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AN ITERATIVE ALGORITHM FOR α-NONEXPANSIVE MAPPINGS IN CAT(0) SPACES

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ABSTRACT

In this paper we obtain some results on strong and Δ-convergence in CAT (0) spaces for an iterative scheme which is both faster than and independent of the Ishikawa scheme.

Keywords: *Iterative process, CAT (0) space, α-Nonexpansive mapping, Δ-convergence, Strong convergence.*

Mathematics Subject Classification: *54E40, 47H09, 47H10.*

1. INTRODUCTION

In 1976, Lim [9] introduced the concept of Δ-convergence in general metric spaces. In 2008, Kirk and Panyanak [8] specialized this concept to CAT(0) spaces and showed that many Banach space results involving weak convergence have precise analogs in this setting. In 2008, Dhompongsa and Panyanak [7] continued to work in this direction. Their results involved the Mann and Ishikawa iteration schemes involving one mapping. In this paper we approximate fixed point of α-nonexpansive mappings by an iteration scheme which is both independent and simpler than the Ishikawa iteration scheme.

Let us recall some basics. A metric space X is called a CAT(0) space if it is geodesically connected and if every geodesic triangle in X is at least as thin as its comparison triangle in Euclidean plane. For a vigorous discussion, see Bridson and Haefliger [3] or Burago-Burago-Ivanov [5].

Let (X, d) be a metric space. A geodesic path joining x ∈ X to y ∈ X (or, more briefly, a geodesic from x to y) is a map c from a closed interval [0, l] ⊂ R to X such that c(0) = x, c(l) = y and d(c(t),c(t')) = |t− t'| for all t, t' ∈ [0, l]. In particular, c is an isometry and d(x, y) = l. The image α of c is called a geodesic (or metric) segment joining x and y. When it is unique this geodesic segment is denoted by [x, y]. The space (X, d) is said to be a geodesic space if every two points of X are joined by a geodesic and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each x, y ∈ X. A subset Y ⊆ X is said to be convex if Y includes every geodesic segment joining any two of its points. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (the vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\bar{\Delta}(x_1, x_2, x_3) = \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3})$ in the Euclidean plane $\mathbb{E}^2$ such that $d_{\mathbb{E}^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for i, j ∈ {1, 2, 3} .

A geodesic space is said to be a CAT (0) space if all geodesic triangles satisfy the following comparison axiom: Let Δ be a geodesic triangle in X and let $\bar{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT (0) inequality if for all x, y ∈ Δ and all comparison points $\bar{x}, \bar{y} \in \bar{\Delta}$, $d(x, y) \le d_{\mathbb{E}^2}(\bar{x}, \bar{y})$.

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \le \frac{1}{2} d(x, y_1)^2 + \frac{1}{2} d(x, y_2)^2 - \frac{1}{4} d(y_1, y_2)^2 \qquad \text{(CN)}$$

This is the (CN) inequality of Bruhat and Tits [4]. In fact, a geodesic space is a CAT(0) space if and only if it satisfy (CN) inequality.

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Let X be a CAT(0) space and let C be a nonempty subset of X and T: C → X be a mapping. Denote F(T) by the set of fixed points of T, i.e., F(T) = {x ∈ C : Tx = x}.

Definition 1.1: T is said to be nonexpansive if d(Tx, Ty) ≤ d(x, y) for all x, y ∈ C and that T is quasi-nonexpansive if F(T) ≠ ∅ and d(Tx, y) ≤ d(x, y) for all x ∈ C and y ∈ F(T).

In 2011, Aoyama and Kohsaka [2] defined α-nonexpansive mappings in Banach spaces and in 2013, Rathee and Ritika [10] introduce the notion of this mapping in CAT(0) spaces.

Definition 1.2: A mapping T: C → X is said to be an α-nonexpansive for some real number $\alpha<1$ if $d(Tx, Ty)^2 \leq \alpha\, d(Tx, y)^2 + \alpha\, d(Ty, x)^2 + (1 - 2\alpha)\, d(x, y)^2$ for all x, y ∈ C.

Clearly, 0-nonexpansive maps are exactly nonexpansive maps.

Lemma 1.3 [7]: Let (X, d) be a CAT(0) space. Then
(i) (X, d) is uniquely geodesic.

(ii) Let p, x, y be points of X, let $\alpha \in [0, 1]$ and let m_1 and m_2 denote, respectively, the points of [p, x] and [p, y] satisfying $d(p, m_1) = \alpha\, d(p, x)$ and $d(p, m_2) = \alpha\, d(p, y)$.Then $d(m_1, m_2) \leq \alpha\, d(x, y)$. (1.1)

(iii) Let x, y ∈ X, x ≠ y and z, w ∈ [x, y] such that d(x, z) = d(x, w). Then z = w.

(iv) Let x, y ∈ X. For each t ∈ [0, 1], there exists a unique point z∈ [x, y] such that d(x, z) = t d(x, y) and d(y, z) = (1 − t) d(x, y). (1.2)

Definition 1.4 [7]: Let $\{x_n\}$ be a bounded sequence in a CAT(0) space X. For x ∈ X,

We set $r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n)$. The asymptotic radius $r(\{x_n\})$ of $\{x_n\}$ is given by $r(\{x_n\}) = \inf\{ r(x, \{x_n\}): x \in X\}$ and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is the set $A(\{x_n\}) = \{x \in X: r(x, \{x_n\}) = r(\{x_n\})\}$.

Remark 1.5: In a CAT(0) space, $A(\{x_n\})$ consists of exactly one point ([6], Proposition 7).

Definition 1.6 [7]: A sequence $\{x_n\}$ in X is said to Δ-converge to x ∈ X if x is the unique asymptotic center of $\{u_n\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta\text{-}\lim x_n = x$ and call x the Δ-limit of $\{x_n\}$.

We denote $\omega_w(x_n) = \cup A(\{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$.

Lemma 1.7 [7]: Let X be a CAT(0) space. Then

$$d((1- t) x \oplus t y, z) \leq (1- t)\, d(x, z) + t\, d(y, z) \text{ for all } x, y, z \in X \text{ and } t \in [0, 1]. \quad (1.3)$$

Lemma 1.8 [7]: Let (X, d) be a CAT(0) space. Then

$$d((1-t)x \oplus ty, z)^2 \leq (1-t)\, d(x, z)^2 + t\, d(y, z)^2 - t(1-t)\, d(x, y)^2 \text{ for all } t \in [0, 1] \text{ and } x, y, z \in X. \quad (1.4)$$

Lemma 1.9 [7]:
(i) Every bounded sequence in a complete CAT(0) space always has a Δ-convergent subsequence.
(ii) If C is a closed convex subset of a complete CAT(0) space and if $\{x_n\}$ is a bounded sequence in C then the asymptotic center of $\{x_n\}$ is in C.
(iii) If C is a closed convex subset of a complete CAT(0) space and if T: C→ X is a nonexpansive mapping then the conditions, $\{x_n\}$ Δ-converges to x and $d(x_n, T(x_n)) \to 0$, imply x ∈ C and T(x) = x.

Lemma 1.10 [10]: Let C be a nonempty subset of a CAT(0) space X. Let T: C→ X be an α-nonexpansive mapping for some real number $\alpha < 1$ such that F(T) ≠ ∅. Then T is quasi-nonexpansive.

The Picard iterative process is defined by the sequence $\{x_n\}$:

$$x_1 = x \in C \text{ and } x_{n+1} = Tx_n, n \in N. \quad (1.5)$$

The Mann iterative process is defined by the sequence $\{x_n\}$: $x_1 = x \in C$ and

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n, n \in N, \text{ where } \{\alpha_n\} \text{ is in } (0, 1). \quad (1.6)$$

The Ishikawa iterative process is defined by the sequence $\{x_n\}$: $x_1 = x \in C$ and

$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Ty_n$,

$y_n = (1 - \beta_n)x_n + \beta_n Tx_n$, $n \in N$, where $\{\alpha_n\}$, $\{\beta_n\}$ are in (0,1). (1.7)

In 2007, Agarwal *et al.* [1] introduced the following iterative process:

$x_1 = x \in C$ and

$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n Ty_n$,

$y_n = (1 - \beta_n)x_n + \beta_n Tx_n$, $n \in N$, where $\{\alpha_n\}$, $\{\beta_n\}$ are in (0, 1). (1.8)

Note that (1.8) is independent of (1.7) and hence of (1.6). Agarwal *et al.* [1] showed that (1.8) converges at a rate same as that of Picard iteration and faster than Mann iteration for contractions and it is not hard to see on similar lines that scheme (1.8) also converges faster than the Ishikawa iteration scheme.

We now modify (1.8) in CAT(0) spaces as follows.

$x_1 = x \in C$ and

$x_{n+1} = (1 - \alpha_n)Tx_n \oplus \alpha_n Ty_n$,

$y_n = (1 - \beta_n)x_n \oplus \beta_n Tx_n$, $n \in N$, where $\{\alpha_n\}$, $\{\beta_n\}$ are in (0,1). (1.9)

Our purpose in this paper is to get some results on strong and Δ-convergence in CAT(0) spaces for (1.9). These results are independent of those proved for (1.7) and hence for (1.6).

2. MAIN RESULTS

We start with proving a key lemma for later use.

Lemma 2.1: Let C be a nonempty closed convex subset of X. Let T be a α-nonexpansive mapping of C. Let $\{\alpha_n\}$, $\{\beta_n\}$ be such that $0 < a \le \alpha_n, \beta_n \le b < 1$ for all $n \in N$ and for some a, b. Let $\{x_n\}$ be defined by the iteration process (1.9). Then $lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$ and $lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Proof: Let $p \in F(T)$. Then

$d(x_{n+1}, p) = d((1 - \alpha_n)Tx_n \oplus \alpha_n Ty_n, p)$

$\le (1 - \alpha_n)d(Tx_n, p) + \alpha_n d(Ty_n, p)$

$\le (1 - \alpha_n)d(x_n, p) + \alpha_n d(y_n, p)$

$= (1 - \alpha_n)d(x_n, p) + \alpha_n d((1 - \beta_n)x_n \oplus \beta_n Tx_n, p)$

$\le (1 - \alpha_n)d(x_n, p) + \alpha_n\{(1 - \beta_n) d(x_n, p) + \beta_n d(Tx_n, p)\}$

$\le (1 - \alpha_n)d(x_n, p) + \alpha_n\{(1 - \beta_n) d(x_n, p) + \beta_n d(x_n, p)\}$

$= (1 - \alpha_n)d(x_n, p) + \alpha_n d(x_n, p)$

$= d(x_n, p)$

This shows that $\{d(x_n, p)\}$ is decreasing and this proves the first part.

Let $lim_{n\to\infty} d(x_n, p) = c$. (2.1)

To prove $lim_{n\to\infty} d(x_n, Tx_n) = 0$, we first prove that $lim_{n\to\infty} d(y_n, p) = c$.

We know that $d(x_{n+1}, p) \le (1 - \alpha_n)d(x_n, p) + \alpha_n d(y_n, p)$.

This gives that $\alpha_n d(x_n, p) \le d(x_n, p) + \alpha_n d(y_n, p) - d(x_{n+1}, p)$

or $d(x_n, p) \leq d(y_n, p) + \frac{1}{\alpha_n}\{ d(x_n, p) - d(x_{n+1}, p) \}$

$\leq d(y_n, p) + \frac{1}{a}\{ d(x_n, p) - d(x_{n+1}, p) \}$

This gives $lim_{n\to\infty} \inf d(x_n, p) \leq lim_{n\to\infty} \inf d(y_n, p) + lim_{n\to\infty} \frac{1}{a}\{ d(x_n, p) - d(x_{n+1}, p)\}$

So that $c \leq lim_{n\to\infty} \inf d(y_n, p)$. (2.2)

But from $d(y_n, p) \leq d(x_n, p)$ and (2.1) , we get

$lim_{n\to\infty} \sup d(y_n, p) \leq c$.

Reading it together with (2.2), we get $lim_{n\to\infty} d(y_n, p) = c$. (2.3)

Now $d(y_n, p)^2 = d((1-\beta_n)x_n \oplus \beta_n Tx_n, p)^2$

$\leq (1-\beta_n)d(x_n, p)^2 + \beta_n d(Tx_n, p)^2 - \beta_n(1-\beta_n) d(x_n, Tx_n)^2$

$\leq d(x_n, p)^2 - \beta_n(1-\beta_n) d(x_n, Tx_n)^2$

Thus $\beta_n(1-\beta_n) d(x_n, Tx_n)^2 \leq d(x_n, p)^2 - d(y_n, p)^2$

So that $d(x_n, Tx_n)^2 \leq \frac{1}{\beta_n(1-\beta_n)} [d(x_n, p)^2 - d(y_n, p)^2]$

$\leq \frac{1}{a(1-b)} [d(x_n, p)^2 - d(y_n, p)^2]$

Now using (2.1), (2.3), $\lim \sup d(x_n, Tx_n) \leq 0$ and hence $lim_{n\to\infty} d(x_n, Tx_n) = 0$.

Theorem 2.2: Let X, C, T, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{x_n\}$ be as in Lemma 2.1, then $\{x_n\}$ Δ-converges to a fixed point of T.

Proof: By lemma 2.1, we have $lim_{n\to\infty} d(x_n, Tx_n) = 0$. Also $lim_{n\to\infty} d(x_n, p)$ exists for all $p \in F(T)$. Thus $\{x_n\}$ is bounded. We now let $\omega_w(x_n) = \cup A(\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. We claim that $\omega_w(x_n) \subset F(T)$. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(\{u_n\}) = \{u\}$. By Lemma 1.9 (i) and (ii) there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ- $\lim_n v_n = v \in C$.

Since $\lim_n d(v_n, T v_n) = 0$, then $v \in F(T)$ by Lemma 1.9 (iii). We claim that u = v. Suppose not, by the uniqueness of asymptotic centers,

$$\begin{aligned} \lim_n \sup d(v_n,v) &< \lim_n \sup d(v_n,u) \\ &\leq \lim_n \sup d(u_n,u) \\ &< \lim_n \sup d(u_n,v) \\ &= \lim_n \sup d(x_n,v) \\ &= \lim_n \sup d(v_n,v), \end{aligned}$$

which is a contradiction, and hence $u = v \in F(T)$. To show that $\{x_n\}$ Δ-converges to a fixed point of T, it suffices to show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$. By Lemma 1.9 (i) and (ii), there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that Δ-$\lim_n v_n = v \in C$. Let $A(\{u_n\}) = \{u\}$ and $A(\{x_n\}) = \{x\}$. We have seen that u = v and $v \in F(T)$. We can complete the proof by showing that x = v. If not, since $\{d(x_n, v)\}$ is convergent, then by the uniqueness of asymptotic centers,

$$\begin{aligned} \lim_n \sup d(v_n,v) &< \lim_n \sup d(v_n, x) \\ &\leq \lim_n \sup d(x_n, x) \\ &< \lim_n \sup d(x_n, v) \\ &= \lim_n \sup d(v_n, v), \end{aligned}$$

which is a contradiction and hence the conclusion follows.

Theorem 2.3: Let X be a complete CAT(0) space and C, T, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{x_n\}$ be as in Lemma 2.1, then $\{x_n\}$ converges strongly to a fixed point of T if and only if $lim_{n\to\infty}$inf d(x_n, F(T)) = 0, where d(x, F(T)) = inf{d(x, p) : p ∈ F(T) }.

Proof: Necessity is obvious. Conversely, suppose that $lim_{n\to\infty}$inf d(x_n, F(T)) = 0. As proved in Lemma 2.1, we have d(x_{n+1}, p) ≤ d(x_n, p) for all p ∈ F(T). This implies that d(x_{n+1}, F(T)) ≤ d(x_n, F(T)) so that $lim_{n\to\infty}$d(x_n, F(T)) exists. Thus by hypothesis $lim_{n\to\infty}$d(x_n, F(T)) = 0.

Next we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\varepsilon > 0$ be arbitrarily chosen. Since $\lim_{n\to\infty}$ d(x_n, F(T)) = 0, there exists a positive integer n_0 such that d(x_n, F(T)) $< \varepsilon/4$ for all n ≥ n_0.

In particular, inf{ d(x_{n_0}, p) : p ∈ F(T)} $< \varepsilon/4$. Thus there must exist p^* ∈ F(T) such that d(x_{n_0}, p^*) $< \varepsilon/2$. Now for all m, n ≥ n_0, we have

$$\begin{aligned} d(x_{n+m}, x_n) &\leq d(x_{n+m}, p^*) + d(p^*, x_n) \\ &\leq 2\, d(x_{n_0}, p^*) \\ &< 2(\varepsilon/2) = \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a complete CAT(0) space and so it must converge to a point q in C and $\lim_{n\to\infty}$ d(x_n, F(T)) = 0 gives that d(q, F(T)) = 0 and closedness of F(T) forces q to be in F(T).

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