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ON THE COMPARATIVE GROWTH ANALYSIS OF A SPECIAL TYPE OF DIFFERENTIAL POLYNOMIAL

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ABSTRACT

In this paper we discuss some comparative growth estimates of composite entire and meromorphic functions and a special type of differential polynomial as considered by Bhooshnurmath and Prasad [4] and generated by one of the factors of the composition.

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1. INTRODUCTION, DEFINITIONS AND NOTATIONS

For any two transcendental entire functions f and g defined in the open complex plane $\mathbb{C}$, Clunie [5] proved that

$$\lim_{r\to\infty}\frac{T(r,f_o g)}{T(r,f)}=\lim_{r\to\infty}\frac{T(r,f_o g)}{T(r,g)}=\infty.$$

Singh [10] studied some comparative growth properties of $\log T(r,f_o g)$ and $T(r,f)$. He [10] also raised the question of investigating the comparative growth of $\log T(r,f_o g)$ and $T(r,g)$ which he was unable to solve. Lahiri [8] proved some results on the comparative growth of $\log T(r,f_o g)$ and $T(r,g)$.

Some mathematicians like H. X. Yi [12] and many more studied the comparative growth of a meromorphic function and its derivatives.

Since the natural extension of a derivative is a differential polynomial, in this paper we extend some earlier results for a special type of linear differential polynomial of the form $F=f^n Q[f]$ where $Q[f]$ is a differential polynomial in f and n = 0, 1, 2,…. as considered by Bhooshnurmath and Prasad [4]. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [11] and [7].

In the sequel we use the following two notations:

$(i)\ \log^{[k]}x=\log(\log^{[k-1]}x)\ \ for\ k=1,2,3,\dots;\ \ \log^{[0]}x=x$

and

$(ii)\ \exp^{[k]}x=\exp\left(exp^{[k-1]}x\right)\ for\ k=1,2,3,\dots;\ \exp^{[0]}x=x.$

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The following definitions are well known:

Definition: 1 *The order ρ_f and lower order λ_f of a meromorphic function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire, one can easily verify that

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition: 2 *The hyper order $\overline{\rho}_f$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function f are defined as*

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log r}.$$

If f is entire, then

$$\overline{\rho}_f = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition: 3 *[9] Let f be a meromorphic function of order zero. Then the quantities ρ_f^*, λ_f^* and $\overline{\rho}_f^*$, $\overline{\lambda}_f^*$ are defined in the following way*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r},$$

$$\lambda_f^* = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log^{[2]} r}$$

and

$$\overline{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}$$

$$\overline{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} T(r,f)}{\log^{[2]} r}.$$

If f is entire then clearly

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r},$$

$$\lambda_f^* = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r}.$$

and

$$\overline{\rho}_f^* = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r},$$

$$\overline{\lambda}_f^* = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log^{[2]} r}.$$

Definition: 4 *The type σ_f of a meromorphic function f is defined as follows*

$$\sigma_f = \limsup_{r\to\infty} \frac{T(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

When f is entire, then

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, 0 < \rho_f < \infty.$$

Definition: 5 *A meromorphic function* $a \equiv a(z)$ *is called small with respect to f if* $T(r,a) = S(r,f)$.

2. LEMMAS

In this section we present some lemmas which will be needed in the sequel.

Lemma: 1 *[5] If f and g be any two entire functions then for all sufficiently large values of r,*

$$M(r, f_o g) \geq M\left(\frac{1}{8} M\left(\frac{r}{2}, g\right) - |g(0)|, f\right).$$

Lemma: 2 *[1] Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T(r, f_o g) \leq \{1 + o(1)\} \frac{T(r,g)}{\log M(r,g)} T(M(r,g), f).$$

Lemma: 3 *[3] Le f be meromorphic and g be entire and suppose that* $0 < \mu < \rho_g \leq \infty$. *Then for a sequence of values of r tending to infinity,*

$$T(r, f_o g) \geq T(\exp(r^\mu), f).$$

Lemma: 4 *[4] Let* $F = f^n Q[f]$ *where Q[f] is a differential polynomial in f. If n ≥ 1 then* $\rho_F = \rho_f$ *and* $\lambda_F = \lambda_f$.

Lemma: 5 *Let* $F = f^n Q[f]$ *where Q[f] is a differential polynomial in f. If n ≥ 1 then*

$$\lim_{r\to\infty} \frac{T(r,\ F)}{T(r,f)} = 1.$$

The proof of Lemma 5 directly follows from Lemma 4.

In the line of Lemma 4 we may prove the following lemma:

Lemma: 6 *Let* $F = f^n Q[f]$ *where Q[f] is a differential polynomial in f. If n ≥ 1 then* $\overline{\rho}_F = \overline{\rho}_f$ *and* $\overline{\lambda}_F = \overline{\lambda}_f$.

Lemma: 7 *Let f be meromorphic and g be transcendental entire such that* $\rho_f = 0$ *and* $\rho_g < \infty$. *Then* $\rho_{f_o g} \leq \rho_f^* . \rho_g$.

Proof: In view of Lemma 2 and the inequality $T(r,g) \leq \log^+ M(r,g)$, we get that

$$\rho_{f_o g} = \limsup_{r\to\infty} \frac{\log T(r, f_o g)}{\log r} \leq \limsup_{r\to\infty} \frac{\log T(M(r,g), f) + o(1)}{\log r}$$

$$= \limsup_{r\to\infty} \frac{\log T(M(r,g),f)}{\log^{[2]} M(r,g)} . \limsup_{r\to\infty} \frac{\log^{[2]} M(r,g)}{\log r} = \rho_f^* . \rho_g.$$

This proves the lemma.

Remark: 1 *The sign '≤' in Lemma 7 cannot be removed by '<' only as we see in the following example.*

Example: 1 *Let f = z and* $g = expz$. *Then* $\rho_{f_o g} = 1, \rho_g = 1$ *and* $\rho_f = 0$. *So*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log^{[2]} r} = \limsup_{r\to\infty} \frac{\log^{[2]} r}{\log^{[2]} r} = 1.$$

Therefore

$$\rho_{f_o g} = \rho_f^* . \rho_g.$$

3. THEOREMS

In this section we present the main results of the paper.

Theorem: 1 *Let f be transcendental meromorphic and g be entire satisfying the following conditions:*
i) ρ_f *and* ρ_g *are both finite,*

ii) ρ_f *is positive and*
iii) let $F = f^n Q[f]$ *for* $n \geq 1$. *Then for* $p' > 0$ *and each* $\alpha \in (-\infty, \infty)$,

$$\liminf_{r\to\infty} \frac{\{\log T(r, f_o g)\}^{1+\alpha}}{\log T(\exp(r^{p'}), F)} = 0 \quad if \ \ p' > (1+\alpha)\rho_g.$$

Proof: If $1+\alpha \leq 0$, the theorem is trivial. So we take $1+\alpha > 0$. Since $T(r,g) \leq \log^+ M(r,g)$, by Lemma 2, we get for all sufficiently large values of *r* that

$$T(r, f_o g) \leq \{1 + o(1)\} T(M(r,g), f)$$

i.e.,

$$\log T(r, f_o g) \leq \log\{1 + o(1)\} + \log T(M(r,g), f)$$

i.e.,

$$\log T(r, f_o g) \leq o(1) + (\rho_f + \varepsilon) \log M(r,g)$$

i.e.,

$$\log T(r, f_o g) \leq o(1) + (\rho_f + \varepsilon) r^{(\rho_g + \varepsilon)}$$

i.e.,

$$\log T(r, f_o g) \leq r^{(\rho_g+\varepsilon)}\{(\rho_f + \varepsilon) + o(1)\}$$

i.e.,

$$\{\log T(r, f_o g)\}^{1+\alpha} \leq r^{(\rho_g+\varepsilon)(1+\alpha)}\{(\rho_f + \varepsilon) + o(1)\}^{1+\alpha}. \tag{1}$$

Again in view of Lemma 4, we have for a sequence of values of *r* tending to infinity and for $\varepsilon > 0$,

$$\log T(\exp(r^{p'}), F) > (\rho_F - \varepsilon) \log(\exp(r^{p'})) = (\rho_f - \varepsilon) r^{p'}. \tag{2}$$

Now combining (1) and (2) we obtain for a sequence of values of *r* tending to infinity that

$$\frac{\{\log T(r, f_o g)\}^{1+\alpha}}{\log T(\exp(r^{p'}), F)} \leq \frac{r^{(\rho_g+\varepsilon)(1+\alpha)}\{(\rho_f + \varepsilon) + o(1)\}^{1+\alpha}}{(\rho_f - \varepsilon) r^{p'}}$$

from which the theorem follows because we can choose ε such that

$$0 < \varepsilon < \min\{\rho_f, \frac{p'}{1+\alpha} - \rho_g\}.$$

This proves the theorem.

Remark: 2 *The condition* $p' > (1+\alpha)\rho_g$ *is essential in Theorem* 1 *as we see in the next example.*

Example: 2 *Let* $f = \exp z$, $g = \exp z$, $\alpha = 0$ *and* $p' = 1$. *Then*

$$\rho_f = 1 = \rho_g$$

and

$$\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2.$$

Also let $F = f^n Q[f]$ *for* $n \geq 1$.

Taking $n = 1, A_j = 1, n_{0j} = 1$ *and* $n_{1j} = \cdots = n_{kj} = 0$; *we see that* $F = \exp(2z)$.
Now we have

$$\log T(r, f_o g) = \log T(r, \exp^{[2]} z) \sim \log\{\frac{\exp r}{(2\pi^3 r)^{\frac{1}{2}}}\} \ (r \to \infty)$$

$$\sim r - \frac{1}{2}\log r + O(1) \ (r \to \infty).$$

Therefore

$$\liminf_{r\to\infty} \frac{\{\log T(r, f_o g)\}^{1+\alpha}}{\log T(\exp(r^{p'}), F)} = \liminf_{r\to\infty} \frac{\log T(r, f_o g)}{\log T(\exp r \ , \exp 2z)}$$

$$= \liminf_{r\to\infty} \frac{r - \frac{1}{2}\log r + O(1)}{\log\{\frac{\exp r}{\pi}\}}$$

$$= \liminf_{r\to\infty} \frac{r - \frac{1}{2}\log r + O(1)}{r + O(1)} = 1.$$

which contradicts Theorem 1.

Theorem: 2 *If f be meromorphic and g be transcendental entire such that $\rho_g < \infty$, $\rho_{f_o g} = \infty$ and for $n \geq 1$, $G = g^n Q[g]$, then for every A > 0,*

$$\limsup_{r\to\infty} \frac{\log T(r, f_o g)}{\log T(r^A, G)} = \infty.$$

Proof: If possible, let there exist a constant β such that for all sufficiently large values of *r*, we have

$$\log T(r, f_o g) \leq \beta \log T(r^A, G). \tag{3}$$

In view of Lemma 4, for all sufficiently large values of *r*, we get that

$$\log T(r^A, G) \leq (\rho_G + \varepsilon) A \log r$$

i.e., $\log T(r^A, G) \leq (\rho_g + \varepsilon) A \log r.$ (4)

Now combining (3) and (4), we obtain for all sufficiently large values of *r* that

$$\log T(r, f_o g) \leq \beta(\rho_g + \varepsilon) A \log r$$

i.e.,

$$\rho_{f_o g} \leq \beta A(\rho_g + \varepsilon),$$

which contradicts the condition $\rho_{f_o g} = \infty$. So for a sequence of values of *r* tending to infinity, it follows that

$$\log T(r, f_o g) > \beta \log T(r^A, G),$$

from which the theorem follows.

Corollary: 1 *Under the assumption of Theorem 2,*

$$\limsup_{r\to\infty} \frac{T(r, f_o g)}{T(r^A, G)} = \infty.$$

Proof: By Theorem 2 we obtain for all sufficiently large values of *r* and for $K_1 > 1$ that

$$\log T(r, f_o g) > K_1 \log T(r^A, G)$$

i.e., $T(r, f_o g) > \{T(r^A, G)\}^{K_1}$,

from which the corollary follows.

Remark: 3 *The condition $\rho_{f_o g} = \infty$ is necessary in Theorem 2 and Corollary 1 which is evident from the following example.*

Example: 3 *Let $f = z$, $g = \exp z$ and A = 1. Then $\rho_g = 1 < \infty$ and $\rho_{f_o g} = 1 < \infty$.*

Let $G = g^n Q[g]$ for $n \geq 1$. Taking $n = 1, A_j = 1, n_{0j} = 1$ and $n_{1j} = \cdots = n_{kj} = 0$; we see that $G = \exp(2z)$. Now we have

$$T(r, f_o g) = T(r, \exp z) = \frac{r}{\pi}$$

and $T(r^A, G) = T(r, \exp 2z) = \frac{2r}{\pi}$.

Therefore

$$\limsup_{r\to\infty} \frac{\log T(r, f_o g)}{\log T(r^A, G)} = \limsup_{r\to\infty} \frac{\log r + O(1)}{\log r + O(1)} = 1$$

and $\limsup_{r\to\infty} \frac{T(r,f_o g)}{T(r^A,G)} = \limsup_{r\to\infty} \frac{(\frac{r}{\pi})}{(\frac{2r}{\pi})} = \frac{1}{2},$

which is contrary to Theorem 2.

Remark: 4 *If we take $\rho_f < \infty$ and $F = f^n Q[f]$ for $n \geq 1$ instead of $\rho_g < \infty$ and $G = g^n Q[g]$ for $n \geq 1$ respectively, then Theorem 2 and Corollary 1 remain valid with G replaced by F in the denominator as we see in the following theorem and corollary.*

Theorem: 3 *If f be transcendental meromorphic and g be entire such that $\rho_f < \infty, \rho_{f_o g} = \infty$ and for $n \geq 1$, $F = f^n Q[f]$, then for every A* > 0,

$$\limsup_{r\to\infty} \frac{\log T(r, f_o g)}{\log T(r^A, F)} = \infty.$$

Proof: If possible let there exist a constant γ such that for all sufficiently large values of r, we have $\log T(r, f_o g) \leq \gamma \log T(r^A, F)$.

In view of Lemma 5, for all sufficiently large values of r we get that

$$\log T(r^A, F) \leq (\rho_f + \varepsilon) A \log r.$$

Now combining the above two inequalities, we get for all sufficiently large values of *r* that

$$\log T(r, f_o g) \leq \gamma(\rho_f + \varepsilon) A \log r$$

i.e., $\rho_{f_o g} \leq \gamma A(\rho_f + \varepsilon)$,

which contradicts the condition $\rho_{f_o g} = \infty$. So for a sequence of values of *r* tending to infinity, it follows that

$\log T(r, f_o g) > \gamma \log T(r^A, F)$,

from which the theorem follows.

Corollary: 2 *Under the assumptions of Theorem 3,*

$$\limsup_{r\to\infty} \frac{T(r, f_o g)}{T(r^A, F)} = \infty.$$

Proof: In view of Theorem 3, we obtain for all sufficiently large values of r and for $K_2 > 1$ that

$$\log T(r, f_o g) > K_2 \log T(r^A, F)$$

i.e., $T(r, f_o g) > \{T(r^A, F)\}^{K_2}$,

from which the corollary follows.

Remark: 5 *The condition $\rho_{f_o g} = \infty$ is necessary in Theorem 3 and Corollary 2 which is evident from the following example.*

Example: 4 *Let $f = \exp z$, $g = z$ and A =1. Then $\rho_f = 1 < \infty$, $\rho_{f_o g} = 1 < \infty$ and for $n \geq 1, F = f^n Q[f]$. Taking $n = 1, A_j = 1, n_{0j} = 1$ and $n_{1j} = \cdots = n_{kj} = 0$; we see that $F = \exp(2z)$. Now we have*

$T(r, f_o g) = T(r, \exp z) = \frac{r}{\pi}$ *and* $T(r^A, F) = T(r, \exp 2z) = \frac{2r}{\pi}$.

Therefore

$\limsup_{r\to\infty} \frac{\log T(r,f_o g)}{\log T(r^A,F)} = \limsup_{r\to\infty} \frac{\log r + O(1)}{\log r + O(1)} = 1$ *and* $\limsup_{r\to\infty} \frac{T(r,f_o g)}{T(r^A,F)} = 1,$

which contradicts Theorem 3.

Theorem: 4 *Let f and g be any two entire functions with $\lambda_f > 0$ and $\rho_f < \lambda_g$. Also let f be transcendental with $F = f^n Q[f]$ for $n \geq 1$. Then*

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, f_o g)}{\log M(r, F)} = \infty.$$

Proof: In view of Lemma 1, we get for all sufficiently large values of *r* that

$$M(r, f_o g) \geq M(\tfrac{1}{16} M\left(\tfrac{r}{2}, g\right), f)$$

i.e., $\log^{[2]} M(r, f_o g) \geq \log^{[2]} M(\frac{1}{16} M\left(\frac{r}{2}, g\right), f)$

i.e., $\log^{[2]} M(r, f_o g) \geq (\lambda_f - \varepsilon) \log \frac{1}{16} + (\lambda_f - \varepsilon) \log M(\frac{r}{2}, g)$

i.e., $\log^{[2]} M(r, f_o g) \geq O(1) + (\lambda_f - \varepsilon)(\frac{r}{2})^{(\lambda_g - \varepsilon)}$. (5)

Again for all sufficiently large values of *r*, we get by Lemma 5 that

$\log M(r, F) \leq r^{(\rho_F + \varepsilon)} = r^{(\rho_f + \varepsilon)}$. (6)

Now combining (5) and (6), it follows for all sufficiently large values of *r* that

$$\frac{\log^{[2]} M(r, f_o g)}{\log M(r, F)} \geq \frac{O(1) + (\lambda_f - \varepsilon)(\frac{r}{2})^{(\lambda_g - \varepsilon)}}{r^{(\rho_f + \varepsilon)}}. \quad (7)$$

Since $\rho_f < \lambda_g$, we can choose $\varepsilon (> 0)$ in such a way that

$\rho_f + \varepsilon < \lambda_g - \varepsilon$. (8)

Thus from (7) and (8) we obtain that

$$\liminf_{r\to\infty} \frac{\log^{[2]} M(r, f_o g)}{\log M(r, F)} = \infty,$$

from which the theorem follows.

Remark: 6 *The condition $\rho_f < \lambda_g$ is necessary in Theorem 4 which is evident from the following two examples.*

Example: 5 *Let $f = \exp z$ and $g = \exp z$. Then $\lambda_f = 1 > 0$, $\rho_f = 1 = \lambda_g$ and $F = f^n Q[f]$ for $n \geq 1$. Taking $n = 1, A_j = 1, n_{0j} = 1$ and $n_{1j} = \cdots = n_{kj} = 0$; we see that $F = \exp(2z)$. Again $M(r, f_o g) = \exp^{[2]} r$ and $M(r, F) = \exp(2r)$.*

Therefore

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, f_o g)}{\log M(r, F)} = \lim_{r\to\infty} \frac{\log^{[2]}(\exp^{[2]} r)}{\log(\exp 2r)} = \lim_{r\to\infty} \frac{r}{2r} = \frac{1}{2},$$

which is contrary to Theorem 4.

Example: 6 *Let $f = \exp z$ and $g = z$. Then $\lambda_f = 1 > 0$, $\rho_f = 1 > 0 = \lambda_g$ and let $F = f^n Q[f]$ for $n \geq 1$. Taking $n = 1, A_j = 1, n_{0j} = 1$ and $n_{1j} = \cdots = n_{kj} = 0$; we see that $F = \exp(2z)$. Again $M(r, f_o g) = \exp r$ and $M(r, F) = \exp(2r)$.*

Therefore

$$\lim_{r\to\infty} \frac{\log^{[2]} M(r, f_o g)}{\log M(r, F)} = \lim_{r\to\infty} \frac{\log^{[2]}(\exp r)}{\log(\exp 2r)} = \lim_{r\to\infty} \frac{\log r}{2r} = 0,$$

which contradicts Theorem 4.

Theorem: 5 *If f be a transcendental meromorphic function and g be entire with $0 < \lambda_f \leq \rho_f < \infty$, $\rho_g < \infty$ and $F = f^n Q[f]$ for $n \geq 1$, then*

$$\lim_{r\to\infty} \frac{T(r, f_o g) T(r, F)}{T(\exp(r^{p'}), F)} = 0,$$

if $p^{'} > \rho_g$.

Proof: Since $T(r,g) \leq \log^{+} M(r,g)$, for all sufficiently large values of r we get from Lemma 2 that

$$T(r, f_o g) \leq \{1+o(1)\} T(M(r,g), f)$$

i.e., $T(r, f_o g) \leq \{1+o(1)\} \exp\{(\rho_f + \varepsilon) r^{(\rho_g+\varepsilon)}\}$. (9)

Again by Lemma 5, we obtain for all sufficiently large values of r that

$T(r,F) \leq r^{(\rho_F+\varepsilon)} = r^{(\rho_f+\varepsilon)}$. (10)

Now combining (9) and (10), it follows for all sufficiently large values of r that

$$T(r, f_o g)\, T(r,F) \leq \{1+o(1)\} r^{(\rho_f+\varepsilon)} \exp\{(\rho_f + \varepsilon) r^{(\rho_g+\varepsilon)}\}. \quad (11)$$

Also in view of Lemma 4, we have for all sufficiently large values of r,

$$\log T(\exp(r^{p^{'}}), F) \geq (\lambda_F - \varepsilon) \log\{\exp(r^{p^{'}})\}$$

i.e., $\log T(\exp(r^{p^{'}}), F) \geq \exp\{(\lambda_f - \varepsilon) r^{p^{'}}\}$. (12)

From (11) and (12) it follows for all sufficiently large values of r that

$$\frac{T(r, f_o g)\, T(r,F)}{\log T(\exp(r^{p^{'}}), F)} \leq \frac{\{1+o(1)\} r^{(\rho_f+\varepsilon)} \exp\{(\rho_f+\varepsilon) r^{(\rho_g+\varepsilon)}\}}{\exp\{(\lambda_f - \varepsilon) r^{p^{'}}\}}. \quad (13)$$

As $p^{'} > \rho_g$ so we can choose $\varepsilon (> 0)$ such that

$p^{'} > \rho_g + \varepsilon$. (14)

Thus the theorem follows from (13) and (14).

Theorem: 6 *Let f be a transcendental meromorphic function and g be a transcendental entire function such that* $0 < \lambda_f \leq \rho_f < \infty$ *and for* $n \geq 1$, $F = f^n Q[f]$. *Then for every A > 0,*

$$\lim_{r\to\infty} \frac{\log T(r, f_o g)}{\log T(r^A, F)} = \infty.$$

If further $\rho_g < \infty$ *and for* $n \geq 1$, $G = g^n Q[g]$, *then*

$$\lim_{r\to\infty} \frac{\log T(r, f_o g)}{\log T(r^A, G)} = \infty.$$

Proof: Since $\lambda_f > 0$, $\lambda_{f_o g} = \infty$ {cf. [2]}. So it follows that for arbitrary large N and for all sufficiently large values of r,

$\log T(r, f_o g) > AN \log r$. (15)

Again since $\rho_f < \infty$, for all sufficiently large values of r we get by Lemma 4 that

$\log T(r^A, F) < A(\rho_f + 1) \log r$. (16)

Now from (15) and (16), it follows for all sufficiently large values of r that

$$\frac{\log T(r, f_o g)}{\log T(r^A, F)} > \frac{AN \log r}{A(\rho_f + 1) \log r}$$

and so $\lim_{r\to\infty} \frac{\log T(r, f_o g)}{\log T(r^A, F)} = \infty$.

Again since $\rho_g < \infty$, then for all sufficiently large values of r we obtain by Lemma 5 that

$\log T(r^A, G) < A(\rho_g + 1) \log r$. (17)

Now from (15) and (17), it follows for all sufficiently large values of r that

$\frac{\log T(r, f_o g)}{\log T(r^A, G)} > \frac{AN \log r}{A(\rho_g + 1) \log r}$. (18)

Thus the theorem follows from (18).

Theorem: 7 *Let f be a transcendental meromorphic function with* $0 < \lambda_f \leq \rho_f < \infty$ *and for* $n \geq 1$, $F = f^n Q[f]$ *and* g *be entire. Then*

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f_o g)}{\log T(\exp(r^{\mu}), F)} = \infty, \text{ where } 0 < \mu < \rho_g.$$

Proof: Let $0 < \mu' < \rho_g$. Then in view of Lemma 3, we get for a sequence of values of *r* tending to infinity that

$$\log T(r, f_o g) \geq \log T(\exp(r^{\mu'}), f)$$

i.e., $\log T(r, f_o g) \geq (\lambda_f - \varepsilon)\log\{\exp(r^{\mu'})\}$

i.e., $\log T(r, f_o g) \geq (\lambda_f - \varepsilon) r^{\mu'}$

i.e., $\log^{[2]} T(r, f_o g) \geq O(1) + \mu' \log r.$

So for a sequence of values of r tending to infinity,

$$\log^{[2]} T(\exp(r^{\rho_g}), f_o g) \geq O(1) + \mu' \log\{\exp(r^{\rho_g})\}$$

i.e., $\log^{[2]} T(\exp(r^{\rho_g}), f_o g) \geq O(1) + \mu' r^{\rho_g}.$ (19)

Again in view of Lemma 4, we obtain for all sufficiently large values of *r* that

$$\log T(\exp(r^{\mu}), F) \leq (\rho_F + \varepsilon)\log\{\exp(r^{\mu})\}$$

i.e., $\log T(\exp(r^{\mu}), F) \leq (\rho_f + \varepsilon) r^{\mu}.$ (20)

Combining (19) and (20), it follows for a sequence of values of *r* tending to infinity that

$$\frac{\log^{[2]} T(\exp(r^{\rho_g}), f_o g)}{\log T(\exp(r^{\mu}), F)} \geq \frac{O(1) + \mu' r^{\rho_g}}{(\rho_f + \varepsilon) r^{\mu}}. \quad (21)$$

Since $\mu < \rho_g$, we get from (21) that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f_o g)}{\log T(\exp(r^{\mu}), F)} = \infty.$$

This proves the theorem.

Remark: 7 *The condition* $\mu < \rho_g$ *in Theorem 7 is essential as we see in the following example.*

Example: 7 *Let* $f = \exp z, g = z$ *and* $\mu = 1$. *Then* $\lambda_f = 1 = \rho_f, \rho_g = 0$ *and let for* $n \geq 1$, $F = f^n Q[f]$. *Taking* $n = 1, A_j = 1, n_{0j} = 1$ *and* $n_{1j} = \cdots = n_{kj} = 0$;

we see that $F = \exp(2z)$. *Also*

$$T(r, \exp z) = \frac{r}{\pi}.$$

So

$$\log^{[2]} T(\exp(r^{\rho_g}), f_o g) = \log^{[2]} T(e, \exp z) = \log^{[2]}\left(\frac{e}{\pi}\right) = O(1)$$

and $\log T(\exp(r^{\mu}), F) = \log T(\exp r, \exp 2z) = \log\{\frac{2\exp r}{\pi}\} = r + O(1).$

Therefore

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(\exp(r^{\rho_g}), f_o g)}{\log T(\exp(r^{\mu}), F)} = \limsup_{r\to\infty} \frac{O(1)}{r + O(1)} = 0,$$

which is contrary to Theorem 7.

Theorem: 8 *Let* f *be rational and* g *be transcendental meromorphic satisfying*

(i) $0 < \overline{\lambda}_{f_o g} \leq \overline{\rho}_{f_o g} < \infty$,

(ii) $0 < \overline{\lambda}_g \leq \overline{\rho}_g < \infty$ *and*

(iii) *for* $n \geq 1$, $G = g^n Q[g]$. *Then for any positive number A,*

$$\frac{\overline{\lambda}_{f_o g}}{A\overline{\rho}_g} \leq \liminf_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \leq \frac{\overline{\lambda}_{f_o g}}{A\overline{\lambda}_g} \leq \limsup_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \leq \frac{\overline{\rho}_{f_o g}}{A\overline{\lambda}_g}.$$

Proof: From the definition of hyper order and hyper lower order and by Lemma 6, we get for arbitrary positive ε and for all sufficiently large values of r that

$$\log^{[2]} T(r, f_o g) \geq \left(\overline{\lambda}_{f_o g} - \varepsilon\right) \log r \tag{22}$$

and $\log^{[2]} T(r^A, G) \leq \left(\overline{\rho}_G + \varepsilon\right) \log r^A$

i.e., $$\log^{[2]} T(r^A, G) \leq A\left(\overline{\rho}_g + \varepsilon\right) \log r. \tag{23}$$

Combining (22) and (23), we obtain for all sufficiently large values of *r* that

$$\frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \geq \frac{\left(\overline{\lambda}_{f_o g} - \varepsilon\right) \log r}{A\left(\overline{\rho}_g + \varepsilon\right) \log r}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \geq \frac{\overline{\lambda}_{f_o g}}{A\overline{\rho}_g}. \tag{24}$$

Again for a sequence of values of *r* tending to infinity,

$$\log^{[2]} T(r, f_o g) \leq \left(\overline{\lambda}_{f_o g} + \varepsilon\right) \log r. \tag{25}$$

Also in view of Lemma 6, we have for all sufficiently large values of *r*,

$$\log^{[2]} T(r^A, G) \geq \left(\overline{\lambda}_G - \varepsilon\right) \log r^A$$

i.e., $$\log^{[2]} T(r^A, G) \geq A\left(\overline{\lambda}_g - \varepsilon\right) \log r. \tag{26}$$

Combining (25) and (26), we get for a sequence of values of *r* tending to infinity that

$$\frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \leq \frac{\left(\overline{\lambda}_{f_o g} + \varepsilon\right) \log r}{A\left(\overline{\lambda}_g - \varepsilon\right) \log r}.$$

As $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \leq \frac{\overline{\lambda}_{f_o g}}{A\overline{\lambda}_g}. \tag{27}$$

Also for a sequence of values of *r* tending to infinity and by Lemma 6,

$$\log^{[2]} T(r^A, G) \leq A\left(\overline{\lambda}_G + \varepsilon\right) \log r$$

i.e., $$\log^{[2]} T(r^A, G) \leq A\left(\overline{\lambda}_g + \varepsilon\right) \log r. \tag{28}$$

Combining (22) and (28), we get for a sequence of values of *r* tending to infinity that

$$\frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \geq \frac{\overline{\lambda}_{f_o g}}{A\overline{\lambda}_g}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \geq \frac{\overline{\lambda}_{f_o g}}{A\overline{\lambda}_g}. \tag{29}$$

Also for all sufficiently large values of *r*,

$$\log^{[2]} T(r, f_o g) \leq \left(\overline{\rho}_{f_o g} + \varepsilon\right) \log r. \tag{30}$$

From (26) and (30), we obtain for all sufficiently large values of *r* that

$$\frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \leq \frac{\left(\overline{\rho}_{f_o g} + \varepsilon\right) \log r}{A\left(\overline{\lambda}_g - \varepsilon\right) \log r}.$$

Since $\varepsilon(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[2]} T(r, f_o g)}{\log^{[2]} T(r^A, G)} \leq \frac{\overline{\rho}_{f_o g}}{A\overline{\lambda}_g}. \tag{31}$$

Thus the theorem follows from (24), (27), (29) and (31).

Theorem: 9 *Let f be meromorphic and g be transcendental entire such that*
(i) $0 < \rho_g < \infty$,
(ii) $\sigma_g > 0$,
(iii) $0 < \rho_{f_o g} < \infty$,
(iv) $\sigma_{f_o g} < \infty$,
(v) $\rho_f^* < 1$ *and*
(vi) *for* $n \geq 1$, $G = g^n Q[g]$. *Then*

$$\liminf_{r\to\infty} \frac{T(r, f_o g)}{T(r, G)} = 0.$$

Proof: From the definition of type, we have for arbitrary positive ε and for all sufficiently large values of *r*,

$$T(r, f_o g) \leq \left(\sigma_{f_o g} + \varepsilon\right) r^{\rho_{f_o g}}. \tag{32}$$

Again in view of Lemma 4, we get for a sequence of values of *r* tending to infinity that

$$T(r, G) \geq (\sigma_G - \varepsilon) r^{\rho_g}$$

i.e., $$T(r, G) \geq \left(\sigma_g - \varepsilon\right) r^{\rho_g}. \tag{33}$$

Since $\rho_{f_o g} < \infty$, it follows that $\rho_f = 0$ {cf. [6]}. So in view of Lemma 7, from (32) and (33), we obtain for a sequence of values of tending to infinity that

$$\frac{T(r, f_o g)}{T(r, G)} \leq \frac{\left(\sigma_{f_o g} + \varepsilon\right) r^{\rho_f^* \rho_g}}{(\sigma_g - \varepsilon) r^{\rho_g}}$$

i.e., $$\frac{T(r, f_o g)}{T(r, G)} \leq \frac{(\sigma_{f_o g} + \varepsilon) r^{(\rho_f^* - 1)\rho_g}}{(\sigma_g - \varepsilon)}.$$

Since $\varepsilon(> 0)$ is arbitrary, in view of condition (*v*), it follows that

$$\liminf_{r\to\infty} \frac{T(r, f_o g)}{T(r, G)} = 0.$$

This proves the theorem.

Remark: 8 *The condition* $\rho_f^* < 1$ *in Theorem 9 is essential which is evident from the following example.*

Example: 8 *Let* $f = z$ *and* $g = \exp z$. *Then* $\rho_g = 1 = \sigma_g$, $\rho_{f_o g} = 1 = \sigma_{f_o g}$, $\rho_f = 0$ *and let* $G = g^n Q[g]$ *for* $n \geq 1$. *Taking* $n = 1, A_j = 1, n_{0j} = 1$ *and*

$n_{1j} = \cdots = n_{kj} = 0$; *we see that* $G = \exp(2z)$. *Also we have*

$$\rho_f^* = \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f)}{\log^{[2]} r} = \limsup_{r\to\infty} \frac{\log^{[2]} r}{\log^{[2]} r} = 1.$$

Again
$T(r, f_o g) = \frac{r}{\pi}$ *and* $T(r, G) = \frac{2r}{\pi}$.

Therefore $\liminf_{r\to\infty} \frac{T(r,f_o g)}{T(r,G)} = \liminf_{r\to\infty} \frac{(\frac{r}{\pi})}{(\frac{2r}{\pi})} = \frac{1}{2}$,

which contradicts Theorem 9.

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