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ON VALUE SHARING OF MEROMORPHIC FUNCTIONS

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ABSTRACT

In this paper, we introduce a new concept of value sharing called additive sharing to prove some uniqueness theorems for meromorphic functions.

Key words: *Meromorphic functions, Order, Additive sharing.*

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1. INTRODUCTION AND DEFINITIONS

Let f and g be two non-constant meromorphic functions defined in the open complex plane C and let $a \in C \cup \{\infty\}$. We say that f and g share the value a CM (counting multiplicities) or IM (ignoring multiplicities) provided $f-a$ and $g-a$ have same zeros CM or IM respectively and f, g share ∞ CM or IM provided that $\frac{1}{f}$ and $\frac{1}{g}$ share 0 CM or IM.

It is assumed that the reader is familiar with the standard notations and definitions of Nevanlinna's theory as found in [5].

In 1979, Gundersen [4] proved the following theorems.

Theorem: A [4] If f and g share four values $\{a_i\}_1^4$ IM and $f \neq g$, then outside a set E of finite linear measure:

$$(a)\lim_{r\to\infty}\frac{T(r,f)}{T(r,g)}=1;$$

$$(b)\lim_{r\to\infty}\sum_{i=1}^{4}\frac{\overline{N}(r,a_i)}{T(r,f)}=\lim_{r\to\infty}\sum_{i=1}^{4}\frac{\overline{N}(r,a_i)}{T(r,g)}=2,$$

where $\overline{N}(r,a_i)=\overline{N}(r,a_i;f)=\overline{N}(r,a_i;g)$ for $i=1,2,3,4$.

Theorem: B [4] If f and g share three values IM, then outside a set E of finite measure,

$$\limsup_{r\to\infty}\frac{T(r,f)}{T(r,g)}\le 3 \text{ and } \limsup_{r\to\infty}\frac{T(r,g)}{T(r,f)}\le 3.$$

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In 1989, Brosch [3] improved Theorem B by proving the following result.

Theorem: C [3] If f and g share three values CM, then

$$\frac{3}{8}T(r,g)(1+o(1)) \le T(r,f) \le \frac{8}{3}T(r,g)(1+o(1)) \text{ as } r \to \infty (r \notin E).$$

Recently Banerjee and Dutta [1] introduced a new idea of value sharing known as relative sharing which runs as follows.

Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We say that f, g share a CM(IM) relatively with respect to a meromorphic function h, provided the functions F and G share a CM(IM) respectively where $F = \frac{f}{h}$ and $G = \frac{g}{h}$.

Using this idea of relative sharing of values of meromorphic functions Banerjee and Dutta proved the followings.

Theorem: D [2] Let f and g be two meromorphic functions. If there is a function h with $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$ such that F, G share four values $\{a_i\}_1^4$ IM, then outside a set E of finite linear measure,

$$(a) \lim_{r\to\infty} \frac{T(r,f)}{T(r,g)} = 1;$$

$$(b) \lim_{r\to\infty} \sum_{i=1}^{4} \frac{\overline{N}(r,a_i)}{T(r,f)} = \lim_{r\to\infty} \sum_{i=1}^{4} \frac{\overline{N}(r,a_i)}{T(r,g)} = 2,$$

where $\overline{N}(r,a_i) = \overline{N}(r,a_i;F) = \overline{N}(r,a_i;G)$ for $i = 1,2,3,4.$

Theorem: E [2] Let f and g be two meromorphic functions. If there is a function h with $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$ such that F, G share three values IM, then outside a set E of finite measure,

$$\limsup_{r\to\infty} \frac{T(r,f)}{T(r,g)} \le 3 \text{ and } \limsup_{r\to\infty} \frac{T(r,g)}{T(r,f)} \le 3.$$

Theorem: F [1] Let f and g be two non-constant meromorphic functions. If there is a function h with $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$ such that F, G share $\{a_i\}_1^3$ IM, then $\rho_f = \rho_g$ where $F = \frac{f}{h}$ and $G = \frac{g}{h}$ and ρ_f denotes the order of f.

In this paper, we introduce another notion of value sharing called `additive sharing' and prove parallel results of Banerjee and Dutta {[1], [2]} using the idea of additive sharing.

First we introduce the following definition.

Definition: 1 Let f and g be two non-constant meromorphic functions and $a \in \mathbb{C} \cup \{\infty\}$. We say that f, g share a CM(IM) additively with respect to a meromorphic function h, provided that F and G share a CM(IM) respectively where $F = f + h$ and $G = g + h$.

Throughout the paper we assume f, g etc. are non-constant meromorphic functions defined in the open complex plane C and $S(r,f)$ any quantity satisfying

$$S(r,f) = o(T(r,f))(r \to \infty, r \notin E).$$

2. THEOREMS

In this section we prove the main results of the paper.

Theorem: 1 Let f and g be two non-constant meromorphic functions. If there is a function h with $T(r,h)=o(T(r,f))$ and $T(r,h)=o(T(r,g))$ such that F, G share three values IM then $\rho_f = \rho_g$ where $F = f+h$ and $G = g+h$.

Proof: We have $T(r,h)=S(r,f)=o(T(r,f))$ as $r\to\infty, r\notin E$ (a set of finite linear measure).

Now $F = f+h$, so $T(r,F)\le T(r,f)+T(r,h)+O(1) = [1+o(1)]T(r,f)$. (1)

On the other hand, $f = F-h$ gives

$$T(r,f)\le T(r,F)+T(r,h)+O(1)=(1+o(1))T(r,F)$$

i.e., $(1+o(1))T(r,f)\le T(r,F)$. (2)

Hence from (1) and (2), $(1+o(1))T(r,f)=T(r,F)$. (3)

Consequently, $\rho_f = \rho_F$. (4)

Applying similar arguments we can also prove that $\rho_g = \rho_G$. (5)

Further since F, G share three values IM, by Theorem B

$$\frac{1}{3}(1+o(1))T(r,G)\le T(r,F)\le 3(1+o(1))T(r,G).$$

So, $\rho_F = \rho_G$. (6)

Combining (4), (5) and (6), we get the result.

Example: 1 Let $f(z)=e^z-e^{-z}, g(z)=3-3e^{-z}$ and $h(z)=e^{-z}$. Then $F(z)=e^z$ and $G(z)=3-2e^{-z}$ share $1,2,\infty$ CM. Here $T(r,h)\ne o(T(r,f))$ and $T(r,h)\ne o(T(r,g))$ but $\rho_f = \rho_g$.

Example: 2 Let $f(z)=z, g(z)=e^{-z}-e^z+z$ and $h(z)=e^z-z$. Then $F(z)=e^z$ and $G(z)=e^{-z}$ share $0,1,-1,\infty$ CM. Here $T(r,h)\ne o(T(r,f))$ and $T(r,h)\ne o(T(r,g))$ and $\rho_f \ne \rho_g$.

Theorem: 2 Let f and g be two non-constant meromorphic functions. If there is a function h with $T(r,h)=o(T(r,f))$ and $T(r,h)=o(T(r,g))$ such that F, G share four values $\{a_i\}_1^4$ IM, then outside a set E of finite linear measure,

$$(a)\lim_{r\to\infty}\frac{T(r,f)}{T(r,g)}=1;$$

$$(b)\lim_{r\to\infty}\sum_{i=1}^{4}\frac{\overline{N}(r,a_i)}{T(r,f)}=\lim_{r\to\infty}\sum_{i=1}^{4}\frac{\overline{N}(r,a_i)}{T(r,g)}=2,$$

where $\overline{N}(r,a_i)=\overline{N}(r,a_i;F)=\overline{N}(r,a_i;G)$ for $i=1,2,3,4$ and $F = f+h$ and $G = g+h$.

Proof: By Second Fundamental theorem, as $r\to\infty$ outside a set of finite linear measure,

$$(3+o(1))T(r,F)\le \sum_{i=1}^{4}\bar{N}(r,a_i)+\bar{N}(r,F).$$

Using (3) and $\bar{N}(r,F)\le T(r,F)$, we get at once

$$(2+o(1))T(r,f)\le\sum_{i=1}^{4}\bar{N}(r,a_i)$$

or, $T(r,f)\le\left(\frac{1}{2}+o(1)\right)\sum_{i=1}^{4}\bar{N}(r,a_i)$. (7)

Similarly for g,

$$T(r,g)\le\left(\frac{1}{2}+o(1)\right)\sum_{i=1}^{4}\bar{N}(r,a_i). \quad (8)$$

Therefore,
$$\begin{aligned}\sum_{i=1}^{4}\bar{N}(r,a_i)&\le\sum_{i=1}^{4}\bar{N}(r,0;F-G)\\&=\bar{N}\left(r,\frac{1}{F-G}\right)\\&\le T\left(r,\tfrac{1}{F-G}\right)\\&\le T(r,F)+T(r,G)+O(1)\\&=[1+o(1)](T(r,f)+T(r,g)),\text{ using (3)} \quad (9)\\&\le(1+o(1))\sum_{i=1}^{4}\bar{N}(r,a_i),\text{ using (7) and (8).}\end{aligned}$$

So outside a set E of finite measure,

$$\lim_{r\to\infty}\frac{T(r,f)+T(r,g)}{\sum_{i=1}^{4}\bar{N}(r,a_i)}=1.$$

Let there is a sequence $r_n\to\infty$ such that

$$\frac{T(r_n,f)}{\sum_{i=1}^{4}\bar{N}(r_n,a_i)}\to c<\frac{1}{2}\text{ and }\frac{T(r_n,g)}{\sum_{i=1}^{4}\bar{N}(r_n,a_i)}\to 1-c$$

where c is a constant.

Then
$$\frac{\sum_{i=1}^{4}\bar{N}(r_n,a_i)}{T(r_n,g)}\to\tfrac{1}{1-c}<2,$$

which contradicts (8).

Hence
$$\lim_{r\to\infty}\frac{\sum_{i=1}^{4}\overline{N}(r,a_i)}{T(r,f)}=\lim_{r\to\infty}\frac{\sum_{i=1}^{4}\overline{N}(r,a_i)}{T(r,g)}=2.$$

This proves (b).

From (9), we have

$$\sum_{i=1}^{4}\bar{N}(r,a_i)\le[1+o(1)](T(r,f)+T(r,g))\le(1+o(1))\sum_{i=1}^{4}\bar{N}(r,a_i)$$

i.e., $\dfrac{\sum_{i=1}^{4}\bar{N}(r,a_i)}{T(r,g)}\le[1+o(1)][1+\dfrac{T(r,f)}{T(r,g)}]\le[1+o(1)]\dfrac{\sum_{i=1}^{4}\bar{N}(r,a_i)}{T(r,g)}$

$$i.e.,\lim_{r\to\infty}\frac{\sum_{i=1}^{4}\overline{N}(r,a_i)}{T(r,g)}\le 1+\lim_{r\to\infty}\frac{T(r,f)}{T(r,g)}\le\lim_{r\to\infty}\frac{\sum_{i=1}^{4}\overline{N}(r,a_i)}{T(r,g)}$$

i.e., $2 \le 1 + \lim_{r\to\infty} \frac{T(r,f)}{T(r,g)} \le 2$

i.e., $\lim_{r\to\infty} \frac{T(r,f)}{T(r,g)} = 1$.

This proves (a).

This completes the proof of the Theorem 2.

Example: 3 Let $f(z) = e^z - z, g(z) = e^{-z} - z$ and $h(z) = z$. Then F, G share 0,-1,1,∞. Again $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$. Also $T(r,f) \sim T(r,g)$.

Example: 4 Let $f(z) = z, g(z) = e^{-z} - e^z + z$ and $h(z) = e^z - z$. Then F, G share 0, -1, 1, ∞. Again $T(r,h) \ne o(T(r,f))$. Also $T(r,f)$ ~/~ $T(r,g)$.

Theorem: 3 Let f and g be two non-constant meromorphic functions. If there is a function h with $T(r,h) = o(T(r,f))$ and $T(r,h) = o(T(r,g))$ such that F, G share three values IM, then outside a set E of finite measure,

$$\limsup_{r\to\infty} \frac{T(r,f)}{T(r,g)} \le 3 \text{ and } \limsup_{r\to\infty} \frac{T(r,g)}{T(r,f)} \le 3, \text{ where } F = f + h \text{ and } G = g + h .$$

Proof: Since F, G share three values IM, so from Theorem B , outside a set E of finite measure,

$$\limsup_{r\to\infty} \frac{T(r,F)}{T(r,G)} \le 3 \text{ and } \limsup_{r\to\infty} \frac{T(r,G)}{T(r,F)} \le 3.$$

i.e., $T(r,F) < 3[1+o(1)]T(r,G)$ and $T(r,G) < 3[1+o(1)]T(r,F)$.

Now using (3), $T(r,f) < 3[1+o(1)]T(r,g)$

i.e., $\frac{T(r,f)}{T(r,g)} < 3 + o(1)$

and hence $\limsup_{r\to\infty} \frac{T(r,f)}{T(r,g)} \le 3$.

Similarly $\limsup_{r\to\infty} \frac{T(r,g)}{T(r,f)} \le 3$.

This proves the Theorem 3.

Example: 5 Let $f(z) = \frac{e^{3z} - 3e^{2z} + 3}{1 - 3e^z}, g(z) = \frac{e^z}{1 - 3e^z}$ and $h(z) = \frac{3}{3e^z - 1}$. Then F, G share three values 0, ∞ CM and 1 IM. Again $T(r,h) \ne o(T(r,g))$ but $T(r,f) \sim 3T(r,g)$.

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