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# SOLUTIONS OF A MULTI-POINT BOUNDARY VALUE PROBLEM FOR HIGHER-ORDER DIFFERENTIAL EQUATIONS

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***ABSTRACT***

*The present paper, is concerned with the existence of solutions of the following multi-point boundary value problem consisting of the higher-order differential equations.*

$$(-1)^{n-1}x^{(2n)} = f\left(t, x(t), x'(t), \dots\dots x^{(2n-1)}(t)\right),\ t\in(0,1), \tag{1}$$

*and the following multi-point boundary value conditions*

$$\begin{cases} x^{(2i-1)}(0) = 0\ , i = 1,2,3,4, \dots\dots\dots, n, \\ x^{(2i-1)}(1) = \sum_{k=1}^{p_i} \alpha_{i,k} x^{(2i-1)}(\xi_i, k), i = 1,2, \dots\dots, n-1, \\ x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i), \end{cases} \tag{2}$$

*are sufficient conditions for existence of at least one solution of the BVP(1).*

## 1. INTRODUCTION

In recent years, the solvability of the multi-point boundary value problems for second order differential equations, arise in many applications, we refer the reader to the monographs [1-3] and the references [4-14]. In [15], Erbe and Tang studied the existence of positive solutions of the following Sturm-Liouvile boundary value problem consisting of the second order differential equation

$$\begin{cases} x''(t) + f(t, x(t), x'(t)) = 0\ , t\in(0,1) \\ \alpha x(0) - \beta x'(0) = \delta x(1) + \gamma x'(1) = 0 \end{cases} \tag{3}$$

where $f$ is continuous and nonnegative, $\alpha \geq 0, \beta \geq 0, \gamma \geq 0$ and $\delta \geq 0$ with $\alpha\delta + \gamma\delta + \alpha\beta > 0$. He proved that, under some assumptions, BVP(3) has at least one or two positive solutions.

In [6], Liu and Yu studied the solvability of the following multi-point boundary value problem consisting of the second-order differential equation

$$\begin{cases} x''(t) = f(t, x(t), x'(t)) + e(t)\ , t\in(0,1) \\ x'(0) = 0\ ,\ x(1) = \sum_{i=1}^{m} \alpha_i x(\xi_i), \end{cases} \tag{4}$$

where $0 < \eta < 1, 0 < \xi < 1, \alpha \geq 0$ and $\beta \geq 0$ and $f$ is continuous and $e \in L^1[0,1]$.

However, the Sturm-Liouvile type boundary value conditions, i.e.., $x(0) = \alpha x'(\xi), x'(1) = \beta x(\eta)$ was not studied in [6].Furthermore, to the best of our knowledge, there has been no paper concerned with the existence of solutions of multi-point boundary value problems for higher-order differential equations at resonance, although there were considerable papers concerned with the existence of positive solutions or solutions of higher-order differential equations at non-resonance cases [1-3, 19, 20].Motivated and inspired by papers [15,16,6], we are concerned with the following fourth-order differential equation

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$$x^4(t) = f\left(t, x(t), -x''(t)\right), t \in (0,1) \tag{5}$$

Or

$$x^4(t) = f\left(t, x(t)\right), t \in (0,1)$$

Subject to the following multi-point boundary value conditions $x(1) = x'(0) = x'(1) = x'''(0) = 0$ as not been studied.

Chyan and Henderson, in [14], studied the following $2m^{th}$-order differential equation

$$x^{(2m)}(t) = f\left(t, x(t), x''(t), \dots\dots\dots, x^{(2m-2)}(t)\right), 0 < t < 1, \tag{6}$$

with either the Lindstone boundary condition

$x^{(2i)}(0) = x^{(2i)}(1) = 0 \quad for \quad i = 0,1,2,3, \dots\dots, m-1$, or the focal boundary value condition (7)

$$x^{(2i+1)}(0) = x^{(2i)}(1) = 0 \quad for \quad i = 0,1,2,3, \dots\dots, m-1, \tag{8}$$

For BVP (1) and (2), the corresponding linear differential equation is

$$(-1)^{n-1} x^{(2n)} = 0 \quad, t \in (0,1). \tag{9}$$

## 2. MAIN RESULTS

In this section, we establish sufficient conditions for the existence of at least one solution of BVP (1)-(2) and one positive solution of BVP(1) and (2). respectively. For convenience, we first introduce some notations and an abstract existence theorem by Gaines and Mawhin[9].

Let $X$ and $Y$ be Banach spaces, $L$: $dom\, L \subset X \to Y$ be a Fredholm operator of index zero, $P: X \to X, Q: Y \to Y$ be projectors such that

$Im\, P = Ker\, L$, $Ker\, Q = Im\, L$, $X = Ker\, L \oplus Ker\, P$, $Y = Im\, L \oplus Im\, Q$. It follows that $L|_{dom\, L \cap Ker\, P}$ : $Ker\, P \to Im\, L$ is invertible, we denote the inverse of that map by $K_P$.

If $\Omega$ is an open bounded subset of $X$ $domL \cap \bar{\Omega} \neq \phi$, the map $N: X \to Y$ will be called L−compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_p(I - Q)N: \bar{\Omega} \to X$ is compact.

**Theorem Gm [9]:** Let $L$ be a Fredholm operator of index zero and let $N$ be $L-$compact on $\Omega$. Assume that the following conditions are satisfied:

(i) $Lx \neq \lambda Nx \; for \; every \; (x, \lambda) \epsilon [(domL/KerL) \cap \partial\Omega] \times (0,1)$;

(ii) $Nx \notin ImL \; for \; every \; x \in KerL \cap \partial\Omega$ ;

(iii)$\deg(\Lambda QN|_{KerL} \;, \Omega \cap KerL, 0) \neq 0, where \Lambda: Y/ImL \to KerL \; is \; the \; isomorphism.$

Then the equation Lx$=Nx$ has at least one solution in $domL \cap \overline{\Omega}$.

We use the classical Banach space $C^k[0,1]$, let $X = C^{2n-1}[0,1]$ $and$ $Y = C^0[0,1]$. $Y$ is endowed with the norm $||y||_\infty = max_{t\in[0,1]}|y(t)|$, $X$ is endowed with the norm $||x|| = max\{||x||_\infty, ||x'||_\infty, \dots, ||x^{(2n-1)}||_\infty\}$. Define the linear operator $L$ and the nonlinear operator $N$ by

$L$: $X \cap domL \to Y$, $\quad Lx(t) = (-1)^{n-1} x^{(2n)}(t) \; for \; x \in X \cap domL$,

$N$: $X \to Y$, $\quad Nx(t) = f(t, x(t), x'(t), \dots, x^{(2n-1)}(t)), \; for \; x \in X$, respectively,

where

$$domL = \{x \in C^{m-1}[0,1], x^{(2i-1)}(0) = 0 \; for \; i = 1, \dots\dots, n$$

$$x^{(2i-1)}(1) = \sum_{k=1}^{p_i} \alpha_{i,k} x^{(2i-1)}\left(\xi_{i,k}\right), for \; i = 1, \dots\dots, n-1,$$

$$x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i)\}$$

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Suppose $\sum_{k=1}^{p_i} \alpha_{i,k}\, \xi_{i,k} \neq 1 \;\; for\; i = 1, \ldots\ldots, n-1$. Let, for $i = 1, \ldots\ldots, n-1$, $G_{i-1}(t,s)$ be the Green's function of the problem

$-u''(t) = \alpha(t)\,,\; u(0) = u(1) - \sum_{k=1}^{p_i} \alpha_{i,k} u\left(\xi_{i,k}\right) = 0$, for some $\alpha$ .Let

$$G(t,s) = \int_0^1 \ldots . \int_0^1 G_1(t,\tau_1) \ldots\ldots . . G_{n-1}(\tau_{n-2},s) d\tau_1 \ldots\ldots\ldots\ldots d\tau_{n-2}.$$

**Lemma 2.1:** The following resultsholds.

(i)There is a $k_i$ so that $\alpha_{i,k} \geq 0$ for $k = 1, \ldots\ldots, k_i$ and $\alpha_{i,k} \leq 0$ for $k = k_i + 1, \ldots, p_i$with $\sum_{k=1}^{p_i} \alpha_{i,k} < 1$;

$$\Delta = \int_0^1 \int_0^1 G(s,\tau) \int_0^\tau u^l du d\tau ds - \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G(s,\tau) \int_0^\tau u^l du d\tau ds \neq 0.$$

Then the following results hold.

(i) $Ker\ L = \{x(t) \equiv c\ , t \in [0,1], c \in R\}$;

(ii) $ImL = y \in Y, \left\{ \begin{array}{l} \int_0^1 \int_0^1 G(s,\tau) \int_0^\tau y(u) du d\tau ds \\ = \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G(s,\tau) \int_0^\tau y(u) du d\tau ds \end{array} \right\}$

(iii) $L$ is Fredholm operator of index zero;

(iv) There are projectors $P: X \to X\ \ and\ \ Q: Y \to Y$ such that $Ker\ L = ImP\ \ and\ KerQ = Im\ L$. Furthermore, let $\Omega \subset X$ be an open bounded subset with $\overline{\Omega} \cap\ dom\ L \neq \phi$, then N is L-compact on $\overline{\Omega}$.

(v) *x(t)* is a solution of BVP(1) and BVP(2) if and only if $x$ is a solution of the operator equation $Lx = Nx\ in\ domL$.

**Proof:** (i) The proof is easy and is omitted.

(ii) If $y \in ImL$, then

$$(-1)^{n-1} x^{(2n)} = y(t)\ \ , t \in (0,1)$$

$$x^{(2i-1)}(0) = x^{(2i-1)}(1) - \sum_{k=1}^{p_i} \alpha_{i,k}\, x^{(2i-1)}\left(\xi_{i,k}\right) = 0\ , i = 1, \ldots\ldots, n-1.,$$

$$x^{(2n-1)}(0) = 0\ , x(1) = \sum_{i=1}^{m} \beta_i x\left(\xi_i\right).$$

This implies $x^{(2n-1)}(t) = (-1)^{n-1} \int_0^t y(u) du$ since $x^{(2n-1)}(0) = 0$ . we get

$$x^{(2n-3)}(t) = (-1)^{n-2} \int_0^1 G_{n-1}(t,\tau) \int_0^\tau y(u) du d\tau,$$

Similarly, we get

$$x'(t) = \int_0^1 G(t,\tau) \int_0^\tau y(u) du d\tau,$$

So,

$$x(t) = c + \int_0^t \int_0^1 G(s,\tau) \int_0^\tau y(u) du d\tau ds.$$

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It follows from $x(1) = \sum_{i=1}^{m} \beta_i x(\xi_i)$ that

$$\int_0^1 \int_0^1 G(s,\tau) \int_0^\tau y(u)du d\tau ds = \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G(s,\tau) \int_0^\tau y(u)du d\tau ds.$$

On the other hand, assume that $\sum_{i=0}^{2n-1} r_i < \frac{1}{2}$ holds. Let

$$x(t) = c + \int_0^t \int_0^1 G(s,\tau) \int_0^\tau y(u)du d\tau ds.$$

Then $x(t)$ satisfies above equation and hence (ii) is complete.

(iii). from (i), $\dim KerL = 1$.On the other hand, for $y \in Y$, let

$y_0 = y - \frac{t^k}{\Delta}\left(\int_0^1 \int_0^1 G(s,\tau) \int_0^\tau y(u)du d\tau ds - \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G(s,\tau) \int_0^\tau y(u)du d\tau ds\right)$. It is easy to check that $y_0 \in ImL$. Let $\overline{R} = \{ct^k : t \in [0,1], c \in R\}$.

We get $Y = \overline{R} + ImL$. It follows from $\overline{R} \cap ImL = \{0\}$ that $Y = \overline{R} \oplus ImL$.

Hence $dim Y/ImL = 1$. On the other hand, $f$ is continuous and $ImL$ is closed. So L is a Fredholm operator of index zero.

(iv). Define the projectors $P: X \to X$ *and* $Q: Y \to Y$ by $Px(t) = x(0) for\ x \in X$,

$Qy(t) = \frac{t^k}{\Delta}\left(\int_0^1 \int_0^1 G(s,\tau) \int_0^\tau y(u)du d\tau ds - \sum_{i=1}^{m} \beta_i \int_0^{\xi_i} \int_0^1 G(s,\tau) \int_0^\tau y(u)du d\tau ds\right)$ for $y \in Y$.

It is easy to check that $KerL = ImP\ and\ ImL = KerQ$. The generalized inverse $K_p: ImL \to domL \cap KerP\ of\ L$ can be written by

$$K_p y(t) = \int_0^t \int_0^1 G(s,\tau) \int_0^\tau y(u)du d\tau ds.$$

(v). The proof is easy and can be omitted.

**Theorem 2.1:** Suppose following conditions hold
**(A1).** There exists functions $a_i (i = 0,1,\ldots,n-1), b$ and $L^1[0,1]$ and a constant $\theta \in [0,1)$ such that for all $x_i \in R (i = 0,1,2,\ldots.n-1)$, the following inequality holds

$$|f(t, x_0, x_1, \ldots\ldots x_{n-1})| \le \sum_{i=0}^{n-1} a_i(t)|x_i| + b(t)|x_{n-1}|^\theta + r(t);$$

**(A2).** There is $M > 0$ such that for any $x \in dom\, L/Ker\, L$, *if* $|x^{(n-1)}(t)| > M$ *for all* $t \in [0,1]$, then

$$\int_0^1 \left(f\left(s, x(s), x'(s), \ldots\ldots, x^{(n-1)}(s)\right) + e(s)\right)ds - \beta \int_0^\eta (\eta - s)\left(f\left(s, x(s), x'(s), \ldots\ldots, x^{(n-1)}(s)\right) + e(s)\right)ds \neq 0;$$

**(A3).** There is $M^* > 0$ such that, for $x(t) = ct^{n-1}$, either

$$c\left[\int_0^1 (f(s, cs^{n-1}, c(n-1)s^{n-2}, \ldots\ldots(n-1)!\,c) + e(s))ds - \beta \int_0^\eta (\eta - s)\int_0^1 (f(s, cs^{n-1}, c(n-1)s^{n-2}, \ldots\ldots(n-1)!\,c) + e(s)ds\right] < 0$$

for all $|c| > M^*$ *or*

$$c\left[\int_0^1 (f(s, cs^{n-1}, c(n-1)s^{n-2}, \ldots\ldots(n-1)!\,c) + e(s))ds - \beta \int_0^\eta (\eta - s)\int_0^1 (f(s, cs^{n-1}, c(n-1)s^{n-2}, \ldots\ldots(n-1)!\,c) + e(s)ds\right] > 0$$

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for all $|c| > M^*$;

**(A4).** $\sum_{i=1}^{n-1} \|a_i\|_1 < 1$.

Then BVP (5) and (6) has atleast one solution.

**Theorem 2.2:** Suppose following conditions hold
**(A′1).** There are continuous functions $h(t, x_0, x_1, \dots\dots, x_{2n-1}), e(t)$ and non negative functions $g_i(t, x)(i = 0,1, \dots\dots, 2n-1)$ and positive numbers $\beta$ and $m$ such that $f$ satisfies

$$(-1)^{n-1} f(t, x_0, x_1, \dots\dots . x_{2n-1}) = e(t) + h(t, x_0, x_1, \dots\dots, x_{2n-1}) + \sum_{i=0}^{2n-1} g_i(t, x_i),$$

and also that $h$ satisfies

$$x_{2n-1} h(t, x_0, x_1, \dots\dots, x_{2n-1}) \leq -\beta |x_{2n-1}|^{m+1}$$

and for all $t \in [0, 1]$ $and$ $(x_0, x_1, \dots\dots . x_{2n-1}) \in R^{2n}$ and

$$\lim_{|x| \to \infty} \sup_{t \in [0,1]} \frac{|g_i(t, x)|}{|x|^m} = r_i, for\ i = 0,1,2, \dots\dots, 2n-1.$$

With $r_i \geq 0\ for\ i = 0,1,2, \dots\dots, 2n-1$ ;

**(A′2).** There exists constants $L \geq 0$ , $\alpha > 0\ and\ \alpha_i \geq 0 (i = 1,2, \dots .2n-2)$ such that

$$|f(t, x_0, x_1, \dots\dots . x_{2n-1})| \geq \alpha |x_0| - \sum_{i=1}^{2n-2} \alpha_i |x_i| - L$$

For all $t \in [0, 1]$ $and$ $(x_0, x_1, \dots\dots . x_{2n-1}) \in R^{2n}$.

Furthermore (A3), (A4) of Theorem 2.1 hold. Then BVP (1) and (2) has atleast one solution provided

$$\left(1 + \frac{\sum_{i=1}^{2n-2} \alpha_i}{\alpha}\right)^m r_0 + \sum_{i=1}^{2n-1} r_i < \beta.$$

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