lambda^2 & 0\\ 0 & i\lambda^2\end{bmatrix}$ and B= $\begin{bmatrix}i\lambda & 0\\ 0 & 0\end{bmatrix}$ are both symmetric, hence con-EP. Here $K_1 = K_2$ and $A+B=\begin{bmatrix}i\left(\lambda^2+\lambda\right) & 0\\ 0 & i\lambda^2\end{bmatrix}$ is polynomial con-EP. But $N(A+B)\not\subseteq N(A)$ or N(B). Thus Theorem 3 fails.

Lemma: 1 Let A and B be polynomial con-EP matrices. Then A and B are p.s. if and only if $N(A+B)\subseteq N(A)$.

Proof: If A and B are p.s. then $N(A+B)\subseteq N(A)$ follows from definition.

Conversely, if $N(A+B)\subseteq N(A)$ then $N(A+B)\subseteq N(B)$. Since A and B are polynomial con-EP matrices by Theorem 1, A+B is polynomial con-EP.

Hence $N(A+B)^T = N(A+B) = N(A)\cap N(B) = N(A^T)\cap N(B^T)$ which implies, $N(A+B)^T = N(A^T)\cap N(B^T)$

Therefore, $N\left(A+B\right)^*\subseteq N\left(A\right)^*$ and $N\left(A+B\right)^*\subseteq N\left(B\right)^*$. By hypothesis $N(A+B)\subseteq N(A)$. Hence A and B are p.s.

In the following Theorem we show that sum and parallel sum of p.s. polynomial con-EP matrices is polynomial con-EP.

Theorem: 4 If A and B are p.s. polynomial con-EP matrices then A: B and A+B are polynomial con-EP.

Proof: Since A and B are p.s. polynomial con-EP matrices, by Lemma 1, $N(A+B)\subseteq N(A)$ and $N(A+B)\subseteq N(B)$. Now, the fact that (A+B) is polynomial con-EP follows from Theorem 1.

Now,
$$\begin{aligned}
R(A:B)^* &= R(A^*:B^*) && \text{(By (2))}\\
&= R(A^*)\cap R(B^*) && \text{(By (4))}\\
&= R(\overline{A})\cap R(\overline{B}) && \text{(A and B are polynomial con-EP)}\\
&= R(\overline{A}:\overline{B}) && \text{(By (4))}\\
&= R(\overline{A:B})
\end{aligned}$$

Which implies $(\overline{A:B})$ is polynomial con-EP and hence A: B is polynomial con-EP. Thus A: B is polynomial con-EP whenever A and B are polynomial con-EP. Hence the Theorem.

Theorem: 5 Let A be polynomial con-EP_{r_1} and B be polynomial con-EP_{r_2} matrices of order n such that $N(A+B)\subseteq N(B)$. Then there exists a $2n\times 2n$ polynomial con-EP_r matrix M such that the schur complement of C in M is polynomial con-EP, where r= r_1+r_2 and C=A+B.

Proof: Since A is polynomial con-EP_{r_1} and B is polynomial con-EP_{r_2}, by Result 2.1 of [5] there exist unitary matrices U and V of order n such that
$A=U^TDU$, and B= V^T EV, where
$D=\begin{bmatrix}H & 0\\ 0 & 0\end{bmatrix}$, H is $r_1\times r_1$ nonsingular and
$E=\begin{bmatrix}K & 0\\ 0 & 0\end{bmatrix}$, H is $r_2\times r_2$ nonsingular.
Let us define $P=\begin{bmatrix}V & 0\\ U & I\end{bmatrix}$, P is nonsingular.

Now, $P^T\begin{bmatrix} E & 0 \\ 0 & D \end{bmatrix}P = \begin{bmatrix} V^T & U^T \\ 0 & I \end{bmatrix}\begin{bmatrix} E & 0 \\ 0 & D \end{bmatrix}\begin{bmatrix} V & 0 \\ U & I \end{bmatrix}$

$$= \begin{bmatrix} V^TEV + U^TDU & U^TD \\ DU & D \end{bmatrix}$$

$$= \begin{bmatrix} A+B & U^TD \\ DU & D \end{bmatrix}$$

$$= \begin{bmatrix} C & AU^* \\ \overline{U}A & \overline{U}AU^* \end{bmatrix} = M.$$

M is $2n\times 2n$ matrix and rk(M) = rk(E)+rk(D)= $r_1 + r_2$ = r.

Let us define $Q = \begin{bmatrix} T_n & 0 \\ UA^\dagger A & I_n \end{bmatrix}$, Q is nonsingular.

Since A is polynomial con-EP $AA^\dagger = \overline{A^\dagger A}$ and by Result 2.2 of [5] $\overline{U}AU^*$ is polynomial con-EP.

We can write M as, M= $Q^T\begin{bmatrix} B & 0 \\ 0 & \overline{U}AU^* \end{bmatrix}Q$. Since B and $\overline{U}AU^*$ are polynomial con-EP, Q is nonsingular, M is polynomial con-EP. Since M is of rank r, M is polynomial con- EP_r. Thus we have proved the existence of the polynomial con-EP_r matrix M. Now C=A+B is polynomial con-EP follows from Theorem 1. Since $N(C) \subseteq N(A) = N(\overline{U}A)$ and $N(C^*) \subseteq N(A^*) = N(AU^*)^*$. By the Lemma in [7], $A = AC^-C = CC^-A$ and $(\overline{U}A)C^-(AU^*)$ is invariant for all choice of C^-. The schur complement of $C^\dagger$ in M is,

$$\begin{aligned} M/C &= \overline{U}AU^* - \overline{U}A\,C^-\,AU^* \\ &= \overline{U}A\,U^* - \overline{U}(A+B)C^-(A\,U^*) + \overline{U}BC^-\,A\,U^* \\ &= \overline{U}A\,U^* - \overline{U}CC^-A\,U^* + \overline{U}BC^-\,A\,U^* \\ &= \overline{U}A\,U^* - \overline{U}A\,U^* + \overline{U}BC\,A\,U^* \\ &= \overline{U}BC^-\,A\,U^* \\ &= \overline{U}(A:B)\,U^* \end{aligned}$$

Since A and B are polynomial con-EP, by Theorem 4, A:B is polynomial con-EP. By Result 2.2 of [5],
M/C = $\overline{U}(A:B)\,U^*$ = P(A:B) P^T, where P= $\overline{U}$ is unitary, is also polynomial con-EP. Hence the Theorem.

Remark: 7 In a special case if A and B are polynomial con-EP matrices such that A+B= I_n then AB= A:B =B:A =BA is polynomial con-EP. However this fails if we relax the conditions on A and B. For instance, A = $\begin{bmatrix} \lambda^3 & 0 \\ 0\lambda & 3 \end{bmatrix}$ is olynomial con-EP_2 and B=$\begin{bmatrix} 0 & 0 \\ i\lambda & 1 \end{bmatrix}$ is not polynomial con-EP. Here AB=BA is not polynomial con-EP, however A+B= I_2.

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