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SOME STUDIES ON ZERO DIVISOR GRAPHS ASSOCIATED WITH CONNECTED RINGS

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ABSTRACT

Anderson and Livingston studied the properties of the zero divisor graph of a commutative ring. In this paper, we present the properties of the zero divisor graph of a connected ring. A connected ring (R, +, . , o) is a ring (R, +, .) with the connected operation o, that is, x o y = x a y for any x, a, y in R. We prove that if R is a commutative connected ring, then the zero divisor graph $\Gamma(R)$ is connected and diam $(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then girth, $gr(\Gamma(R)) \leq 7$. Also if R is a commutative connected Artinian ring and $\Gamma(R)$ contains a cycle, then $gr(\Gamma(R)) \leq 4$.

***Key words**: connected ring, Artinian ring, zero divisor graph, diameter, cycle, girth.*

1. INTRODUCTION

Anderson and Livingston [1] studied the properties of the zero divisor graph of a commutative ring. In this paper we present some properties of zero-divisor graphs associated with connected rings. Throughout this paper R denotes a commutative connected ring with identity element 1 and Z (R) be its set of zero-devisor. Γ(R) denotes a graph associated to R such that the vertices of Γ(R) of the elements of Z(R)*, where Z(R)* is the set of non-zero zero-divisors of R. The vertices x and y are adjacent if and only if *x* o *y* = *x* a *y* =0 for any a in R. This Γ(R) is called a zero-divisor graph of R. Γ(R) is an empty graph if and only if R is an integral domain. A ring R is Artinian if it satisfies the descending chain condition on ideals. A path whose origin and terminus vertices are same is called a cycle.

The diameter of a graph G is the sup {d(*x*, *y*) / *x* and *y* are distinct vertices in G}, where d (*x*, *y*) is the length of shortest path from *x* to *y* in G. The girth of G denoted by gr (G) is the length of the shortest cycle in G.

2. EXAMPLES

We give some examples of zero-divisor graphs.

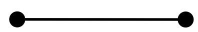
Let Z_4 be the connected ring of integers modulo 4.

Then Z_4 = {0, 1, 2, 3,}.

.	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

(a) We have the following zero-divisor graphs Γ(*R*) with |Γ(*R*)| ≤ 3.

Z_4 (or) $Z_2[x]/(x^2)$



Z_9, $Z_2 \times Z_2$ (or) $Z_3[x]/(x^2)$

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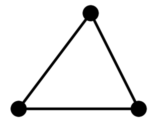
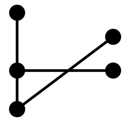
Z_6, Z_8 (or), $Z_2[x] / (x^3)$
$Z_4[x] / (2x, x^2-2)$

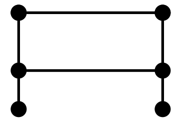


$Z_2[x, y] / (x^2, xy, y^2)$ or $F_4[x] / (x^2)$

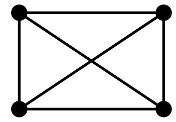
(b) We have eleven graphs with four vertices out of which only six are connected. These are given by



(i) $Z_2 \times F_4$



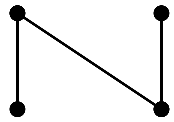
(ii) $Z_3 \times Z_3$



(iii) Z_{25} (or) $Z_5[x] / (x^2)$

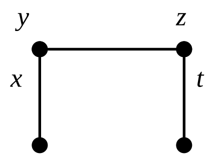
(iv)

(v)



(vi)

From the above six only three (i), (ii), (iii) are zero-divisor graphs $\Gamma(R)$. We prove that the graph Γ(R) given in (v) with vertices $\{x, y, z, t\}$ and edges xy, yz, zt can not be a zero-divisor graph $\Gamma(R)$.



Suppose a ring R with zero – divisors $Z(R) = \{0, x, y, z, t\}$. Then $x+z \in Z(R)$, since $(x+z) \cdot y = (x+z)ay = 0$. Hence $(x+z)$ must be 0, x, y, z or t. But there is only possibility $x + z = y$. Similarly $y+t = z$. Hence $y = x+z = x+y+t$. So $x+t = 0 \Rightarrow t = -x$. Thus $y \circ t = yat = y \circ (-x) = 0$, a contradiction. Hence (v) is not a zero – divisor graph. Similarly it can be proved that the graph in (iv) and (vi) are not zero-divisor graphs.

(c) Now clearly $\Gamma(R)$ can not be a triangle or a square. But sub $\Gamma(R)$ can not be an n-gon for $n \geq 5$. So there is a zero-divisor graph for each $n \geq 3$ with an n-cycle.

Let $R_n = I_z[x_1, \ldots\ldots, x_n]/I = Z_2[x_1, \ldots\ldots, x_n]$,

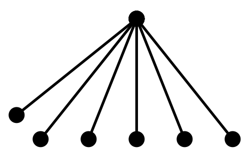
where $I = (x_1^2 \ldots\ldots, x_n^2, \quad x_1x_2, x_2x_3 \ldots\ldots\ldots x_nx_1]$.

Then $\Gamma(R_n)$ is finite and has a cycle of length n. i.e., $x_1 - x_2 \ldots\ldots\ldots\ldots x_n - x_1$.

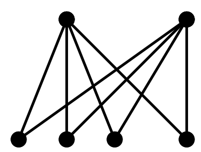
(d) Let $R = A$xB, where A and B be integral domains. Suppose $\Gamma(R)$ can be partitioned into two disjoint vertex sets $V_1 = \{(a,0) / a \in A^*\}$ and $V_2 = \{(0,b)/b \in B^*\}$, and two vertices x and y are adjacent if and only if they are in distinct vertex sets. Then $\Gamma(R)$ is a complete bipartite graph with $|\Gamma(R)| = |A| + |B| - 2$. A complete bipartite graph with vertex sets having m and n elements is denoted by $K_{m,n}$. If $A = Z_2$, Then $\Gamma(R)$ is a star graph with $|\Gamma(R)| = |B|$.

For example, $\Gamma(F_p$x$F_q) = K_{p-1,q-1}$.

and $\Gamma(F_2 \times F_q) = K_{1,q-1}$



$Z_2 \times Z_7$



$Z_3 \times Z_5$

The set of zero divisors of Z_4 is {0}.

$\Gamma(R)$ may be infinite because a connected ring may have an infinite number of zero-divisors. But $\Gamma(R)$ is of most interest when it is finite, for we can diam $\Gamma(R)$

3. MAIN THEOREMS

We now discuss some properties of $\Gamma(R)$, where R is a commutative connected ring.

Theorem: 3.1 Let R be a commutative connected ring. Then $\Gamma(R)$ is connected and diam $(\Gamma(R)) \leq 3$. Moreover, if $\Gamma(R)$ contains a cycle, then $g(\Gamma(R)) \leq 7$.

Proof: Let R be a commutative connected ring, $\Gamma(R)$ is connected implies that there is a path between any two distinct vertices in $\Gamma(R)$. Let $x, y \in Z(R)^*$ and $x \neq y$. Let $d(x, y)$ be the length of shortest path from x to y.

If $x \circ y = xay = 0$, then $d(x, y) = 1$.

Suppose that $x \circ y = xay$ is non-zero.

If $xax = yay = 0$, then $x - xay - y$ is a path of length 2, i.e., $d(x, y) = 2$. If $x \circ x = xax = 0$ and $y \circ y = yay \neq 0$, then there is $ab \in Z(R)^* - \{x, y\}$ with $b \circ y = bay = 0$. If $b \circ x = bax = 0$, then $x - b - y$ is a path of length 2. If $b \circ x = bax \neq 0$, then $x - bax - y$ is a path of length 2. In either case we have $d(x, y) = 2$.

Similarly if $yay = 0$ and $xax \neq 0$, we can show that $d(x, y) = 2$. So we assume that $xay \neq 0$, $xax \neq 0$ and $yay \neq 0$. Hence there are $s, t \in Z(R)^* - \{x, y\}$ with $sax = tay = 0$. If $s = t$, then $x - s - y$ is a path of length 2. So we assume that $s \neq t$.

If $sat = 0$ then $x - s - t - y$ is a path of length 3. Hence $d(x, y) \leq 3$.

If $sat = 0$ then $x - sat - y$ is a path of length 2.

Thus $d(x, y) \leq 2$.

Hence $d(x, y) \leq 3$. Thus diam $(\Gamma(R)) \leq 3$. If $\Gamma(R)$ contains a cycle, then $gr\ (\Gamma(R)) \leq 2$ diam $\Gamma + 1$.

So $gr\ (\Gamma(R)) \leq 2.3 + 1 = 7$.

If we consider the graphs given in Example, it is clear that diam $(\Gamma(R)) = 0, 1, 2$.

If $R = Z_2 \times Z_4$ then the path $(\bar{0},\bar{1}) - (\bar{1},\bar{0}) - (\bar{0},\bar{2}) - (\bar{1},\bar{2})$ gives that diam $(\Gamma(R)) = 3$.

Remark: The ring R given in Example (a) have diam $(\Gamma(R)) = 0$, 1, or 2. If $R = Z_2 \times Z_4$, then the path $(\bar{0},\bar{1}) - (\bar{1},\bar{0}) - (\bar{0},\bar{2}) - (\bar{1},\bar{2})$ shows that diam $(\Gamma(R)) = 3$.

Further the rings given in Example have $gr\ (\Gamma(R)) \leq 3$, 4 or ∞.

If R is Artinnian then we can improve the value of $g(\Gamma(R))$ as $g(\Gamma(R)) \leq 4$.

This can be seen as follows:

Theorem: 3.2 Let R be a commutative connected Artinian ring (in particular, R is a finite commutative connected ring).

If $\Gamma(R)$ contains a cycle, then $gr(\Gamma(R)) \leq 4$.

Proof: Let R be a commutative connected Artinian ring. Suppose $\Gamma(R)$ contains a cycle. Then R is a finite direct product of Artinian Local rings [3].

Suppose that R is a local ring with non-zero maximal ideal M. So $M = Ann\ x$ for some $x \in M^*$ [5].

If $y \neq z$ and $y, z \in M^*$-$\{x\}$ with $yaz = 0$, then $y - x - z - y$ is a triangle, otherwise, $\Gamma(R)$ contains no cycles, a contradiction to $\Gamma(R)$ contains a cycle.

Therefore in this case $gr\ (\Gamma(R)) = 3$.

Suppose that $R = R_1 \times R_2$.

If R_1, R_2 are such that $|R_1| \geq 3$ and $|R_2| \geq 3$, then we choose $a_i \in R_i\ \{0, 1\}$. Then $(1,0) - (0,1) - (a_1,0) - (0,a_2) - (1,0)$ is a square. So in this case, $gr\ (\Gamma(R)) \leq 4$.

Thus we may assume that $R_1 = Z_2$.

If $|Z(R_2)| \leq 2$, then $\Gamma(R)$ contains no cycles, a contradiction. Hence $|Z\ (R_2)| \geq 3$.

Since $\Gamma(R)$ is connected, there are distinct $xy \in Z(R_2)^*$ with $xay = 0$.
Thus $(\bar{0}, x) - (\bar{1}, 0) - (\bar{0}, y) - (\bar{0}, x)$ is a triangle. Hence in this case $gr\ (\Gamma(R)) = 3$.

Thus in all cases, $gr\ (\Gamma(R)) \leq 4$.

The proof of the above theorem shows that a finite commutative connected ring 'R' has $gr\ (\Gamma(R)) = 4$ if and only if either $R \cong F \times K$ where F and K are finite fields with $|F| \geq 3$, $|K| \geq 3$ or $R \cong F \times A$ where F is a finite field with $|F| \geq 3$ and A is a finite commutative connected ring with $|Z(A)| = 2$ i.e., in this case $A = Z_4$ or $A \cong Z_2(x)/(x^2)$.

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