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CHARACTERIZATION OF A PARTIAL ORDER RELATION ON PRE A* -ALGEBRA

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ABSTRACT

This manuscript is a classification on Pre-A-algebra A in sight of it is like a partially ordered set. Using a binary operation in Pre-A*-algebra, an observation is made on Pre A*-Algebra as a partially ordered set with respect to binary operation $\wedge$ and obtained consequent results. It is also grant access to an equivalent condition for a Pre A*-algebra become a Boolean algebra.*

Key words: *A*-algebra, Pre-A*-algebra, Boolean algebra, partially ordered set, Ada, homomorphism.*

AMS subject classification (2000): *06E05, 06E25, 06E99, 06B10.*

INTRODUCTION

In a draft manuscript entitled "The Equational theory of Disjoint Alternatives", E. G. Manes (1989) introduced the concept of Ada **Error! Bookmark not defined.**(Algebra of disjoint alternatives) (A, $\wedge$, $\vee$, $(-)^{\prime}$, $(-)_{\pi}$, 0, 1, 2) which is however differs from the definition of the Ada of E. G. Manes (1993) later paper entitled "Adas and the equational theory of if-then-else". While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras and the later concept is based on C-algebras (A, $\wedge$, $\vee$, $(-)^{\sim}$) introduced by Fernando Guzman and Craig C. Squir (1990) . P. Koteswara Rao (1994) first introduced the concept of A*-algebra (A, $\wedge$, $\vee$, $*$, $(-)^{\sim}$, $(-)_{\pi}$, 0, 1, 2) not only studied the equivalence with Ada, C-algebra, Ada's connection with 3-Ring, Stone type representation but also introduced the concept of A*-clone, the If-Then-Else structure over A*-algebra and Ideal of A*-algebra.

J.Venkateswara Rao (2000) introduced the concept Pre A*-algebra (A, $\wedge$, $\vee$, $(-)^{\sim}$) analogous to C-algebra as a reduct of A*- algebra. Venkateswara Rao.J, Praroopa.Y (2006) made a structural study on Boolean algebras and Pre A*-Algebras.

Boolean algebra depends on two element logic. C-algebra, Ada, A*- algebra and our Pre A*-algebra are regular extensions of Boolean logic to 3 truth values, where the third truth value stands for an undefined truth value. The Pre A*- algebra structure is denoted by (A, $\wedge$, $\vee$, $(-)^{\sim}$) where A is non-empty set $\wedge$, $\vee$, are binary operations and $(-)^{\sim}$ is a unary operation.

In this paper we define a relation $\leq$ on Pre A*-algebra with respect to the binary operation $\vee$ and we discuss the properties of a Pre A*-algebra like a poset. We find the necessary conditions for a poset to become a lattice. We also present a equivalent condition for a Pre A*-algebra become a Boolean algebra. For any $a \in A$ define $A_a = \{x \in A \,/\, a \vee x = x\}$ and $x^a = a \vee x^{\sim}$ then (A_a, $\wedge, \vee, {}^a$) is a Pre A*-algebra. We also define a mapping $\alpha_{a,b}$ from A_a to A_b by $\alpha_{a,b}$ (x) = $b \vee x$ for all $x \in A_a$ is a homomorphism of Pre A*-algebras.

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1. PRELIMINARIES

1.1. Definition: The relation R on a set A is called a partial order on A when R(≤) is reflexive, anti-symmetric, and transitive. Under these conditions, the set A is called a partially ordered set or a poset. Frequently we write (A, R) or (A, ≤) to denote that A is partially ordered by the relation R(≤). Since the relation ≤ on the set of real numbers is the prototype of a partial order it is common to write ≤ to represent an arbitrary partial order can be described as follows:

1. For all $a \in A$, $a \le a$ (symmetry)
2. For all $a, b \in A$, $a \le b$, $b \le a$, then $a = b$ (anti symmetry)
3. For all $a, b, c \in A$, $a \le b$ and $b \le c$, then $a \le c$ (transitivity)

Two elements a and b in A are said to be comparable under ≤ if either $a \le b$ or $b \le a$; otherwise they are incomparable. If every pair of elements of A are comparable, then we say that the poset is totally ordered.

1.2. Definition: An algebra $(A, \wedge, \vee, (-)^{\sim})$ where A is a non-empty set with $1, \wedge, \vee$ are binary operations and $(-)^{\sim}$ is a unary operation satisfying

(a) $x^{\sim\sim} = x \quad \forall x \in A$
(b) $x \wedge x = x, \forall x \in A$
(c) $x \wedge y = y \wedge x, \forall x, y \in A$
(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim} \quad \forall x, y \in A$
(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z, \ \forall x, y, z \in A$
(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z), \forall x, y, z \in A$
(g) $x \wedge y = x \wedge (x^{\sim} \vee y), \ \forall x, y \in A$ is called a Pre A*-algebra.

1.1. Example: 3 = {0, 1, 2} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1	2		$\vee$	0	1	2		x	$x^{\sim}$
0	0	0	2		0	0	1	2		0	1
1	0	1	2		1	1	1	2		1	0
2	2	2	2		2	2	2	2		2	2

1.1. Note: The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$ (b) $1 \wedge x = x$ for all $x \in$ **3**
(c) $0 \vee x = x$ for all $x \in$ **3** (d) $2 \wedge x = 2 \vee x = 2$ for all $x \in$ **3**.

1.2. Example: 2 = {0, 1} with operations $\wedge, \vee, (-)^{\sim}$ defined below is a Pre A*-algebra.

$\wedge$	0	1		$\vee$	0	1		x	$x^{\sim}$
0	0	0		0	0	1		0	1
1	0	1		1	1	1		1	0

1.2. Note:

(i) $\left(2, \vee, \wedge, (-)^{\sim}\right)$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.

(ii) The identities 1.2(a) and 1.2(d) imply that the varieties of Pre A*-algebras satisfies all the dual statements of 1.2(b) to 1.2(g).

1.3. Definition: Let A be a Pre A*-algebra. An element $x \in A$ is called a central element of A if $x \vee x^{\sim} = 1$ and the set $\{x \in A / x \vee x^{\sim} = 1\}$ of all central elements of A is called the centre of A and it is denoted by B (A).

1.1. Theorem:[Satyanarayana.A, (2012)] Let A be a Pre A*-algebra with 1, then B (A) is a Boolean algebra with the induced operations $\wedge, \vee, (-)^{\sim}$

1.1. Lemma: [Satyanarayana.A, (2012),] Every Pre A*-algebra with 1 satisfies the following laws

(a) $x \vee 1 = x \vee x^{\sim}$ (b) $x \wedge 0 = x \wedge x^{\sim}$

1.2. Lemma: [7] Every Pre A*-algebra with 1 satisfies the following laws.

(a) $x \wedge (x^{\sim} \vee x) \quad x \veebar (x^{\sim} \wedge x) = x$
(b) $(x \vee x^{\sim}) \wedge \mathrm{y} = (\mathrm{x} \wedge \mathrm{y}) \vee (\mathrm{x}^{\sim} \wedge \mathrm{y})$
(c) $(x \vee y) \wedge \mathrm{z} = (x \wedge \mathrm{z}) \vee (x^{\sim} \wedge \mathrm{y} \wedge \mathrm{z})$

1.4. Definition: Let $(A_1, \vee, \wedge, (-)^{\sim})$ and $(A_2, \vee, \wedge, (-)^{\sim})$ be a two Pre A*- algebras. A mapping $f : A_1 \to A_2$ is called a Pre A*-homomorphism if

(i) $f(a \wedge b) = f(a) \wedge f(b)$ (ii) $f(a \vee b) = f(a) \vee f(b)$ (iii) $f(a^{\sim}) = (f(a))^{\sim}$

The homomorphism $f : A_1 \to A_2$ is onto, then f is called epimorphism.

The homomorphism $f : A_1 \to A_2$ is one-one then f is called monomorphism

The homomorphisn $f : A_1 \to A_2$ is one-one and onto then f is called an isomorphism, and A_1, A_2 are isomorphic, denoted in symbol $A_1 \cong A_2$.

2. Pre A*-algebra as a poset with respect to Binary Operation $\vee$

2.1 Definition: Let A be a Pre A*-algebra. Define ≤ on A by x ≤ y if and only if y∨x = x∨y = y.

2.1 Lemma: If A is a Pre A*-algebra, then (A, ≤) is a poset.

Proof: Since x∨x = x, x≤x for all x∈A.

Therefore ≤ is reflexive.

Suppose that x, y, z ∈ A, x ≤ y and y ≤ z.

Then we have y ∨ x = x ∨ y = y and z ∨ y = y ∨ z = y.

Now z = y ∨ z = x ∨ y ∨ z = x ∨ z.

Therefore x ∨ z = z ∨ x = z, i.e., x ≤ z.

This shows that ≤ is transitive.

Suppose that x, y ∈ A, x ≤ y and y ≤ x.

Then we have y ∨ x = x∨y = y and x ∨ y = y ∨ x = y.

This shows that x = y.

Therefore ≤ is antisymmetric.

Therefore (A, ≤) is poset.

2.1. Note: If A is a Pre A*-algebra with 1, 0, 2 then 0 ≤ x (0 ∨ x = x ∨ 0 = x), for all x ∈ A and x ≤ 2 (2 ∨ x = x ∨ 2 = 2). This shows that 2 is the greatest element and 0 is the least element of the poset.

The Hasse diagram of the poset $(A, \leq)$ is given by

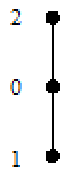

Diagram: 2.1

We know that $A \times A$ is a Pre A*-algebra under point wise operation. The Hasse diagram is given below
$A \times A = \{a_1 = (1, 1), a_2 = (1, 0), a_3 = (1, 2), a_4 = (0, 1), a_5 = (0, 0), a_6 = (0, 2), a_7 = (2, 1), a_8 = (2, 0), a_9 = (2, 2)\}$

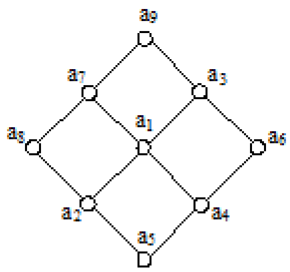

Diagram: 2.2

Observe that, $x \leq a_9$, i.e., $(x \vee a_9 = a_9 \vee x = a_9)$ and $a_5 \leq x$ $(x \vee a_5 = a_5 \vee x = x)$ for all $x \in A \times A$. This shows that a_9 is the greatest element and a_5 is the least element of $A \times A$.

We have $\mathbf{2} \times \mathbf{3} = \{a_1 = (1,1), a_2 = (0,0), a_3 = (1,0), a_4 = (0,1), a_5 = (0,2), a_6 = (1,2)\}$ is Pre A*-algebra under point wise operation having four central elements two non-central elements and no element is satisfying the property that $a^{\sim} = a$.

The Hasse diagram for $(\mathbf{2} \times \mathbf{3}, \leq)$ as given below

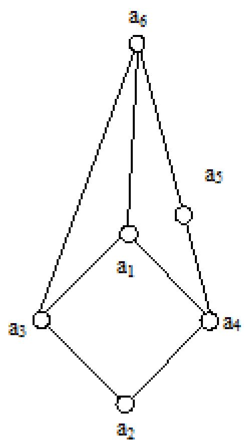

Diagram: 2.3

Observe that, $x \leq a_6$, i.e., $x \vee a_6 = a_6 \vee x = a_6$ and $a_2 \leq x$ $(x \vee a_2 = a_2 \vee x = x)$ for all $x \in \mathbf{2} \times \mathbf{3}$. This shows that a_6 is the greatest element and a_2 is the least element of $\mathbf{2} \times \mathbf{3}$.

2.1. Theorem: In the poset $(A, \leq)$, for any $x \in A$, Supremum $\{x, x^{\sim}\} = x \vee x^{\sim}$ infimum $\{x, x^{\sim}\} = x \wedge x^{\sim}$.

Proof: We have $(x \vee x^{\sim}) \vee x = x \vee x^{\sim}$ and $x^{\sim} \vee (x \vee x^{\sim}) = x \vee x^{\sim}$

Therefore, $x \leq x \vee x^{\sim}$ and $x^{\sim} \leq x \vee x^{\sim}$.

Hence $x \vee x^{\sim}$ is an upper bound of $\{x, x^{\sim}\}$

Suppose n is an upper bound of $\{x. x^{\sim}\}$

That is, $x \le n, x^{\sim} \le n \Rightarrow n \vee x = n$ and $n \vee x^{\sim} = n$

Now $n \vee (x \vee x^{\sim}) = (n \vee x) \vee x^{\sim} = n \vee x^{\sim} = n$

This shows that $x \vee x^{\sim} \le n$

Therefore $x \vee x^{\sim}$ is a least upper bound of $\{x, x^{\sim}\}$

This shows that supremum of $\{x, x^{\sim}\} = x \vee x^{\sim}$

Again we have $(x \wedge x^{\sim}) \vee x = x$ and $(x \wedge x^{\sim}) \vee x^{\sim} = x^{\sim}$

Therefore $x \wedge x^{\sim} \le x$ and $x \wedge x^{\sim} \le x^{\sim}$

Hence $x \wedge x^{\sim}$ is a lower bound of $\{x, x^{\sim}\}$

Suppose m is a lower bound of $\{x, x^{\sim}\}$

That is, $m \le x, m \le x^{\sim} \Rightarrow m \vee x = x$ and $m \vee x^{\sim} = x^{\sim}$

Now $m \vee (x \wedge x^{\sim}) = (m \vee x) \wedge (m \vee x^{\sim}) = x \wedge x^{\sim}$

This shows that $m \le x \wedge x^{\sim}$

Therefore $x \wedge x^{\sim}$ is greatest lower bound of $\{x, x^{\sim}\}$.

This shows that infimum of $\{x, x^{\sim}\} = x \wedge x^{\sim}$.

2.2. Theorem: In a poset $(A, \le)$ with 1, for any $x, y \in A$, $\sup\{x, y\} = x \vee y$.

Proof: We have $(x \vee y) \vee x = x \vee y$ and $(x \vee y) \vee y = x \vee y$

Therefore, $x \le x \vee y$ and $y \le x \vee y$.

Hence $x \vee y$ us an upper bound of $\{x, y\}$

Suppose m is an upper bound of $\{x, y\}$

That is, $x \le m, y \le m \Rightarrow m \vee x = m$ and $m \vee y = m$

Now $m \vee (x \vee y) = (m \vee x) \vee y = m \vee y = m$.

This shows that $x \vee y \le m$

Therefore $x \vee y$ is a least upper bound of $\{x, y\}$

This shows that supremum of $\{x, y\} = x \vee y$.

In general for a Pre A*-algebra with 1, $x \wedge y$ need not be the greatest lower bound of $\{x, y\}$ in $(A, \le)$. For example $2 \vee x = 2 \wedge x = 2, \forall x \in A$ is not a greatest lower bound. However we have the following.

2.3. Theorem: In a poset $(A, \le)$ with 1, for any $x, y \in B(A)$, $\mathrm{Inf}(x, y) = x \wedge y$

Proof: If $x, y \in B(A)$, then we have $x \vee (x \wedge y)$ and $y \vee (x \wedge y) = y$

This shows that, $x \wedge y \le x$ and $x \wedge y \le y$.

Hence $x \wedge y$ is a lower bound of $\{x, y\}$

Suppose m is a lower bound of $\{x, y\}$, then $m \vee x = x$, $m \vee y = y$.

Now $m \vee (x \wedge y) = (m \vee x) \wedge (m \vee y) = x \wedge y$

Therefore $m \leq x \wedge y$.

Hence $\text{Inf}\{x, y\} = x \wedge y$.

2.4. Theorem: In the poset $(A, \leq)$, if $x,y \in B(A)$, then $x \wedge y \leq x \wedge x^{\sim}$

Proof: $$(x \wedge x^{\sim}) \vee (x \wedge y) = \{(x \wedge x^{\sim}) \vee x\} \wedge \{(x \wedge x^{\sim}) \vee y\}$$
$$= x \wedge (0 \vee y)$$
$$= x \wedge y$$
Therefore $x \wedge y \leq x \wedge x^{\sim}$

2.5. Theorem: In the poset $(A, \leq)$, if $x \leq y$, then for any $z \in A$,
(a) $z \wedge x \leq z \wedge y$
(b) $z \vee x \leq z \vee y$

Proof: If $x \leq y$, then $x \vee y = y$

(a) $(z \wedge x) \vee (z \wedge y) = z \wedge (x \vee y) = z \wedge y$.

Therefore $z \wedge x \leq z \wedge y$

(b) $(z \vee x) \vee (z \vee y) = z \vee (x \vee y) = z \vee y$.

Therefore $z \vee x \leq z \vee y$

Now we are giving the following equivalent conditions for $x \leq y$.

2.2. Lemma: In a Pre A*-algebra
(i) $x \leq y \Leftrightarrow x \vee (x^{\sim} \wedge y) = (x^{\sim} \wedge y) \vee x = y$
(ii) $x \leq y \Leftrightarrow y \vee (y^{\sim} \wedge x) = (y^{\sim} \wedge x) \vee y = y$

Proof:

(i) If $x \leq y$, $\Leftrightarrow x \vee y = y$
$\Leftrightarrow x \vee (x^{\sim} \wedge y) = (x^{\sim} \wedge y) \vee x = y$

(ii) If $x \leq y$ $\Leftrightarrow y \vee x = y$
$\Leftrightarrow y \vee (y^{\sim} \wedge x) = (y^{\sim} \wedge x) \vee y = y$

Now we prove modular type results in the following

2.3. Lemma: In the poset $(A, \leq)$, if $x \leq y \Rightarrow x \vee (y \wedge z) = y \wedge (x \vee z)$

Proof: Suppose $x \leq y$, then $x \vee y = y$.

Now $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) = y \wedge (x \vee z)$

If $x, y \in B(A)$ then by theorem 2.3 $\text{Inf}\{x, y\} = x \wedge y$. In general $x \wedge y$ need not be an upper bound of $\{x, y\}$ in poset $(A, \leq)$. If $x \wedge y$ is an upper bound of $\{x, y\}$ in poset $(A, \leq)$, then A becomes Boolean algebra.

Now we have the following theorem.

2.6. Theorem: If A is a Pre A*-algebra and $x \wedge (x \vee y) = x$ for all $x, y \in A$ then $(A, \leq)$ is a lattice.

Proof: By theorem 2.2, we have every pair of elements have l.u.b and if $x \vee (x \wedge y) = x$ for all $x, y \in A$, then by theorem 2.3 we have every pair of elements have g.l.b. Hence $(A, \leq)$ is a lattice.

Now we present a equivalent condition for a Pre A*-algebra become a Boolean algebra.

2.7. Theorem: The following conditions are equivalent for any Pre A*-algebra $(A,\wedge, \vee, (-)^{\sim})$.
(1) A is a Boolean Algebra
(2) $x\wedge y \leq x$ for all $x, y\in A$
(3) $x \wedge y \leq y$ for all $x, y \in A$
(4) $x\wedge y$ is a lower bound of $\{x, y\}$ in $(A, \leq)$ for all $x, y \in A$
(5) $x \wedge y$ is a infimum of $\{x, y\}$ in $(A, \leq)$ for all $x, y \in A$
(6) $x \vee x^{\sim}$ is the least element in $(A, \leq)$ for every $x \in A$

Proof: (1) $\Rightarrow$ (2) Suppose A be a Boolean algebra

Now $x \vee (x \wedge y) = x$ (by absorption law)

Therefore $x \wedge y \leq x$.

(2) $\Rightarrow$ (3) Suppose $x \wedge y \leq x$ then $x \vee (x \wedge y) = x$

Now $y \vee (x \wedge y) = y$.

Hence $x \wedge y \leq y$.

(3) $\Rightarrow$ (4) suppose that $x \wedge y \leq y \Rightarrow y \vee (x \wedge y)= y$

Since $x \wedge y \leq y$ then $x \wedge y$ is lower bound of y

Now $x \vee (x \wedge y) = x$ (by supposition)

Therefore $x \wedge y \leq x$

$\Rightarrow$ $x \wedge y$ is a lower bound of x

$x \wedge y$ is a lower bound of $\{x, y\}$.

(4) $\Rightarrow$ (5) suppose $x \wedge y$ is a lower bound of $\{x, y\}$

Suppose z is a lower bound of $\{x, y\}$ then $z\leq x, z\leq y$ that is

$x \vee z = x, y \vee z = y$

Now $z \vee (x \wedge y) = (z \vee x) \wedge (z \vee y) = x \wedge y$

Therefore $z\leq x \wedge y$.

$x \wedge y$ is the greatest lower bound of $\{x, y\}$

Hence $\text{Inf}\,\{x, y\} = x \wedge y$.

(5) $\Rightarrow$ (6) Suppose $\text{Inf}\,\{x, y\} = x \wedge y$ then $x,y \in B(A)$

Now $\text{Inf}\,\{x \wedge x^{\sim}, y\} = x \wedge x^{\sim} \wedge y = x \wedge x^{\sim}$

$\Rightarrow x \wedge x^{\sim} \leq_{\oplus} y$

Therefore $x \wedge x^{\sim}$ is the least element in $(A, \leq)$.

(6) $\Rightarrow$ (1) suppose $x \wedge x^{\sim}$ is the least element in A then $x \wedge x^{\sim} \leq y$,

for $y \in A$

$\Rightarrow$ $(x \wedge x^{\sim}) \vee y = y$

Now $y \wedge (x \vee y) = [(x \wedge x^{\sim}) \vee y] \vee (x \vee y)$

$= [(x \wedge x^{\sim}) \vee x] \vee y$

$= (x \wedge x^{\sim}) \vee y = y$ (by supposition)

Therefore absorption law holds hence A is a Boolean algebra.

2.8. Theorem: Let A be a pre A*-algebra $x \vee x^{\sim}$ is the greatest element in (A, $\leq$) for every $x \in A$ then A is a Boolean algebra.

Proof: Suppose $x \vee x^{\sim}$ is the greatest element in (A, $\leq$) then

$y \leq x \vee x^{\sim}$ for any $y \in A$

$\Rightarrow (x \vee x^{\sim}) \vee y = x \vee x^{\sim}$

Now $x \vee (x \wedge y) = [x \wedge (x^{\sim} \vee x)] \vee (x \wedge y)$

$= x \wedge [(x \vee x^{\sim}) \vee y]$

$= x \wedge (x \vee x^{\sim})$ (by supposition)

$= x$

Therefore $x \vee (x \wedge y) = x$, absorption law holds.

Hence A is a Boolean algebra.

2.9. Theorem: Let A be a Pre A*-algebra and $a \in A$. Let $A_a = \{x \in A \,/\, a \vee x = x\}$.Then A_a is closed under the operations $\wedge$ and $\vee$. Also for any $x \in A_a$ define, $x^a = a \vee x^{\sim}$. Then $(A_a, \wedge, \vee, {}^a)$ is a Pre A*-algebra with 1(here a is itself is the identity for $\vee$ in A_a; that is 1 in A_a).

Proof: Let $x, y \in A_a$ .Then $a \vee x = x$ and $a \vee y = y$.

Now $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) = x \wedge y \Rightarrow x \wedge y \in A_a$

Also $a \vee (x \vee y) = (a \vee x) \vee y = x \vee y \Rightarrow x \vee y \in A_a$

Therefore A_a is closed under the operation $\wedge$ and $\vee$.

$a \vee x^a = a \vee (a \vee x^{\sim}) = a \vee x^{\sim} = x^a \Rightarrow x^a \in A_a$

Thus A_a is closed under ^{a}.

Now for any $x, y, z \in A_a$

(1) $x^{aa} = (a \vee x^{\sim})^a = a \vee (a \vee x^{\sim})^{\sim} = a \vee (a^{\sim} \wedge x) = a \vee x = x$

(2) $x \wedge x = (a \vee x) \wedge (a \vee x) = x \wedge x = x$

(3) $x \wedge y = (a \vee x) \wedge (a \vee y) = (a \vee y) \wedge (a \vee x) = y \wedge x$

(4) $(x \wedge y)^a = a \vee (x \wedge y)^{\sim} = a \vee (x^{\sim} \vee y^{\sim})$

$= (a \vee x^{\sim}) \vee (a \vee y^{\sim})$

$= x^a \vee y^b$

(5) $x \wedge (y \wedge z) = (a \vee x) \wedge \{(a \vee y) \wedge (a \vee z)\}$
$= a \vee \{x \wedge (y \wedge z)\}$
$= a \vee \{(x \wedge y) \wedge z\}$ (since $x, y, z \in A$)
$= (x \wedge y) \wedge z$

(6) $x \wedge (y \vee z) = (a \vee x) \wedge \{(a \vee y) \vee (a \vee z)\}$
$= \{(a \vee x) \wedge (a \vee y)\} \vee \{(a \vee x) \wedge (a \vee z)\}$
$= \{a \vee (x \wedge y)\} \vee \{(a \vee (x \wedge z)\}$
$= (x \wedge y) \vee (x \wedge z)$

(7) $x \wedge (x^a \vee y) = x \wedge \{ (a \vee x^{\sim}) \vee y\}$
$= \{x \wedge (a \vee x^{\sim})\} \vee (x \wedge y)$
$= (x \wedge x^{\sim}) \vee (x \wedge y)$ (since $a \vee x = x$)
$= x \wedge (x^{\sim} \vee y)$
$= x \wedge y$

Finally $x \in A_a$ implies that $a \vee x = x = x \vee a$. Thus (A_a, $\wedge, \vee, ^a$) is a Pre A*-algebra with a as identity for $\vee$.

2.10. Theorem: Let a, b be elements in a Pre A*-algebra A such that $a \leq b$ .Then the following hold.
(1) $a \vee b = b$
(2) The map $\alpha_{a,b} : A_a \to A_b$ defined by $\alpha_{a,b}$ (x) = $b \vee x$ for all $x \in A_a$ is a homomorphism of Pre A*-algebras.
(3) $\alpha_{a,b}$ (B(A_a)) $\subseteq$ B(A_b)
(4) If $a \leq b \leq c$ then $\alpha_{a,b} \, o \, \alpha_{b,c} = \alpha_{a,c}$
(5) $\alpha_{a,a}$ is the identity map on A_a

Proof: Suppose that $a \leq b$
(1) We have $a \leq b \Rightarrow a \vee b = b$
(2) Let x, y $\in A_a$ .Then $\alpha_{a,b}$ $(x \wedge y) = b \vee (x \wedge y)$
$= (b \vee x) \wedge (b \vee y)$
$= \alpha_{a,b}(x) \wedge \alpha_{a,b}(y)$

and $\alpha_{a,b}$ $(x \vee y) = b \vee (x \vee y)$
$= (b \vee x) \vee (b \vee y)$
$= \alpha_{a,b}(x) \vee \alpha_{a,b}(y)$

Also $\alpha_{a,b}(x^a) = b \vee x^a$
$= b \vee (a \vee x^{\sim})$
$= (b \vee a) \vee x^{\sim}$
$= b \vee x^{\sim}$
$= b \vee (b^{\sim} \wedge x^{\sim})$
$= b \vee (b \vee x)^{\sim}$
$= (b \vee x)^b$
$= (\alpha_{a,b}(x))^b$

Therefore $\alpha_{a,b}$ is a homomorphism of Pre A*-algebras.
(3) Let $x \in$ B(A_a) .

Then $x \vee x^a$ =a (since a is identity in A_a) and therefore $a = x \vee (a \vee x^{\sim})$ (i)

Now $\alpha_{a,b}(x) \vee [\alpha_{a,b}(x)]^b = (b \vee x) \vee (b \vee x)^b$
$= (b \vee x) \vee [b \vee (b \vee x)^{\sim}]$
$= (b \vee x) \vee [b \vee (b^{\sim} \wedge x^{\sim})]$
$= (b \vee x) \vee (b \vee x^{\sim})$

$= b \vee (x \vee x^{\sim})$

$= (a \vee b) \vee (x \vee x^{\sim})$
$= b \vee [a \vee (x \vee x^{\sim})]$
$= b \vee a$ (by (i))
$= b$, which is 1 in A_b

Therefore $\alpha_{a,b}(x) \in B(A_b)$

Thus $\alpha_{a,b}(B(A_a)) \subseteq B(A_b)$

(4) Let $a \le b \le c$

$[\alpha_{a,b}\, o\, \alpha_{b,c}](x) = \alpha_{a,b}[\alpha_{b,c}(x)]$
$= \alpha_{a,b}[c \vee x]$
$= b \vee c \vee x$
$= c \vee x$
$= \alpha_{a,c}(x)$

Therefore $\alpha_{a,b}\, o\, \alpha_{b,c} = \alpha_{a,c}$

(5) $\alpha_{a,a}(x) = a \vee x = x$ for all $x \in A_a$

Then $\alpha_{a,a}$ is identity map on A_a.

CONCLUSION

This manuscript point ups the character of the Pre-A*-algebra like a partially ordered set. With respect to binary operation $\vee$, able to define a relation $\le$ on a Pre-A*-algebra and observed that such a Pre-A*-algebra as a partially ordered set with respect to the relation $\le$ and derived corresponding results. It has been observed a necessary condition a Pre-A*-algebra to become a lattice. We also present a equivalent condition for a Pre A*-algebra become a Boolean algebra. For any $a \in A$ defined a set $A_a = \{x \in A \,/\, a \vee x = x\}$, $x^a = a \vee x^{\sim}$ and observed that $(A_a, \wedge, \vee, ^a)$ is a Pre A*-algebra. Also defined a mapping $\alpha_{a,b}$ from A_a to A_b by $\alpha_{a,b}(x) = b \vee x$ for all $x \in A_a$ and confirmed a homomorphism of Pre A*-algebras.

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