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SOLUTIONS OF A MULTI-POINT BOUNDARY VALUE PROBLEM FOR HIGHER-ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT

The present paper, is concerned with the existence of solutions of the following multi-point boundary value problem consisting of the higher-order differential equations.

$$(-1)^{n-1}x^{(2n)} = f\left(t, x(t), x'(t), \ldots\ldots x^{(2n-1)}(t)\right), \ t\in(0,1), \tag{1}$$

and the following multi-point boundary value conditions

$$\begin{cases} x^{(2i-1)}(0) = 0, i = 1,2,3,4, \ldots\ldots\ldots, n, \\ x^{(2i-1)}(1) = \sum_{k=1}^{p_i} \alpha_{i,k} x^{(2i-1)}(\xi_i, k), i = 1,2, \ldots\ldots, n-1, \\ x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i), \end{cases} \tag{2}$$

are sufficient conditions for existence of at least one solution of the BVP(1).

1. INTRODUCTION

In recent years, the solvability of the multi-point boundary value problems for second order differential equations, arise in many applications, we refer the reader to the monographs [1-3] and the references [4-14]. In [15], Erbe and Tang studied the existence of positive solutions of the following Sturm-Liouvile boundary value problem consisting of the second order differential equation

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0, t\in(0,1) \\ \alpha x(0) - \beta x'(0) = \delta x(1) + \gamma x'(1) = 0 \end{cases} \tag{3}$$

where f is continuous and nonnegative, $\alpha \geq 0, \beta \geq 0, \gamma \geq 0$ and $\delta \geq 0$ with $\alpha\delta + \gamma\delta + \alpha\beta > 0$. He proved that, under some assumptions, BVP(3) has at least one or two positive solutions.

In [6], Liu and Yu studied the solvability of the following multi-point boundary value problem consisting of the second-order differential equation

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t), t\in(0,1) \\ x'(0) = 0, \ x(1) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \end{cases} \tag{4}$$

where $0 < \eta < 1, 0 < \xi < 1, \alpha \geq 0$ and $\beta \geq 0$ and f is continuous and $e \in L^1[0,1]$.

However, the Sturm-Liouvile type boundary value conditions, i.e.., $x(0) = \alpha x'(\xi)$, $x'(1) = \beta x(\eta)$ was not studied in [6].Furthermore, to the best of our knowledge, there has been no paper concerned with the existence of solutions of multi-point boundary value problems for higher-order differential equations at resonance, although there were considerable papers concerned with the existence of positive solutions or solutions of higher-order differential equations at non-resonance cases [1-3, 19, 20].Motivated and inspired by papers [15,16,6], we are concerned with the following fourth-order differential equation

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$x^4(t) = f\left(t, x(t), -x''(t)\right), t \in (0,1)$ (5)

Or

$x^4(t) = f\left(t, x(t)\right), t \in (0,1)$

Subject to the following multi-point boundary value conditions $x(1) = x'(0) = x'(1) = x'''(0) = 0$ as not been studied.

Chyan and Henderson, in [14], studied the following $2m^{th}$-order differential equation

$$x^{(2m)}(t) = f\left(t, x(t), x''(t), \ldots\ldots\ldots, x^{(2m-2)}(t)\right), 0 < t < 1, \tag{6}$$

with either the Lindstone boundary condition

$x^{(2i)}(0) = x^{(2i)}(1) = 0 \quad for \ i = 0,1,2,3, \ldots\ldots, m-1$, or the focal boundary value condition (7)

$$x^{(2i+1)}(0) = x^{(2i)}(1) = 0 \quad for \ i = 0,1,2,3, \ldots\ldots, m-1, \tag{8}$$

For BVP (1) and (2), the corresponding linear differential equation is

$$(-1)^{n-1} x^{(2n)} = 0 \ , t \in (0,1). \tag{9}$$

2. MAIN RESULTS

In this section, we establish sufficient conditions for the existence of at least one solution of BVP (1)-(2) and one positive solution of BVP(1) and (2). respectively. For convenience, we first introduce some notations and an abstract existence theorem by Gaines and Mawhin[9].

Let X and Y be Banach spaces, L: $dom\, L \subset X \to Y$ be a Fredholm operator of index zero, $P: X \to X, Q: Y \to Y$ be projectors such that

$Im\, P = Ker\, L,\ Ker\, Q = Im\, L,\ X = Ker\, L \oplus Ker\, P,\ Y = Im\, L \oplus Im\, Q$. It follows that $L|_{dom\ L \cap Ker\ P}$: $Ker\, P \to Im\, L$ is invertible, we denote the inverse of that map by K_P.

If Ω is an open bounded subset of X $domL \cap \bar{\Omega} \neq \phi$, the map $N: X \to Y$ will be called L−compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N: \bar{\Omega} \to X$ is compact.

Theorem Gm [9]: Let L be a Fredholm operator of index zero and let N be L−compact on Ω. Assume that the following conditions are satisfied:

(i) $Lx \neq \lambda Nx \ for\ every\ (x, \lambda) \epsilon [(domL/KerL) \cap \partial\Omega] \times (0,1)$;

(ii) $Nx \notin ImL \ for\ every\ x \in KerL \cap \partial\Omega$;

(iii)$\deg(\Lambda QN|_{KerL}\ , \Omega \cap KerL, 0) \neq 0, where \Lambda: Y/ImL \to KerL\ is\ the\ isomorphism.$

Then the equation Lx$=Nx$ has at least one solution in $domL \cap \overline{\Omega}$.

We use the classical Banach space $C^k[0,1]$, let $X = C^{2n-1}[0,1]$ and $Y = C^0[0,1]$. Y is endowed with the norm $||y||_\infty = max_{t\in[0,1]}|y(t)|$, X is endowed with the norm $||x|| = max\{||x||_\infty, ||x'||_\infty, \ldots, ||x^{(2n-1)}||_\infty\}$. Define the linear operator L and the nonlinear operator N by

$L:\ X \cap domL \to Y, \qquad Lx(t) = (-1)^{n-1} x^{(2n)}(t)\ for\ x \in X \cap domL,$

$N:\ X \to Y, \qquad Nx(t) = f(t, x(t), x'(t), \ldots, x^{(2n-1)}(t)),\ for\ x \in X$, respectively,

where

$$domL = \{x \in C^{m-1}[0,1], x^{(2i-1)}(0) = 0\ for\ i = 1, \ldots\ldots, n$$

$$x^{(2i-1)}(1) = \sum_{k=1}^{p_i} \alpha_{i,k} x^{(2i-1)}\left(\xi_{i,k}\right), for\ i = 1, \ldots\ldots, n-1,$$

$$x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i)\}$$

Suppose $\sum_{k=1}^{p_i} \alpha_{i,k}\, \xi_{i,k} \neq 1 \;\; for\; i = 1, \ldots\ldots, n-1.$ Let, for $i = 1, \ldots\ldots, n-1$, $G_{i-1}(t,s)$ be the Green's function of the problem

$-u''(t) = \alpha(t)\,,\; u(0) = u(1) - \sum_{k=1}^{p_i} \alpha_{i,k} u\left(\xi_{i,k}\right) = 0,$ for some α .Let

$$G(t,s) = \int_0^1 \ldots.\int_0^1 G_1(t,\tau_1)\ldots\ldots..G_{n-1}(\tau_{n-2},s)d\tau_1 \ldots\ldots\ldots\ldots d\tau_{n-2}.$$

Lemma 2.1: The following resultsholds.

(i)There is a k_i so that $\alpha_{i,k} \geq 0$ for $k = 1, \ldots\ldots, k_i$ and $\alpha_{i,k} \leq 0$ for $k = k_i + 1, \ldots, p_i$with $\sum_{k=1}^{p_i} \alpha_{i,k} < 1$;

$$\Delta = \int_0^1\int_0^1 G(s,\tau)\int_0^\tau u^l du d\tau ds - \sum_{i=1}^m \beta_i \int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau u^l du d\tau ds \neq 0.$$

Then the following results hold.

(i) $Ker\ L = \{x(t) \equiv c\ , t \in [0,1], c \in R\}$;

(ii) $ImL = y \in Y, \left\{ \begin{array}{c} \int_0^1\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds \\ = \sum_{i=1}^m \beta_i \int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds \end{array} \right\}$

(iii) L is Fredholm operator of index zero;

(iv) There are projectors $P: X \to X\ \ and\ \ Q: Y \to Y$ such that $Ker\ L = ImP\ \ and\ KerQ = \ Im\ L.$ Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap\ dom\ L \neq \phi$, then N is L-compact on $\overline{\Omega}$.

(v) *x(t)* is a solution of BVP(1) and BVP(2) if and only if x is a solution of the operator equation $Lx = Nx\ in\ domL$.

Proof: (i) The proof is easy and is omitted.

(ii) If $y \in ImL$, then

$$(-1)^{n-1}x^{(2n)} = y(t)\ \ , t \in (0,1)$$

$$x^{(2i-1)}(0) = \ x^{(2i-1)}(1) - \sum_{k=1}^{p_i} \alpha_{i,k}\, x^{(2i-1)}\left(\xi_{i,k}\right) = 0\ , i = 1, \ldots\ldots, n-1.,$$

$$x^{(2n-1)}(0) = 0\ , x(1) = \sum_{i=1}^m \beta_i x\left(\xi_i\right).$$

This implies $x^{(2n-1)}(t) = (-1)^{n-1}\int_0^t y(u)du$ since $x^{(2n-1)}(0) = 0$. we get

$$x^{(2n-3)}(t) = (-1)^{n-2}\int_0^1 G_{n-1}(t,\tau)\int_0^\tau y(u)du d\tau,$$

Similarly, we get

$$x'(t) = \int_0^1 G(t,\tau)\int_0^\tau y(u)du d\tau,$$

So,

$$x(t) = c + \int_0^t\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds.$$

It follows from $x(1)=\sum_{i=1}^{m}\beta_i x(\xi_i)$ that

$$\int_0^1\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds=\sum_{i=1}^{m}\beta_i\int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds.$$

On the other hand, assume that $\sum_{i=0}^{2n-1} r_i<\frac{1}{2}$ holds. Let

$$x(t)=c+\int_0^t\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds.$$

Then $x(t)$ satisfies above equation and hence (ii) is complete.

(iii). from (i), $\dim KerL=1$.On the other hand, for $y\in Y$, let

$y_0=y-\frac{t^k}{\Delta}\left(\int_0^1\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds-\sum_{i=1}^{m}\beta_i\int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds\right)$. It is easy to check that $y_0\in ImL$. Let $\overline{R}=\{ct^k: t\in[0,1], c\in R\}$.

We get $Y=\overline{R}+ImL$. It follows from $\overline{R}\cap ImL=\{0\}$ that $Y=\overline{R}\oplus ImL$.

Hence $dimY/ImL=1$. On the other hand, f is continuous and ImL is closed. So L is a Fredholm operator of index zero.

(iv). Define the projectors $P: X\to X$ and $Q: Y\to Y$ by $Px(t)=x(0) for\ x\in X$,

$Qy(t)=\frac{t^k}{\Delta}\left(\int_0^1\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds-\sum_{i=1}^{m}\beta_i\int_0^{\xi_i}\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds\right)$ for $y\in Y$.

It is easy to check that $KerL=ImP$ and $ImL=KerQ$. The generalized inverse $K_p: ImL\to domL\cap KerP$ of L can be written by

$$K_p y(t)=\int_0^t\int_0^1 G(s,\tau)\int_0^\tau y(u)du d\tau ds.$$

(v). The proof is easy and can be omitted.

Theorem 2.1: Suppose following conditions hold
(A1). There exists functions $a_i(i=0,1,\ldots,n-1), b$ and $L^1[0,1]$ and a constant $\theta\in[0,1)$ such that for all $x_i\in R(i=0,1,2,\ldots.n-1)$, the following inequality holds

$$|f(t,x_0,x_1,\ldots\ldots x_{n-1})|\le\sum_{i=0}^{n-1}a_i(t)|x_i|+b(t)|x_{n-1}|^\theta+r(t);$$

(A2). There is $M>0$ such that for any $x\in dom\,L/Ker\,L$, if $|x^{(n-1)}(t)|>M$ for all $t\in[0,1]$, then

$$\int_0^1\left(f\left(s,x(s),x'(s),\ldots\ldots,x^{(n-1)}(s)\right)+e(s)\right)ds-\beta\int_0^\eta(\eta-s)\left(f\left(s,x(s),x'(s),\ldots\ldots,x^{(n-1)}(s)\right)+e(s)\right)ds\neq0;$$

(A3). There is $M^*>0$ such that, for $x(t)=ct^{n-1}$, either

$$c\left[\int_0^1(f(s,cs^{n-1},c(n-1)s^{n-2},\ldots.(n-1)!\,c)+e(s))ds-\beta\int_0^\eta(\eta-s)\int_0^1(f(s,cs^{n-1},c(n-1)s^{n-2},\ldots.(n-1)!\,c)+e(s)ds\right]<0$$

for all $|c|>M^*$ or

$$c\left[\int_0^1(f(s,cs^{n-1},c(n-1)s^{n-2},\ldots.(n-1)!\,c)+e(s))ds-\beta\int_0^\eta(\eta-s)\int_0^1(f(s,cs^{n-1},c(n-1)s^{n-2},\ldots.(n-1)!\,c)+e(s)ds\right]>0$$

for all $|c| > M^*$;

(A4). $\sum_{i=1}^{n-1} \|a_i\|_1 < 1$.

Then BVP (5) and (6) has atleast one solution.

Theorem 2.2: Suppose following conditions hold
(A′1). There are continuous functions $h(t, x_0, x_1, \ldots\ldots, x_{2n-1}), e(t)$ and non negative functions $g_i(t, x)(i = 0,1, \ldots\ldots, 2n-1)$ and positive numbers β and m such that f satisfies

$$(-1)^{n-1} f(t, x_0, x_1, \ldots\ldots x_{2n-1}) = e(t) + h(t, x_0, x_1, \ldots\ldots, x_{2n-1}) + \sum_{i=0}^{2n-1} g_i(t, x_i),$$

and also that h satisfies

$$x_{2n-1} h(t, x_0, x_1, \ldots\ldots, x_{2n-1}) \le -\beta |x_{2n-1}|^{m+1}$$

and for all $t \in [0, 1]$ *and* $(x_0, x_1, \ldots\ldots x_{2n-1}) \in R^{2n}$ and

$$\lim_{|x|\to\infty} \sup_{t \in [0,1]} \frac{|g_i(t, x)|}{|x|^m} = r_i, for\ i = 0,1,2, \ldots\ldots, 2n-1.$$

With $r_i \ge 0\ for\ i = 0,1,2, \ldots\ldots, 2n-1$;

(A′2). There exists constants $L \ge 0$, $\alpha > 0$ *and* $\alpha_i \ge 0 (i = 1,2, \ldots .2n-2)$ such that

$$|f(t, x_0, x_1, \ldots\ldots x_{2n-1})| \ge \alpha |x_0| - \sum_{i=1}^{2n-2} \alpha_i |x_i| - L$$

For all $t \in [0, 1]$ *and* $(x_0, x_1, \ldots\ldots x_{2n-1}) \in R^{2n}$.

Furthermore (A3), (A4) of Theorem 2.1 hold. Then BVP (1) and (2) has atleast one solution provided

$$\left(1 + \frac{\sum_{i=1}^{2n-2} \alpha_i}{\alpha}\right)^m r_0 + \sum_{i=1}^{2n-1} r_i < \beta.$$

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