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SOME EXACT SOLUTIONS OF SECOND GRADE ALIGNED MAGNETO HYDRODYNAMIC FLOW IN POROUS MEDIA

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ABSTRACT

The aim of the present paper in to find some exact solutions of unsteady two dimensional electrically conducting incompressible second grade, MHD aligned fluid flow which undergoes isochoric motion. Governing equations are first reformulated in terms of magnetic flux function ϕ. Study and unsteady solutions have been obtained via inverse method, when current density distribution is proportional to the magnetic flux function ϕ, perturbed by a quadratic term.

Keywords and Phrases: *Unsteady flow, Second grade fluid, Exact solution MHD.*

1. INTRODUCTION:

It is well known that Newtonian fluid flows are governed by Navier-Stokes equations. But flowing behavior of many real fluids like blood, dyes, polymer melts, synovial fluids, paints, etc., departs from that of a Newtonian fluid, so that the rate of shear is not proportional to the corresponding stress and hence these are classified as non-Newtonian fluid.

The flow of non-Newtonian fluids occur in a wide range of practical applications and have gained a lot of importance in recent years because of its numerous technological applications including plastic manufacture, performance of lubricants, application of paints, processing of food and movement of biological fluids. Most biologically important fluids contain higher molecular weight components and are, therefore, non-Newtonian. The unusual properties of polymer melts and solutions, together with the desirable attributes of many polymeric solids, have given rise to the world wide industry of polymer processing. Geophysical applications concerning ice and magna flows are based on non-Newtonian constitutive behaviours.

However it is not possible to assign a single model to Non-Newtonian fluids as the are themselves varied in nature. For this reason, many models or constitutive equations have been proposed and one of the simplest type of model to account for the rheological effect of non-Newtonian fluids is the second grade model. The governing equation of non-Newtonian fluids are highly non-linear and one order higher than Navier-Stokes equations and hence we face more difficulty in solving them exactly.

Navier-Stokes equations are inherently non-linear partial differential equation has non general solution, and only a small number of exact solutions have been found because the nonlinear inertial terms do not disappear automatically. Exact solutions are very important no only because they are solutions of some fundamental flows but also they serve as accuracy checks for experimental, numerical and asymptotic methods. So in order to perform this task one adopt transformations, inverse or semi-inverse method for the reformulation of equations in solvable form. Following the Martin's formulation [1], some researchers [3,4] have used hodograph transformation [2] in order to linearized the system of governing equations and successfully got some exact solutions. Some authors [21, 13] have used inverse method [18] where some a priory condition is assumed about the flow variables and have found some exact solutions.

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Taking the vorticity to be proportional to the stream function, Taylor [17], obtained the solution of the problem of a double infinite array of vortices decaying exponentially with time. After Taylor this method has been extensively used by many researchers for the first grade fluid such as Kovaznay [6], Wang [18, 12], Lin and Tobak [14], Hui [15], Jeffrey [9], Riabouchinsky [11], Chandna [21] and others. They assumed that the vorticity is proprtional to the stream function perturbed by a uniform stream and derived several classes of exact solutions. In case of second grade fluid Rajgopal [16] and Siddiqui [19], following Nemenyi [18], applied this method to find some exact solutions. Benharbit and Siddiqui [20] used this method to study the steady and unsteady second grade fluid flow by taking vorticity function of the form $\nabla^2\psi = K(\psi - Uy)$. Further this work was extended by Chandna and Ukpong [22] in the study of aligned second grade fluid flow. Recently Labropulu [25] studied steady and unsteady hydrodynamic flow by taking different vorticity distribution. Further Labropulu [23, 26] obtained more exact solutions of second grade fluid flows assuming different forms of vorticity and stream function. C thakur and B. Singh [24] by formulating the governing equations of second grade aligned flow in terms of magnetic flux function and then found some exact solutions. Furthermore this method has been recently applied by Asghar [24] et. al in the study of unsteady Riabouchinsky flows of second grade fluid. In the study of second grade aligned fluid flow by Hayat [28] et.al.

In the present paper we have studied second grade electrically conducting fluid flow under the presence of magnetic field in porous space, with assumption that velocity and magnetic vector field are parallel to each other. To find the exact solutions we have gone through the alternate formulation of the governing equations in terms of magnetic flux function rather than introducing stream function and then we have considered current density proportional to the magnetic flux function perturbed by a quadratic stream $B(C_x+D_y+E_y^2)$. At last we have found some exact solutions for finitely and infinitely electrically conducting fluid under certain possible cases.

2. EQUATIONS OF MOTION:

The governing equation of unsteady plane flow of an incompressible electrically conducting second grade fluid, under the presence of magnetic field are given as

$$\nabla.\vec{V} = 0, \tag{1}$$

$$\rho\left[\vec{V}_t + (\vec{V}.\nabla)\vec{V}\right] = DiuT + \mu^*\vec{J}\times\vec{H} - \frac{\phi^*}{K}\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\vec{V}, \tag{2}$$

$$\frac{\partial H}{\partial t} = Curl(\vec{V}\times\vec{H}) - \frac{1}{\mu^*\sigma}Curl\,Curl\,\vec{H}), \tag{3}$$

$$\nabla.\vec{H} = 0, \tag{4}$$

The constitutive equation for stress is

$$T = -pI + \mu A_1 + \alpha_1 A_2 + \alpha_2 A_1^2, \tag{5}$$

where

$$A_1 = \left(grad\,\overline{V}\right) + \left(grad\,\overline{V}\right)^T, \tag{6}$$

$$A_2 = A_{1t} + A \quad \left(grad \quad \overline{V}\right) + \left(grad \quad \overline{V}\right)^T$$

and

$$\mu \geq 0,\ \alpha_1 \geq 0, a_1 + a_2 = 0.$$

In above system, $\vec{V}$ is the velocity vector, $\vec{H}$ the magnetic field intensity, T the stress tensor, p the fluid pressure, ρ the fluid density, μ^* the magnetic permeability, σ the electrical conductivity, μ the constant viscosity, α_1 and α_2 the constant normal stress moduli, A_1 and A_2 the Rivlin-Ericksen tensors.

Now since we have considered the two dimensional MHD flow, so we must have

$$\vec{V} = (u, v, 0), \vec{H} = (H_1, H_2, 0),\ \vec{J} = Curl\ \vec{H} = (0, 0, H_{2x} - H_{1y}) \tag{9}$$

Now in view of equation (9) we have the two dimensional form of governing equations as

$$u_x + u_y = 0 \tag{10}$$

$$\rho[u_t + uu_x + uu_y] + p_x = \mu\nabla^2 u + \alpha_1\Big[\nabla^2 u_t + \big\{2uu_{xx} + 2vu_{xy} + 4u_x^2 + 2u_x(v_x + u_y)\big\}_x + \left\{\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(v_x + u_y) + 2u_x u_y + 2v_x v_y\right\}_y\Big] \tag{11}$$

$$\rho[v_t + uv_x + vv_y] + p_y = \mu\nabla^2 v + \alpha_1\Big[\nabla^2 u_t + \big\{2vv_{yy}, +2uv_{xy} + 4v_y^2 + 2u_x(v_x + u_y)\big\}_y + \left\{\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(v_x + u_y) + 2u_x u_y + 2v_x v_y\right\}_x\Big] \tag{12}$$

$$+\alpha_2[4v_y^2 + (v_x + u_y)^2]_y + \mu^* H_1(H_{2x} - H_{1y}) - \frac{\phi^*}{k}\left(\mu + a_1\frac{\partial}{\partial t}\right)v,$$

$$(H_{2x} - H_{1y})_t = \frac{1}{\mu^*\sigma}\nabla^2(H_{2x} - H_{1y}) + \nabla^2(vH_1 - uH_2) \tag{13}$$

and

$$H_{1x} + H_{2y} = 0 \tag{14}$$

Now introducing the vorticity, current density and generalized energy function as

$$\omega = (v_x - u_y), \tag{15}$$

$$\Omega = (H_{2x} - H_{1y}), \tag{16}$$

and

$$h = p + \frac{1}{2}\rho(u^2 + v^2) - a_1(u\nabla^2 u + v\nabla^2 v) - \frac{(3\alpha_1 + 2\alpha_2)}{2}[2u_x^2 + 2v_y^2 + (v_x + u_y)^2] \tag{17}$$

Now using equation (15)-(17), the above equations can be rewritten as

$$v_x + u_y = 0, \tag{18}$$

$$h_x = -\mu\omega_y + \rho(v\omega - u_t) - \mu^*\Omega H_2 - \alpha_1(\omega_{yt} + v\nabla^2\omega) - \frac{\phi^*}{k}\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)u, \tag{19}$$

$$h_y = -\mu\omega_x + \rho(u\omega - u_t) - \mu^*\Omega H_1 - \alpha_1(\omega_{xt} + u\nabla^2\omega) - \frac{\phi^*}{k}\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)v, \tag{20}$$

$$\Omega_t = \frac{1}{\mu^*\sigma}\nabla^2\Omega + \nabla^2\left(vH_1 - uH_2\right), \tag{21}$$

$$H_{1x} + H_{2y} = 0, \tag{22}$$

$$H_{2x} - H_{1y} = \Omega\,. \tag{23}$$

Thesolenoidal condition of equation (22) leads to the existence of magnetic flux function $\phi(x,y,t)$ such that

$$H_1 = \phi_y,\ \ H_2 = -\ \phi_x. \tag{24}$$

We, now study the aligned flow i.e

$$u = fH_1, \quad v = fH_2, \tag{25}$$

where $f = f(x,y,t) \neq 0,$ is an arbitrary scalar function and equations (24), (25) implies

$$u = f\phi_y, \qquad v = -f\phi_x, \tag{26}$$

$$\Omega = -\nabla^2\phi, \tag{27}$$

and

$$\omega = -\left\{f\,\nabla^2\phi + f_x\phi_x + f_y\phi_y\right\}. \tag{28}$$

Again using these equations we have

$$\begin{aligned} h_x &= \mu\left\{f\nabla^2\phi + f_x\phi_x + f_y\phi_y\right\}_y + \rho\left\{\left(f\phi_x\right)\left(f\nabla^2\phi + f_x\phi_x + f_y\phi_y\right) - \left(f\phi_y\right)_t\right\} - \mu^*\phi_x\nabla^2\phi \\ &- \alpha_1\left[-\left\{f\nabla^2\phi + f_x\phi_x + f_y\phi_y\right\}_{yt} + \left(f\phi_x\right)\nabla^2\left\{f\nabla^2\phi + f_x\phi_x + f_y\phi_y\right\}\right] \\ &- \frac{\phi^*}{k}\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(f\phi_y\right), \end{aligned} \tag{29}$$

$$\begin{aligned} h_y &= \mu\left\{f\,\nabla^2\phi + f_x\phi_x + f_x\phi_y\right\}_y + \rho\left\{\left(f\phi_y\right)\left(f\,\nabla^2\phi + f_x\phi_x + f_y\phi_y\right) + \left(f\phi_x\right)_t\right\} - \mu^*\phi_y\nabla^2\phi \\ &- \alpha_1\left[-\left\{f\,\nabla^2\phi + f_x\phi_x + f_y\phi_y\right\}_{xt} - \left(f\phi_y\right)\nabla^2\left\{f\,\nabla^2\phi + f_x\phi_x + f_y\phi_y\right\}\right] \\ &- \frac{\phi^*}{k}\left(\mu + \alpha_1\frac{\partial}{\partial t}\right)\left(f\phi_x\right). \end{aligned} \tag{30}$$

Now using the integrability criteria $h_{xy} = h_{yx},$ we must have

$$\begin{aligned} &\mu\nabla^2\left\{f\,\nabla\phi + f_x\phi_x + f_x\phi_y\right\} - \mu^*\frac{\partial\left(\phi, \nabla^2\phi\right)}{\partial(x,y)} \\ &+ \rho\left[\left\{\left(f\phi_x\right)\left(f\,\nabla^2\phi + f_x\phi_x + f_y\phi_y\right) - \left(f\phi_y\right)_t\right\}_y - \left\{\left(f\phi_y\right)\left(f\,\nabla^2\phi + f_x\phi_x + f_y\phi_y\right) - \left(f\phi_x\right)_t\right\}_x\right] \end{aligned}$$

$$-\alpha_1\left[-\left(f\nabla^2\phi+f_x\phi_x+f_y\phi_y\right)_{yt}+\left(f\phi_x\right)\nabla^2\left(f\nabla^2\phi+f_x\phi_x+f_y\phi_y\right)\right]_y$$
$$-\alpha_1\left[-\left(f\nabla^2\phi+f_x\phi_x+f_y\phi_y\right)_{xt}+\left(f\phi_y\right)\nabla^2\left(f\nabla^2\phi+f_x\phi_x+f_y\phi_y\right)\right]_x$$
$$-\frac{\phi^*}{k}\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(f\nabla^2\phi+f_x\phi_x+f_y\phi_y\right)=0, \tag{31}$$

$$\frac{\partial(f,\phi)}{\partial(x,y)}=0 \tag{32}$$

and the diffusion equation (21) gives

$$\left(\nabla^2\phi\right)_t=\frac{1}{\mu\sigma}\nabla^4\phi, \tag{33}$$

equation (32) implies

$$f=f(\phi) \tag{34}$$

3. SOLUTION:

The above system (31) to (34) is coupled system of non-linear PDEs in two unknowns f,ϕ depending on three independent variables x,y and t. It is quite complex to solve. So in order to solve system exactly we must assume $f=f_0$ a constant, so we have

$$\mu f_0\left(\nabla^4\phi\right)+\rho\left[f_0^2\frac{\partial\left(\phi,\nabla^2\phi\right)}{(x,y)}-f_0\left(\nabla^2\phi\right)_t\right]-\mu^*\frac{\partial\left(\phi,\nabla^2\phi\right)}{(x,y)}$$
$$+\alpha_1\left[f_0\left(\nabla^4\phi\right)_t-f_0^2\frac{\partial\left(\phi,\nabla^4\phi\right)}{(x,y)}\right]-\frac{\phi^*}{K}\left(\mu+\alpha_1\frac{\partial}{\partial t}\right)\left(f_0\nabla^2\phi\right)=0. \tag{35}$$

Now we find the solution of above equation for finitely and infinitely conducting case

3.1 SOLUTION FOR INFINITELY CONDUCTING FLUID:

In this case we consider the fluid of infinite conductivity i.e., $\alpha\to\infty$, under such condition equation (33) implies

$$\left(\nabla^2\phi\right)_t=0 \tag{36}$$

and hence $\phi_t=0$. Equation (32) is identically satisfied and equation (35) becomes

$$\mu f_0\nabla^4\phi+\rho f_0^2\frac{\partial\left(\phi,\nabla^2\phi\right)}{(x,y)}-\mu^*\frac{\partial\left(\phi,\nabla^2\phi\right)}{(x,y)}-\alpha_1 f_0^2\frac{\partial\left(\phi,\nabla^4\phi\right)}{(x,y)}-\frac{\phi^*}{K}\mu f_0\nabla^2\phi=0. \tag{37}$$

Now we assume the current density proportional to magnetic flux function ϕ perturbed by the quadratic stream and is given by

$$\nabla^2\phi=A\left[\phi-B\left(Cx+Dy+Ey^2\right)\right]. \tag{38}$$

Again introducing the following transformations

$$\begin{aligned} &\Phi = \phi - B\left(Cx + Dy + Ey^2\right) \\ &\xi = \left(Cx + Dy + ey^2\right) \\ &\eta = y. \end{aligned} \tag{39}$$

Now using (38), (39) in (37) and (39) in (38) we have

$$\mu f_0\left(A - \frac{\phi^*}{K}\right)\Phi + B\left(\rho f_0^2 - \mu^* - \alpha_1 A f_0^2\right)C\Phi_\eta = 2\mu f_0 BE \tag{40}$$

and

$$\left\{C^2 + (D + 2E\eta)^2\right\}\Phi_{\xi\xi} + 2(D + 2E\eta)\Phi_{\xi\eta} + 2E\Phi_\xi + \Phi_{\eta\eta} = A\Phi - 2BE. \tag{41}$$

One of the solution of (40) , when $\left(\rho f_0^2 - \mu^* - \alpha_1 A f_0^2\right) = 0,$ is

$$\Phi = \frac{2BE}{(A - \frac{\phi^*}{K})} \tag{42}$$

which is consistent with equation (41) if $E = 0$. So in this case we must have the trivial solution $\Phi = 0$ and hence using equation (39) we have the magnetic flux function

$$\phi = B(Cx + Dy). \tag{43}$$

Equation (43), when combined with equation (26) shows that this solution corresponds to the uniform flow inclined to the axes.

Now solving equation (40) we get

$$\Phi = \frac{2BE}{(A - \frac{\phi^*}{K})} + g(\xi)e^{-\delta\eta}, \tag{44}$$

where

$$\delta = \frac{\mu f_0 (A - \frac{\phi^*}{K})}{BC\,(\rho f_0^2 - \mu^* - \alpha_1 A\, f_0^2)}.$$

Now using equation (44) in (41) we get

$$e^{-\delta\eta}\left[\left\{c^2 + (D\_2E\eta)^2\right\}g''(\xi) + \left\{-2\delta(D + 2E\eta) + 2E\right\}g'(\xi) + (\delta^2 - A)g(\xi)\right] = \frac{2BE\phi^*}{K(A - \frac{\phi^*}{K})} \tag{45}$$

Since ξ, η are independent variables, we must have E = o and then we get the equation in $g(\xi)$ as

$$(C^2 + D^2)g''(\xi) - 2\delta D g'(\xi) + (\delta^2 + A)\,g(\xi) = 0\,. \tag{46}$$

Now solving aboveequation for $g(\xi)$ and then combining with equations (39), (46) and taking $E=0$, we get the solution for ϕ as under

$$\phi(x,y)=\begin{cases} B(Cx+Dy)+A_1e^{(m_1(Cx+Dy)-\delta_y)}+A_2e^{(m_2(Cx+Dy)-\delta_y)}; & for\, M>0, \\ B(Cx+Dy)+(B_1+B_2(Cx+Dy))\,e^{(m_3(Cx+Dy)-\delta_y)}; & for\, M=0, \\ B(Cx+Dy)+C_1e^{(\alpha(Cx+Dy)-\delta_y)}\cos(\beta(Cx+Dy)+C_2); & for\, M=0, \end{cases} \tag{47}$$

Where

$$\begin{aligned} &M=\delta^2D^2-(C^2+D^2)(\delta^2-A^2), \\ &m_1=\frac{\delta D+\sqrt{M}}{(C^2+D^2)},\; m_2=\frac{\delta D-\sqrt{M}}{(C^2+D^2)},\, m_2=\frac{\delta D}{(C^2+D^2)} \\ &\alpha=\frac{\delta D}{(C^2+D^2)},\; \beta=\frac{\sqrt{M}}{(C^2+D^2)}. \end{aligned} \tag{48}$$

and A_1, A_2, B_1, B_2, C_1 and C_2 are arbitrary constants.

In the above equation if we take $D=0$, $C=1$ and $\phi^* \to 0$ i.e. in the absence of porous media, then the equation (47) reduces to

$$\phi(x,y)=\begin{cases} Bx+A_1e^{(\xi x-\delta y)}+A_2e^{-(\xi x-\delta y)}; & for\, \delta^2-A^2=-\xi^2<0, \\ BC+(B_1+B_2x)\,e^{-\delta y}; & for\, \delta^2-A^2=0, \\ Bx+C_1e^{-\delta y}\cos(\xi x-C_2)\,; & for\, \delta^2-A^2=\xi^2>0, \end{cases} \tag{49}$$

Which is identical with the result of C thakur and B. Singh (2002).

Now we have the magnetic field, velocity field, vorticity distribution and current density as

$$\begin{aligned} H_1&=\left[BD+A_1(m_1D+\delta)\,e^{(m_1Cx+(m_1D-\delta)y)}+A_2(m_2D-\delta)e^{(m_2Cx+(m_2D-\delta)y)}\right], \\ H_2&=-\left[BD+A_1m_1\,Ce^{(m_1Cx+(m_1D-\delta)y)}+A_2m_2Ce^{(m_2Cx+(m_2D-\delta)y)}\right] \\ u&=f_0\left[BD+A_1(m_1\,D-\delta)e^{(m_1Cx+(m_1D-\delta)y)}+A_2(m_2D-\delta)e^{(m_2Cx+(m_2D-\delta)y)}\right] \\ v&=-f_0\left[BC+A_1m_1Ce^{(m_1Cx+(m_1D-\delta)y)}+A_2m_2Ce^{(m_2Cx+(m_2D-\delta)y)}\right] \end{aligned} \tag{50}$$

$$\omega=-\left[A_1\left\{m_1^2\,C^2+(m_1D-\delta)^2\right\}e^{(m_1Cx+(m_1D-\delta)y)}+A_2\left\{m_2^2\,C^2+(m_2D-\delta)^2\right\}e^{(m_2Cx+(m_2D-\delta)y)}\right]$$

$$\Omega=f_0\left[A_1\left\{m_1^2\,C^2+(m_1D-\delta)^2\right\}e^{(m_1Cx+(m_1D-\delta)y)}+A_2\left\{m_2^2\,C^2+(m_2D-\delta)^2\right\}e^{(m_2Cx+(m_2D-\delta)y)}\right].$$

Now, proceeding in the similar way we can have for solutions in other two cases as

$$H_1=\left[BD+B_2De^{(m_3Cx+(m_3D-\delta)y)}+\{B_1+B_2(Cx+Dy)\}(m_3D-\delta)e^{(m_3Cx+(m_3D-\delta)y)}\right]$$

$$H_2=-\left[BC+B_2Ce^{(m_3Cx+(m_3D-\delta)y)}+\{B_1+B_2(Cx+Dy)\}m_3Ce^{(m_3Cx+(m_3D-\delta)y)}\right]$$

$$u=f_0\left[BD+B_2De^{(m_3Cx+(m_3D-\delta)y)}+\{B_1+B_2(Cx+Dy)\}(m_3D-\delta)e^{(m_3Cx+(m_3D-\delta)y)}\right]$$

$$v=-f_0\left[BC+B_2Ce^{(m_3Cx+(m_3D-\delta)y)}+\{B_1+B_2(Cx+Dy)\}m_3Ce^{(m_3Cx+(m_3D-\delta)y)}\right] \tag{51}$$

$$\Omega = -\left[2B_2\left\{m_3C^2 + D(m_3D-\delta\right\} + \left\{B_1 + B_2(Cx+Dy)\right\}(m_3D-\delta)^2 e^{(m_3 Cx+(m_3D-\delta)y)}\right]$$

$$\omega = -f_0\left[2B_2\left\{m_3C^2 + D(m_3D-\delta\right\} + \left\{B_1 + B_2(Cx+Dy)\right\}(m_3D-\delta)^2 e^{(m_3 Cx+(m_3D-\delta)y)}\right].$$

$$H_1 = \left[BD + C_1(\alpha D-\delta)e^{(\alpha Cx+(\alpha D-\delta)y)}\cos(\beta(Cx+Dy)+C_2) - C_1\beta De^{(\alpha Cx+(\alpha D-\delta)y)}\sin(\beta(Cx+Dy)+C_2)\right]$$

$$H_2 = -\left[BC + C_1\alpha Ce^{(\alpha Cx+(\alpha D-\delta)y)}\cos(\beta(Cx+Dy)+C_2) - C_1e^{(\alpha Cx+(\alpha D-\delta)y)}\sin(\beta(Cx+Dy)+C_2)\right]$$

$$u = f_0\left[BD + C_1(\alpha D-\delta)e^{(\alpha Cx+(\alpha D-\delta)y)}\cos(\beta(Cx+Dy)+C_2) - C_1\beta De^{(\alpha Cx+(\alpha D-\delta)y)}\sin(\beta(Cx+Dy)+C_2)\right]$$

$$v = -f_0\left[BC + C_1\alpha Ce^{(\alpha Cx+(\alpha D-\delta)y)}\cos(\beta(Cx+Dy)+C_2) - C_1e^{(\alpha Cx+(\alpha D-\delta)y)}\sin(\beta(Cx+Dy)+C_2)\right] \tag{52}$$

$$\Omega = -\left[C_1\left\{\alpha^2C^2 + (\alpha D-\delta)^2 - \beta^2(C^2+D^2)\right\}e^{(\alpha Cx+(\alpha D-\delta)y)}\cos(\beta(Cx+Dy)+C_2)\right.$$
$$\left. - 2C_1\alpha\beta(C^2+D^2)e^{(\alpha Cx+(\alpha D-\delta)y)}\sin(\beta(Cx+Dy)+C_2)\right]$$

$$\omega = -f_0\left[C_1\left\{\alpha^2C^2 + (\alpha D-\delta)^2 - \beta^2(C^2+D^2)\right\}e^{(\alpha Cx+(\alpha D-\delta)y)}\cos(\beta(Cx+Dy)+C_2)\right.$$
$$\left. - 2C_1\alpha\beta(C^2+D^2)e^{(\alpha Cx+(\alpha D-\delta)y)}\sin(\beta(Cx+Dy)+C_2)\right]$$

3.2 SOLUTION FOR FINITELY CONDUCTING FLUID:

In this case we consider the fluid of finite conductivity. Taking $f = f_0$ a constant, equation (32) is identically satisfied and using equation (39), equations (31) and (33) becomes

$$-\left[\rho f_0 - \alpha_1 f_0 A + \frac{\phi^*}{K}\alpha_1 f_0\right]\Phi_t + \mu f_0(A - \frac{\phi^*}{K})\Phi + B(\rho f_0^2 - \mu^* - \alpha_1 A f_0^2)\left[-(2Ey+D)\Phi_x + C\Phi_x\right] = 2\mu f_0 BE, \tag{53}$$

$$\Phi_t = \frac{A}{\mu^*\sigma}\Phi - \frac{-BE}{\mu^*\sigma} \tag{54}$$

We now find out the solution in steady and unsteady cases separately

Case (I) : Steady flow

For steady flow we must have, $\Phi_t = 0.$ Now for $(\rho f_0^2 - \mu^* - \alpha_1 A f_0^2 = 0$ equation (53) gives

$$\Phi = -\frac{2BE}{\left(A - \frac{\phi^*}{K}\right)}, \tag{55}$$

which is consistent with equation (54) if and only if $E = 0$. Thus we have only trivial solution for Φ as $\Phi = 0$, which on combining with equation (35) gives the magnetic flux function as

$$\phi = Cx + Dy \tag{56}$$

and this when combining with equation (26), leads to the uniform flow inclined to the axes.

Case (II): Unsteady flow

for unsteady flow equation (54) implies

$$\Phi = \frac{2BE}{A} + F(x,y)e^{\frac{A}{\mu^*\sigma}t}, \tag{57}$$

which on putting in equation (53) gives

$$-\frac{f_0}{\mu^*\sigma}\left[A\left\{\rho-\alpha_1 A+\frac{\mu^*}{K}\alpha_1\right\}+\mu\mu^*\sigma\left(a-\frac{\phi^*}{K}\right)\right]F$$
$$+B\left\{\rho f_0^2-\mu^*-\alpha_1 A f_0^2\right\}\left[-(2Ey+D)F_x+CF_y\right]=2\mu f_0 BE\left(\frac{\phi^*}{KA}\right)e^{\frac{A}{\mu^*\sigma}t}. \tag{58}$$

In the above equation RHS is a function of *t* alone so we must have *E = 0* . Now we have the equation (57) and (58) as

$$\Phi = F(x,y)e^{\frac{A}{\mu^*\sigma}t} \tag{59}$$

and

$$-\frac{f_0}{\mu^*\sigma}\left[A\left\{\rho-\alpha_1 A+\frac{\phi^*}{K}\alpha_1\right\}+\mu\mu^*\sigma\left(A-\frac{\phi^*}{K}\right)\right]F$$
$$+B\left\{\rho f_0^2-\mu^*-\alpha_1 A f_0^2\right\}\left[-DF_x+CF_y\right]=0. \tag{60}$$

We now introduced the following transformation

$$\xi = Cx+Dy,\ \eta = y. \tag{61}$$

In the view of above transformation equations using (59) in (38), (58) becomes

$$(C^2+D^2)F_{\xi\xi}++2DF_{\xi\eta}+F_{\eta\eta}=AF, \tag{62}$$

$$-\frac{f_0}{\mu^*\sigma}\left[a\left\{\rho-\alpha_1 A+\frac{\phi^*}{K}\alpha_1\right\}+\mu\mu^*\sigma\left(A-\frac{\phi^*}{K}\right)\right]F+B\left\{\rho f_0^2-\mu^*-\alpha_1 A f_0^2\right\}CF_\eta=0. \tag{63}$$

Now solving the above equation (63) we get

$$F = h(\xi)e^{\lambda\eta}, \tag{64}$$

where

$$\lambda = \frac{f_0\left[A\left\{\rho-\alpha_1 A+\frac{\phi^*}{K}\alpha_1\right\}+\mu\mu^*\sigma\left(A-\frac{\phi^*}{K}\right)\right]}{CB\left\{\rho f_0^2-\mu^*-\alpha_1 A f_0^2\right\}} \tag{65}$$

and h (ξ) is an unknown function to be determined. Now using equation (62) and (64) we have

$$(C^2+D^2)h''(\xi)+2\lambda Dh'(\xi)+(\lambda^2-A)h(\xi)=0. \tag{66}$$

Now solving (66) and using equation (59), (61) and (39) we get the magnetic flux function as under

$$\phi(x,y)=\begin{cases} B(Cx+Dy)+A_1e^{(m_1(Cx+Dy)+\lambda y+\frac{A}{\mu^*\sigma}t)}+A_2e^{(m_2(Cx+Dy)+\lambda y+\frac{A}{\mu^*\sigma}t)}; & for\, N>0, \\ B(Cx+Dy)+(B_1+B_2(Cx+Dy))e^{(m_3(Cx+Dy)+\lambda y+\frac{A}{\mu^*\sigma}t)}; & for\, N=0, \\ B(Cx+Dy)+C_1e^{(\alpha(Cx+Dy)+\lambda y+\frac{A}{\mu^*\sigma}t)}\cos(\beta(Cx+Dy)+C_2); & for\, N=0, \end{cases} \tag{67}$$

where

$$\begin{aligned} &N=\lambda^2D^2-(C^2+D^2)(\lambda^2-A^2) \\ &m_1=\frac{-\lambda D+\sqrt{N}}{(C^2+D^2)},\ m_2=\frac{-\lambda D-\sqrt{N}}{(C^2+D^2)},\quad \frac{-\lambda D}{(C^2+D^2)} \\ &\alpha=\frac{-\lambda D}{(C^2+D^2)},\quad \beta=\frac{\sqrt{N}}{(C^2+D^2)}. \end{aligned} \tag{68}$$

Now we have

$$H_1=\left[BD+A_1(m_1D+\delta)e^{(m_1Cx+(m_1D-\delta)y+\frac{A}{\mu^*\sigma}t)}+A_2(m_2D-\delta)e^{(m_2Cx+(m_2D-\delta)y+\frac{A}{\mu^*\sigma}t)}\right],$$

$$H_2=-\left[BC+A_1m_1Ce^{(m_1Cx+(m_1D-\delta)y+\frac{A}{\mu^*\sigma}t)}+A_2m_2Ce^{(m_2Cx+(m_2D-\delta)y)}\right]$$

$$u=f_0\left[BD+A_1(m_1D-\delta)e^{(m_1Cx+(m_1D+\delta)y+\frac{A}{\mu^*\sigma}t)}+A_2(m_2D-\delta)e^{(m_2Cx+(m_2D+\delta)y+\frac{A}{\mu^*\sigma}t)}\right]$$

$$v=-f_0\left[BC+A_1m_1Ce^{(m_1Cx+(m_1D+\delta)y+\frac{A}{\mu^*\sigma}t)}+A_2m_2Ce^{(m_2Cx+(m_2D+\delta)y\frac{A}{\mu^*\sigma}t)}\right] \tag{69}$$

$$\omega=-\left[\begin{aligned} &A_1\left\{m_1^2C^2+(m_1D+\delta)^2\right\}e^{(m_1Cx+(m_1D+\delta)y+\frac{A}{\mu^*\sigma}t)} \\ &+A_2\left\{m_2^2C^2+(m_2D+\delta)^2\right\}e^{(m_2Cx+(m_2D+\delta)y+\frac{A}{\mu^*\sigma}t)} \end{aligned}\right]$$

$$\Omega=f_0\left[A_1\left\{m_1^2C^2+(m_1D+\delta)^2\right\}e^{(m_1Cx+(m_1D+\delta)y+\frac{A}{\mu^*\sigma}t)}\right.$$

$$\left.+A_2\left\{m_2^2C^2+(m_2D+\delta)^2\right\}e^{(m_2Cx+(m_2D+\delta)y+\frac{A}{\mu^*\sigma}t)}\right]$$

Now, proceeding in the similar way we can have for solution in other two cases as

$$H_1=\left[BD+B_2De^{(m_3Cx+(m_3D+\delta)y+\frac{A}{\mu^*\sigma}t)}+\left\{B_1+B_2(Cx+Dy)\right\}(m_3D+\delta)e^{(m_3Cx+(m_3D+\delta)y)}\right]$$

$$H_2 = -\left[BC + B_2Ce^{(m_3 Cx+(m_3D+\delta)y+\frac{A}{\mu^*\sigma}t)} + \{B_1 + B_2(Cx+Dy)\}m_3Ce^{(m_3 Cx+(m_3D+\delta)y+\frac{A}{\mu^*\sigma}t)} \right]$$

$$u = f_0\left[BD + B_2De^{(m_3 Cx+(m_3D+\delta)y)} + \{B_1 + B_2(Cx+Dy)\}(m_3D+\delta)e^{(m_3 Cx+(m_3D+\delta)y+\frac{A}{\mu^*\sigma}t)} \right]$$

$$v = -f_0\left[BC + B_2Ce^{(m_3 Cx+(m_3D+\delta)y+\frac{A}{\mu^*\sigma}t)} + \{B_1 + B_2(Cx+Dy)\}m_3Ce^{(m_3 Cx+(m_3C+\delta)y+\frac{A}{\mu^*\sigma}t)} \right] \tag{70}$$

$$\Omega = -\left[2B_2\{m_3C^2 + D(m_3D+\delta\} + \{B_1 + B_2(Cx+Dy)\}(m_3D+\delta)^2 e^{(m_3 Cx+(m_3D+\delta)y+\frac{A}{\mu^*\sigma}t)} \right]$$

$$\omega = -f_0\left[2B_2\{m_3C^2 + D(m_3D+\delta\} + \{B_1 + B_2(Cx+Dy)\}(m_3D+\delta)^2 e^{(m_3 Cx+(m_3D+\delta)y+\frac{A}{\mu^*\sigma}t)} \right].$$

$$H_1 = \left[BD + C_1(\alpha D+\delta)e^{(\alpha Cx+(\alpha D+\delta)y+\frac{A}{\mu^*\sigma}t)} \cos(\beta(Cx+Dy)+C_2) \right.$$
$$\left. - C_1\beta De^{(\alpha Cx+(\alpha D+\delta)y)} \sin(\beta(Cx+Dy)+C_2) \right]$$

$$H_2 = -\left[BC + C_1\alpha Ce^{(\alpha Cx+(\alpha D+\delta)y+\frac{A}{\mu^*\sigma}t)} \cos(\beta(Cx+Dy)+C_2) \right.$$
$$\left. - C_1e^{(\alpha Cx+(\alpha D+\delta)y+\frac{A}{\mu^*\sigma}t)} \sin(\beta(Cx+Dy)+C_2) \right]$$

$$u = f_0\left[BD + C_1(\alpha D+\delta)e^{(\alpha Cx+(\alpha D+\delta)y)} \cos(\beta(Cx+Dy)+C_2) - \right.$$
$$\left. C_1\beta De^{(\alpha Cx+(\alpha D+\delta)y+\frac{A}{\mu^*\sigma}t)} \sin(\beta(Cx+Dy)+C_2) \right]$$

$$v = -f_0\left[BC + C_1\alpha Ce^{(\alpha Cx+(\alpha D+\delta)y+\frac{A}{\mu^*\sigma}t)} \cos(\beta(Cx+Dy)+C_2) \right.$$
$$\left. - C_1e^{(\alpha Cx+(\alpha D+\delta)y+\frac{A}{\mu^*\sigma}t)} \sin(\beta(Cx+Dy)+C_2) \right] \tag{71}$$

$$\Omega = -\left[C_1\{\alpha^2C^2 + (\alpha D+\delta)^2 - \beta^2(C^2+D^2)\}e^{(\alpha Cx+(\alpha D+\delta)y+\frac{A}{\mu^*\sigma}t)} \cos(\beta(Cx+Dy)+C_2) - \right.$$
$$\left. 2C_1\alpha\beta(C^2+D^2)e^{(\alpha Cx+(\alpha D+\delta)y)} \sin(\beta(Cx+Dy)+C_2) \right]$$

$$\omega = -f_0\left[C_1\{\alpha^2C^2 + (\alpha D+\delta)^2 - \beta^2(C^2+D^2)\}e^{(\alpha Cx+(\alpha D+\delta)y+\frac{A}{\mu^*\sigma}t)} \cos(\beta(Cx+Dy)+C_2) - \right.$$
$$\left. 2C_1\alpha\beta(C^2+D^2)e^{(\alpha Cx+(\alpha D+\delta)y+\frac{A}{\mu^*\sigma}t)} \sin(\beta(Cx+Dy)+C_2) \right].$$

4 CONCLUDING REMARK:

In this paper, we have found the exact solutions of the governing equations of second grade MHD aligned flow in porous media when the current density is proportional to the magnetic flux function perturbed by the quadratic stream. The magnetic vector field and velocity vector field are assumed parallel to each other. In the paper, we observe that obtained solutions consist of uniform flow perturbed by exponential and trigonometric functions in both finitely and infinitely conducting fluid cases, see equations (47), (67). We notice that there is only steady solutions in case of infinitely conducting fluid flow. It is also remarkable that there is no solution corresponding to $E \neq 0$, i.e quadratic perturbation term. But when porous media is absent i.e the term $\frac{\phi^*}{K} \to 0$, in all the equations and then on solving the revised equations following the reference [28], we can easily get some solutions containing quadratic perturbation term i.e Ey^2. As for physical aspect of the problem is concern, our problem is more general than that of references[22, 24, 28) as we have considered the fluid flow in porous media. Moreover neglecting the porous parameter in our result and taking $D = E = 0$, we can get the results of B. Singh and C. Thakur [24).

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