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DOUBLE DIFFUSIVE CONVECTIVE INSTABILITY DRIVEN BY THERMAL AND SOLUTAL GRADIENTS IN AN INCLINED SLOT

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ABSTRACT

The stability of double diffusive buoyancy driven convection in a tilted slot is investigated using linear stability analysis. Using the perturbation method with angle of inclination as perturbation parameter the critical Rayleigh number and the wave number at the critical point are determined. The results yield a unique flow pattern and the flow sets in becomes unstable when the temperature and salinity differences exceeds certain critical values. Some similarity can be noted between this problem and the problem when the planes are exactly horizontal and in fact some of the results are almost identical. However, owing to the existence of disturbances due to the inclined surfaces the system imparts a definite structure to the undisturbed system. The graphs presented for the velocity, temperature and solutal distributions for various parameter values will determine results in the secondary flow pattern and convection results in the form transverse rolls problem under investigation.

Keywords *Double-diffusive convection· Buoyancy-Driven convection · Convection in an inclined slot*

NOMENCLATURE

Cp_0: heat capacity, evaluated at the temperature T_0
D: depth of fluid layer
κ_0: thermalconductivity of fluid, evaluated at temperature T_0
P': pressure associated with the basic flow, i.e, undisturbed flow $P = \frac{P'}{\rho_0 U_c^2}$
T_1, T_2: the temperatures at respectively, the lower and upper plane
T_0 : the arithmetic mean temperature $\frac{T_1+T_2}{2}$
x, y, z: dimensionless Cartesian coordinates
u , v , w: dimensionless velocity components in the x , y , z direction respectively
U_c: characteristic velocity $= \frac{\kappa_0}{\rho_0 cp_0 d}$
U(y): the basic; i.e, velocity profile (dimensionless)
R_T: Thermal Rayleigh number $= \frac{\rho c_p g \alpha_T (T_1 - T_0) d^3}{\nu\kappa}$
g: gravitational acceleration
α_T is the coefficient of volumetric expansion of temperature
R_S: Solutal Rayleigh number $= \frac{\rho c_p g \alpha_S (S_1 - S_0) d^3}{\nu\kappa}$
α_S is the coefficient of volumetric expansion of concentration
α & β are wave numbers in the x & z direction, respectively
θ: dimensionless temperature$= \frac{T-T_0}{T_1-T_0}$
φ: dimensionless concentration $= \frac{S-S_0}{S_1-S_0}$
ν: is the kinematic viscosity
k_T: thermal diffusivity
k_S: solutal diffusivity
ρ: fluid density
σ: Prandtl number = $\frac{c_p \rho \nu}{\kappa}$ (physical property evaluated at T_0)
ϕ: angle of inclination of the slot with respect to horizontal
γ : $\frac{k_S}{k_T}$ is the ratio of diffusivities

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1. INTRODUCTION

Doubly diffusive instabilities have been observed in a variety of fluid systems and have been hypothesized to occur in still others (Turner 1974; Schechter, Verlde & Platten 1974). These problems are of practical importance in mixing of different water masses and general mixing processes, crystallization processes, design of solar ponds, engineering systems, oceanography, metallurgy etc. Baines and Gill (1969) solved the linear thermohaline stability problem for constant vertical gradients of temperature and salinity. They found in addition to the salt finger instability of Stern(1960), the overstability wave instability alluded by Stommel (1962) and demonstrated experimentally by Turner and Stommel (1964).

In the early years the focus was on one dimensional problems in which both the thermal and solutal gradients are in the vertical directions. Convective motions can occur either in the finger regime in which the more slowly diffusing component is heavy on top, or in the diffusive regime in which the component with the larger diffusivity is heavy on top. In either case, there is a tendency for horizontal convecting layers to develop. Later these studies were extended to two dimensional.

Brakke (1955) observed, and correctly explained, a doubly diffusive instability that occurs when a solution of a slowly diffusing protein is layered over a denser solution of more rapidly diffusing sucrose. Thorpe, Hutt & Souls by (1969) and Hart (1971) considered a two dimensional configuration of a fluid with vertical salinity gradient confined within a narrow slot whose two walls are held at different temperatures. Paliwal & Chen (1980) considered the slot inclined at an angle to the vertical. By using the linear stability theory he predicts for the observed critical Rayleigh number and wavelength of the steady convection he showed that there is no over stability. However, the motion consists of convection rolls with alternating directions of rotation. Experimental evidence indicates that all convection rolls have the same sense of rotation, rising along the hot wall and descending along the cold wall. A nonlinear treatment of the problem (Thangam, Zebib & Chen 1982) revealed that the cells with wrong sense of rotation (descending along the hot wall) are quickly squeezed in to interfaces between cells with correct sense of rotation.

Liang & Acrivos (1969) investigated the buoyancy driven convection in a slot and in a fluid layer bounded by the infinite parallel surfaces, tilted at a small angle ϕ, with respect to the horizontal. Here the instability sets in whenever the temperature difference between the two planes exceeds a certain critical value. The similarity between this and the usual case in which the planes are exactly is of course evident; in fact, both the method of solution and some of the principal results of the linear stability analysis are almost identical. However, it will be seen that, although the critical wave number will remain unaffected by tilting the planes small amount, a preferred mode will emerge in the form of rolls having their axes along the direction of the mean motion. Hence, owing to the existence of this basic flow which imparts a definite structure to the undisturbed system, the degeneracy usually associated with convection problems of this type will be removed.

In this paper the investigation is to study double diffusive convection driven by both temperature and salinity gradients in an inclined slot bounded by two infinite parallel plates inclined at an angle ϕ to the horizontal. The critical Rayleigh number expression using the linear stability analysis and the perturbation method is obtained on the lines of weakly nonlinear theory. Within the transition range of the angle of inclination when it is small but finite lead to a rather complicated dependence of the critical Rayleigh numbers R_T and R_S on $\sigma\ and\ \gamma$ which leads to longitudinal rolls with their axis aligned in the direction of the mean flow. This is in contrast to the problem in a vertical slot in which the secondary flow pattern is known to consist of transvers rolls, their axes normal to the mean motion.

2. MATHEMATICAL FORMULATION

We consider two-component Newtonian fluid-saturated horizontal layer confined between two boundary surfaces $y' = 0\ , d$ are taken to be free and tilted at a small angle ϕ with respect to the horizontal. The layer is heated and salted from below. The configuration is as shown in figure 1. The boundary surfaces are maintained at constant temperatures T_1 and T_2 and solutal concentrations S_1 and S_2 respectively. The basic governing equations of the problem under the Boussinesq approximations are the following.

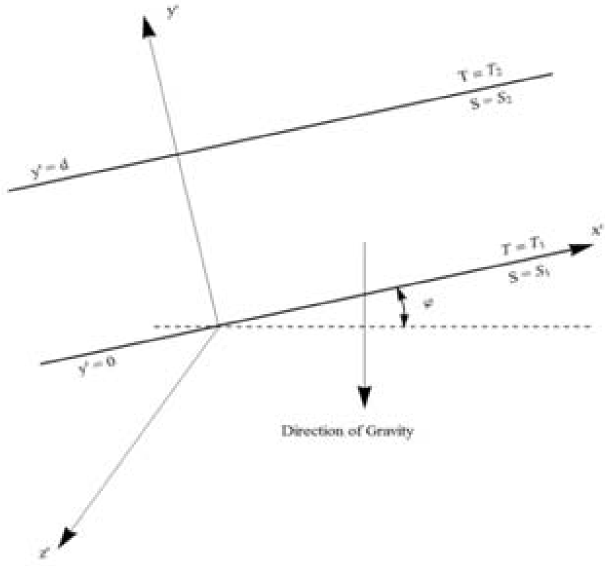

Fig 1. Physical configuration of the system

Conservation of mass: $$\nabla \cdot \vec{q} = 0 \tag{1}$$

Momentum equations: $$\rho_0 cp_0 \left[\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q}\right] = -\nabla \mathrm{p} + \rho \mathrm{g} + \mu \nabla^2 \vec{q} \tag{2}$$

Energy equation: $$\rho_0 cp_0 \left[\frac{\partial T}{\partial t} + (\vec{q} \cdot \nabla)T\right] = \kappa_T \nabla^2 T \tag{3}$$

Concentration equation: $$\rho_0 cp_0 \left[\frac{\partial S}{\partial t} + (\vec{q} \cdot \nabla)S\right] = \kappa_S \nabla^2 S \tag{4}$$

Equation of state: $$\rho = \rho_0[1 - \alpha_T(T - T_0) + \alpha_S(S - S_0)]. \tag{5}$$

The following non-dimensional quantities are introduced into equations (1) to (5) and using Boussinesq approximations

$$(x, y, z) = \left(\frac{x'}{d}, \frac{y'}{d}, \frac{z'}{d}\right), \quad \vec{q} = (u, v, w) = \left(\frac{u'}{U_c}, \frac{v'}{U_c}, \frac{w'}{U_c}\right) = \left(\frac{\vec{q'}}{U_c}\right), \quad P = \frac{P'}{\rho_0 U_c^2},$$

$$t = \frac{t' U_c}{d}, \theta = \frac{T - T_0}{T_1 - T_0}, \phi = \frac{S - S_0}{S_1 - S_0}, T_0 = \frac{T_1 + T_2}{2}$$

in which a prime refers to a dimensional variables and a script '0' to a physical quantity evaluated at the temperature T_0.

$$\nabla \cdot \vec{q} = 0 \tag{6}$$

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} = -\nabla \mathrm{p} + \sigma R_T \theta \mathrm{g} - \sigma R_S \phi \mathrm{g} + \sigma \nabla^2 \vec{q} \tag{7}$$

$$\frac{\partial \theta}{\partial t} + (\vec{q} \cdot \nabla)\theta = \nabla^2 \theta \tag{8}$$

$$\frac{\partial \phi}{\partial t} + (\vec{q} \cdot \nabla)\phi = \gamma \nabla^2 \phi \tag{9}$$

where $R_T = \frac{\rho c_p g \alpha_T (T_1 - T_0) d^3}{\nu \kappa}$ is the Thermal Rayleigh number,

$R_s = \frac{\rho c_p g \alpha_S (S_1 - S_0) d^3}{\nu \kappa}$ Solutal Rayleigh number

And $\gamma = \frac{k_S}{k_T}$ is the ratio of diffusivities i.e Lewes number

3. BASIC STATE SOLUTION OF THE PROBLEM

Using the basic state u = U(y) , v = w = 0 and $\theta = \theta(y)$ and boundary conditions $T = T_1$ $at\ y = 0$ $and\ T = T_2$ $at\ y = 1$ on equation (8) we get

$$\theta = 1 - 2y \tag{10}$$

and $\phi = \phi(y)$ and boundary conditions $S = S_1 \;\; at\; y = 0\; and\; S = S_2 \;\; at\; y = 1$ on equation (8), we get

$$\phi = 1 - 2y \tag{11}$$

The two boundary surfaces are free, we have $\frac{\partial U(y)}{\partial y} = 0 \;\; at\; y = 0\; \&\; 1$

$$U(y) = (R_T - R_S)\sin\varphi\left\{\frac{y^3}{3} - \frac{y^2}{2} + \frac{1}{12}\right\} \tag{12}$$

$$P = P_0 + \sigma(R_T - R_S)\cos\varphi\,(y - y^2) = P(y) \tag{13}$$

where P_0 is a constant, R_T is the thermal Rayleigh number, R_S is the solute Rayleigh number and σ the Prandtl number.

This solution indicates that no matter how small the inclined angle ϕ, a shear-like flow in the x-direction [u = U(y)] will always be established, and that even in the presence of such a motion, the transport of heat from lower to the upper plane will be due to conduction alone provided no lateral boundaries exist.

4. LINEAR STABILITY ANALYSIS

On the basic state, we superpose small perturbations about the basic state in the form,

$$u = U(y) + \hat{u}(x, y, z),\; v = \hat{v}(x, y, z),\; w = \hat{w}(x, y, z),\; P = P(y) + \hat{p}(x, y, z),$$
$$\theta = 1 - 2y + \hat{\theta}(x, y, z) \quad and\; \phi = 1 - 2y + \quad \hat{\phi}\,(x, y, z) \tag{14}$$

where the caret quantities indicate small perturbations. Substituting Eq. (14) into Eqs. (7)- (9), and neglecting the non-linear terms and dropping the carets yields

$$\frac{\partial u}{\partial t} + U(y)\frac{\partial u}{\partial x} + vDU(y) = -\frac{\partial p}{\partial x} + \sigma R_T\theta sin\varphi - \sigma R_S\, \phi sin\varphi + \sigma\nabla^2 u \tag{15}$$

$$\frac{\partial v}{\partial t} + U(y)\frac{\partial v}{\partial x} = -\frac{\partial p}{\partial y} + \sigma R_T\theta cos\varphi - \sigma R_S\phi cos\varphi + \sigma\nabla^2 v \tag{16}$$

$$\frac{\partial w}{\partial t} + U(y)\frac{\partial w}{\partial x} = -\frac{\partial p}{\partial z} + \sigma\nabla^2 w \tag{17}$$

$$\frac{\partial \theta}{\partial t} + U(y)\frac{\partial \theta}{\partial x} - 2v = \nabla^2\theta \tag{18}$$

$$\frac{\partial \phi}{\partial t} + U(y)\frac{\partial \phi}{\partial x} - 2v = \gamma\nabla^2\phi \tag{19}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{20}$$

where $D = \frac{d}{dy} \quad and\;\; \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

Cross differentiating and eliminating the pressure term, and further eliminating u and w from the equations give

$$\sigma\nabla^4 v - \frac{\partial}{\partial t}(\nabla^2 v) - U(y)\frac{\partial}{\partial x}(\nabla^2 v) + \frac{\partial v}{\partial x}D^2U(y) = -\,\sigma cos\varphi(R_T\nabla_1^2\theta - R_S\nabla_1^2\phi) + \sigma sin\varphi\left(R_T\frac{\partial^2\theta}{\partial x\partial y} - R_S\frac{\partial^2\phi}{\partial x\partial y}\right) \tag{21}$$

$$\left\{\nabla^2 - \frac{\partial}{\partial t} - U(y)\frac{\partial}{\partial x}\right\}\theta + 2v = 0 \tag{22}$$

$$\left\{\gamma\nabla^2 - \frac{\partial}{\partial t} - U(y)\frac{\partial}{\partial x}\right\}\phi + 2v = 0 \tag{23}$$

where$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$

Using the normal mode analysis, the dependent variables are assumed in the following form

$$v(x, y, z, t) = V(y)\exp\{i(\alpha x + \beta z - ct)\}$$
$$\theta(x, y, z, t) = \theta(y)\exp\{i(\alpha x + \beta z - ct)\} \tag{24}$$
$$\phi(x, y, z, t) = \phi(y)\exp\{i(\alpha x + \beta z - ct)\}$$

whose real parts represent the actual physical quantities. The wave numbers α *and* β are real and the growth rate c, is generally complex. Substituting equations (24) into equations (21) to (23) yields the following equations

$$\left(\sigma(D^2-\alpha^2-\beta^2)^2 - i\alpha\left[\left\{U(y)-\frac{c}{\alpha}\right\}(D^2-\alpha^2-\beta^2) - D^2U(y)\right]\right)V(y) = \sigma sin\varphi\, i\alpha\{R_T D\theta(y) - R_S D\phi(y)\} + \mathrm{M}\sigma(\alpha^2+\beta^2)cos\varphi\{R_T\theta(y) - R_S\phi(y)\} \quad (25)$$

$$[(D^2-\alpha^2-\beta^2) + ic - i\alpha U(y)]\theta(y) + 2V(y) = 0 \quad (26)$$

$$[\gamma(D^2-\alpha^2-\beta^2) + ic - i\alpha U(y)]\phi(y) + 2V(y) = 0 \quad (27)$$

The boundary conditions of the problem are

$$V(y) = D^2V(y) = \theta(y) = \phi(y) = 0 \;\; at\; y = 0\,,1 \quad (28)$$

Equations (25) to (27) are the familiar Orr-Sommerfeld equation coupled with the energy and concentration equations. Here, since the present study is restricted to small inclined angles ϕ, the above equation will be solved by using perturbation technique with $sin\phi$ as a small parameter perturbation quantity. Thus, expanding all the following quantities in terms of the perturbation expansions

$$\begin{Bmatrix} c \\ \alpha \\ \beta \\ R_T \\ R_S \\ V(y) \\ \theta(y) \\ \phi(y) \end{Bmatrix} = \begin{Bmatrix} c_0 \\ \alpha_0 \\ \beta_0 \\ R_{T0} \\ R_{S0} \\ V_0(y) \\ \theta_0(y) \\ \phi_0(y) \end{Bmatrix} + \begin{Bmatrix} c_1 \\ \alpha_1 \\ \beta_1 \\ R_{T1} \\ R_{S1} \\ V_1(y) \\ \theta_1(y) \\ \phi_1(y) \end{Bmatrix} sin\varphi + \begin{Bmatrix} c_2 \\ \alpha_2 \\ \beta_2 \\ R_{T2} \\ R_{S2} \\ V_2(y) \\ \theta_2(y) \\ \phi_2(y) \end{Bmatrix} sin^2\varphi + \cdots\dots\dots\dots\dots\dots \quad (29)$$

At zero th order the system off equations are

$$\left[\sigma\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)^2 + ic_0\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)\right]V_0(y) = \sigma\left(\alpha_0{}^2+\beta_0{}^2\right)\left(R_{T0}\theta_0 - R_{S0}\phi_0\right) \quad (30)$$

$$\left[\left(D^2-\alpha_0{}^2-\beta_0{}^2\right) + ic_0\right]\theta_0(y) + 2V_0(y) = 0 \quad (31)$$

$$\left[\left(D^2-\alpha_0{}^2-\beta_0{}^2\right) + ic_0\right]\phi_0(y) + 2V_0(y) = 0 \quad (32)$$

with the boundary conditions $V(y) = D^2V(y) = \theta(y) = \phi(y) = 0 \;\; at\; y = 0\,,1$As the principle of the exchange of stabilities is valid, c_0 is real and the marginal state is characterized by $c_0 = 0$.

The solution for the system of equations (30) to (32) are given by

$$V_0(y) = sin\pi y,\;\; \theta_0(y) = \frac{4}{3\pi^2}sin\pi y\,,\;\; \phi_0(y) = \frac{4}{\gamma 3\pi^2}sin\pi y,$$
$$\alpha_0{}^2+\beta_0{}^2 = \frac{\pi^2}{2}\,,\quad R_{T0} = \frac{1}{\gamma}R_{S0} + \frac{27\pi^4}{8} \quad (33)$$

Prior to solving the higher order equations, it is first necessary to solve the homogeneous adjoint problem.

$$\left[\sigma\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)^2\right]V^*(y) + 2\theta^*(y) + 2\phi^*(y) = 0$$
$$\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)\theta^*(y) - \sigma\left(\alpha_0{}^2+\beta_0{}^2\right)R_{T0}V^*(y) = 0 \quad (34)$$
$$\gamma\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)\phi^*(y) + \sigma\left(\alpha_0{}^2+\beta_0{}^2\right)R_{S0}V^*(y) = 0$$

With the boundary conditions are the same as before

$$V(y) = D^2V(y) = \theta(y) = \phi(y) = 0 \quad at\;\; y = 0, 1$$

Hence

$$V^*(y) = V_0(y) = \sin\pi y,\quad \theta^*(y) = -\frac{\sigma}{3}R_{T0}\sin\pi y\,,\;\; \phi^*(y) = \frac{\sigma}{3\gamma}R_{S0}\sin\pi y \quad (35)$$

Substituting equation (29) in to equations (25) to (27), we next obtain for the first order equations

$$\left[\sigma\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)^2+ic_0\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)\right]V_1(y)$$
$$+\left[\sigma\left\{-4(\alpha_0\alpha_1+\beta_0\beta_1)\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)\right\}-i\left\{\begin{matrix}2c_0(\alpha_0\alpha_1+\beta_0\beta_1)+\\ [-c_1+h(y)\alpha_0(R_{T0}-R_{S0})]\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)-\alpha_0(R_{T0}-R_{S0})D^2h\end{matrix}\right\}\right]V_0(y)$$

$$=i\sigma\alpha_0\left(R_{T0}D\theta_0-R_{S0}D\phi_0\right)+\sigma cos\varphi\left\{\begin{matrix}\left(\alpha_0{}^2+\beta_0{}^2\right)\left(R_{T0}\theta_1+R_{T1}\theta_0-R_{S0}\phi_1-R_{S1}\phi_0\right)\\ +2(\alpha_0\alpha_1+\beta_0\beta_1)\left(R_{T0}\theta_0-R_{S0}\phi_0\right)\end{matrix}\right\} \tag{36}$$

$$\left[\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)+ic_0\right]\theta_1(y)\ +2V_1(y)=\{2(\alpha_0\alpha_1+\beta_0\beta_1)-ic_1+i\alpha_0(R_{T0}-R_{S0})h\}\theta_0 \tag{37}$$

$$\left[\gamma\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)+ic_0\right]\phi_1(y)+2V_1(y)\quad=\{2\gamma(\alpha_0\alpha_1+\beta_0\beta_1)-ic_1+i\alpha_0(R_{T0}-R_{S0})h\}\phi_0 \tag{38}$$

On using $c_0 = 0$, and $\alpha_0+\beta_0=\frac{\pi^2}{2}$ we get

$$\left[\sigma\left(D^2-\frac{\pi^2}{2}\right)^2\right]V_1(y)-\sigma\left\{\frac{\pi^2}{2}\left(R_{T0}\theta_1-R_{S0}\phi_1\right)\right\}=\sigma\left\{4(\alpha_0\alpha_1+\beta_0\beta_1)\left(D^2-\frac{\pi^2}{2}\right)V_0(y)\right\}-i\alpha_0(R_{T0}-R_{S0})V_0D^2h$$
$$+i\alpha_0(R_{T0}-R_{S0})h\left(D^2-\frac{\pi^2}{2}\right)V_0-ic_1\left(D^2-\frac{\pi^2}{2}\right)V_0+i\alpha_0\sigma\left(R_{T0}D\theta_0-R_{S0}D\phi_0\right)$$
$$+\sigma\frac{\pi^2}{2}\left(R_{T1}\theta_0-R_{S1}\phi_0\right)+2\sigma(\alpha_0\alpha_1+\beta_0\beta_1)\left(R_{T0}\theta_0-R_{S0}\phi_0\right) \tag{39}$$

$$\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)\theta_1(y)\ +2V_1(y)\ i\alpha_0(R_{T0}-R_{S0})h\theta_0-ic_1\theta_0+2(\alpha_0\alpha_1+\beta_0\beta_1)\theta_0 \tag{40}$$

$$\gamma\left(D^2-\alpha_0{}^2-\beta_0{}^2\right)\phi_1(y)+2V_1(y)=i\alpha_0(R_{T0}-R_{S0})h\phi_0-ic_1\phi_0+2\gamma(\alpha_0\alpha_1+\beta_0\beta_1)\phi_0 \tag{41}$$

where $h(y)=\frac{U(y)}{(R_T-R_S)sin\varphi}=\frac{y^3}{3}-\frac{y^2}{2}+\frac{1}{12}$ (42)

$$\sigma\left(D^2-\frac{\pi^2}{2}\right)^2V_1(y)-\sigma R_{T0}\frac{\pi^2}{2}\theta_1+\sigma R_{S0}\frac{\pi^2}{2}\phi_1$$
$$=3\sigma\pi^2(\alpha_0\alpha_1+\beta_0\beta_1)sin\pi y-i\alpha_0(R_{T0}-R_{S0})sin\pi yD^2h\ -i\alpha_0\frac{3\pi^2}{2}h(R_{T0}-R_{S0})sin\pi y$$
$$-ic_1\frac{3\pi^2}{2}sin\pi y\ +i\alpha_0\sigma\frac{9\pi^3}{2}cos\pi y+\sigma\frac{2}{3}\left(R_{T1}-\frac{1}{\gamma}R_{S1}\right)sin\pi y \tag{43}$$

$$\left(D^2-\frac{\pi^2}{2}\right)\theta_1(y)\ +2V_1(y)\ =i\alpha_0\frac{4}{3\pi^2}h(R_{T0}-R_{S0})sin\pi y-ic_1\frac{4}{3\pi^2}sin\pi y+\frac{8}{3\pi^2}(\alpha_0\alpha_1+\beta_0\beta_1)sin\pi y \tag{44}$$

$$\gamma\left(D^2-\frac{\pi^2}{2}\right)\phi_1(y)+2V_1(y)=i\alpha_0\frac{4}{3\pi^2\gamma}h(R_{T0}-R_{S0})sin\pi y-ic_1\frac{4}{3\pi^2\gamma}sin\pi y+\frac{8}{3\pi^2\gamma}(\alpha_0\alpha_1+\beta_0\beta_1)sin\pi y \tag{45}$$

Since the inhomogeneous part of equation must be orthogonal to the homogeneous adjoint solution, the eigen value R_1, can be computed as follows: Multiplying the equation (43) by V^*, (44) by θ^* and equation (45) by ϕ^*, summing and then integrating from y = 0 to y = 1, yields

$$R_{T1}-\frac{1}{\gamma}R_{S1}=-ic_1\left\{\frac{9\pi^2}{4}\frac{1}{\sigma}+\frac{2}{3\pi^2}\left(\frac{1}{\gamma}R_{T0}-\frac{1}{\gamma^2}R_{S0}\right)\right\} \tag{46}$$

$$R_{T1}-\frac{1}{\gamma}R_{S1}=-ic_1\left\{\frac{9\pi^2}{4}\frac{1}{\sigma}+\frac{2}{3\pi^2}\left(\frac{1}{\gamma}R_{S0}+\frac{27\pi^4}{8}-\frac{1}{\gamma^2}R_{S0}\right)\right\} \tag{47}$$

Since R_{T1}&R_{S1} are real, C_1 must be imaginary. Thus to this order there is no oscillatory motion and the neutral state C_1 and hence $R_{T1}-\frac{1}{\gamma}R_{S1}$ must equal to zero.

$$R_{T1}-\frac{1}{\gamma}R_{S1}=0 \tag{48}$$

In view of equation (34) and the fact that R_{T1} , R_{S1} and C_1 $\quad R_{T1}-\frac{1}{\gamma}R_{S1}=0$, equations (44) to (46) becomes

$$\sigma\left(D^2-\frac{\pi^2}{2}\right)^2 V_1(y)-\sigma R_{T0}\frac{\pi^2}{2}\theta_1+\sigma R_{S0}\frac{\pi^2}{2}\phi_1 = 3\sigma\pi^2(\alpha_0\alpha_1+\beta_0\beta_1)sin\pi y+i\alpha_0\sigma\frac{9\pi^3}{2}cos\pi y-i\alpha_0(R_{T0}-R_{S0})\left(\frac{3\pi^2}{2}h+D^2h\right)sin\pi y \tag{49}$$

$$\left(D^2-\frac{\pi^2}{2}\right)\theta_1(y)+2V_1(y)=i\alpha_0\frac{4}{3\pi^2}h(R_{T0}-R_{S0})sin\pi y+\frac{8}{3\pi^2}(\alpha_0\alpha_1+\beta_0\beta_1)sin\pi y \tag{50}$$

$$\gamma\left(D^2-\frac{\pi^2}{2}\right)\phi_1(y)+2V_1(y)=i\alpha_0\frac{4}{3\pi^2\gamma}h(R_{T0}-R_{S0})sin\pi y+\frac{8}{3\pi^2\gamma}(\alpha_0\alpha_1+\beta_0\beta_1)sin\pi y \tag{51}$$

Further elimination of θ_1 and ϕ_1 yields

$$\left[\left(D^2-\frac{\pi^2}{2}\right)^3+\frac{27\pi^6}{8}\right]\bar{V}_1(y)=\frac{1}{\sigma}\begin{bmatrix}(R_{T0}-R_{S0})\left(\frac{y^3}{3}-\frac{y^2}{2}+\frac{1}{12}\right)\left\{\sigma\frac{2}{3}\left(R_{T0}-\frac{1}{\gamma^2}R_{S0}\right)+\frac{9\pi^4}{4}\right\}sin\pi y\\ -\left\{(R_{T0}-R_{S0})3\pi^3\left(y-y^2-\frac{4}{\pi^2}\right)-\sigma\frac{27\pi^5}{4}\right\}cos\pi y\end{bmatrix} \tag{52}$$

where $\bar{V}_1(y)=\frac{1}{i\alpha_0}V_1(y)$ (53)

With boundary conditions

$\bar{V}_1(y)=D^2\bar{V}_1=0$at y = 0, 1

Equation (50) and (51) can be simplified by substituting

$$\theta_1(y)=i\alpha_0\bar{\theta}_1(y)+2(\alpha_1\alpha_0+\beta_1\beta_0)\tilde{\theta}_1(y) \tag{54}$$

$$\phi_1(y)=i\alpha_0\bar{\phi}_1(y)+2(\alpha_1\alpha_0+\beta_1\beta_0)\tilde{\phi}_1(y) \tag{55}$$

Then

$$\left(D^2-\frac{\pi^2}{2}\right)\bar{\theta}_1(y)=-2\bar{V}_1(y)+\frac{4}{3\pi^2}(R_{T0}-R_{S0})\left(\frac{y^3}{3}-\frac{y^2}{2}+\frac{1}{12}\right)sin\pi y \tag{56}$$

$$\left(D^2-\frac{\pi^2}{2}\right)\tilde{\theta}_1(y)=\frac{4}{3\pi^2}sin\pi y \tag{57}$$

Clearly, $\tilde{\theta}_1(y)=-\frac{8}{9\pi^4}sin\pi y$ (58)

$$\gamma\left(D^2-\frac{\pi^2}{2}\right)\bar{\phi}_1(y)=-2\bar{V}_1(y)+\frac{4}{3\pi^2\gamma}(R_{T0}-R_{S0})\left(\frac{y^3}{3}-\frac{y^2}{2}+\frac{1}{12}\right)sin\pi y \tag{59}$$

$$\gamma\left(D^2-\frac{\pi^2}{2}\right)\tilde{\phi}_1(y)=\frac{4}{3\pi^2}sin\pi y \tag{60}$$

Clearly, $\tilde{\phi}_1(y)=-\frac{8}{9\pi^4\gamma}sin\pi y$ (61)

As for $\bar{V}_1(y)$, $\bar{\theta}_1(y)$ and $\bar{\phi}_1(y)$, these had to be obtained via a numerical solution of equations (52), (56) and (59) are shown in figures. As required by their governing equations and the associated boundary conditions, the functions are anti-symmetric conditions with respect to mid-point y = 0.5. Also, it is apparent from figure that, for Prandtl number higher than 1.0.

Considering the second order terms in $sin^2\varphi$ we get,

$$\begin{aligned}\sigma\left(D^2-\frac{\pi^2}{2}\right)^2 V_2-\sigma\frac{\pi^2}{2}R_{T0}\theta_2+\sigma\frac{\pi^2}{2}R_{S0}\phi_2\\ = \alpha_0^2\left[(R_{T0}-R_{S0})\bar{V}_1D^2h+(R_{T0}-R_{S0})h\left(D^2-\frac{\pi^2}{2}\right)\bar{V}_1-R_{T0}\sigma D\bar{\theta}_1+R_{S0}\sigma D\bar{\phi}_1\right]\\ +i\alpha_0\xi\left[R_{T0}\sigma D\tilde{\theta}_1-R_{S0}\sigma D\tilde{\phi}_1-(R_{T0}-R_{S0})hV_0+2\sigma\left(D^2-\frac{\pi^2}{2}\right)\bar{V}_1+\sigma R_{T0}\bar{\theta}_1-\sigma R_{S0}\bar{\phi}_1\right]\end{aligned}$$

$$-i\frac{3\pi^2}{2}c_2 sin\pi y+\xi^2\left[\sigma R_{T0}\tilde{\theta}_1-\sigma R_{S0}\tilde{\phi}_1-\sigma V_0\right]+\sigma\frac{\pi^2}{2}R_{T2}\theta_0-\sigma\frac{\pi^2}{2}R_{S2}\phi_0$$
$$+i\alpha_1\left[(R_{T0}-R_{S0})h\left(D^2-\frac{\pi^2}{2}\right)V_0-(R_{T0}-R_{S0})V_0D^2h+\sigma R_{T0}D\theta_0-\sigma R_{S0}D\phi_0\right]$$
$$+\zeta\left[2\sigma\left(D^2-\frac{\pi^2}{2}\right)V_0+\sigma R_{T0}\theta_0-\sigma R_{S0}\phi_0\right]+i\alpha_0\begin{bmatrix}(R_{T1}-R_{S1})h\left(D^2-\frac{\pi^2}{2}\right)V_0-(R_{T1}-R_{S1})V_0D^2h+\\ \sigma R_{T1}D\theta_0-\sigma R_{S1}D\phi_0+\sigma\frac{\pi^2}{2}R_{T1}\bar{\theta}_1-\sigma\frac{\pi^2}{2}R_{S1}\bar{\phi}_1\end{bmatrix}$$
$$+\xi\left[\sigma\frac{\pi^2}{2}R_{T1}\tilde{\theta}_1-\sigma\frac{\pi^2}{2}R_{S1}\tilde{\phi}_1+\sigma R_{T1}\theta_0-\sigma R_{S1}\phi_0\right] \tag{62}$$

$$\left(D^2-\frac{\pi^2}{2}\right)\theta_2+2V_2=-ic_2\theta_0+\xi^2\tilde{\theta}_1+\zeta\theta_0+i\alpha_0\xi\,(R_{T0}-R_{S0})\,h\,\tilde{\theta}_1-i\alpha_0\xi\bar{\theta}_1$$
$$-\alpha_0^2(R_{T0}-R_{S0})h\,\bar{\theta}_1+i\alpha_1(R_{T0}-R_{S0})\,h\,\theta_0+i\alpha_0\,(R_{T1}-R_{S1})h\,\theta_0 \tag{63}$$

$$\gamma\left(D^2-\frac{\pi^2}{2}\right)\phi_2+2V_2=-ic_2\phi_0+\gamma\xi^2\tilde{\phi}_1+\gamma\zeta\phi_0+i\alpha_0\xi\,(R_{T0}-R_{S0})\,h\,\tilde{\phi}_1-i\alpha_0\gamma\xi\bar{\phi}_1$$
$$-\alpha_0^2(R_{T0}-R_{S0})h\,\bar{\phi}_1+i\alpha_1(R_{T0}-R_{S0})\,h\,\phi_0+i\alpha_0\,(R_{T1}-R_{S1})h\,\phi_0 \tag{64}$$

where $\xi=2\alpha_1\alpha_0+2\beta_1\beta_0$, $\quad\zeta=2\alpha_2\alpha_0+\alpha_1^2+2\beta_2\beta_0+\beta\beta_1^2$ (65)

Again, multiplying the first equation by V^*, the second by θ^* and the third by ϕ^*, summing and integrating, yields we gets

$$R_{T2}-\frac{1}{\gamma}R_{S2}=\zeta\left[\frac{9\pi^2}{2}+\frac{4}{3\pi^2}\left(R_{T0}-\frac{1}{\gamma}R_{S0}\right)\right]+\frac{9}{2}\xi^2-\alpha_0^2R_{T0}k_1(\sigma)+\alpha_0^2R_{S0}k_2(\sigma)-3i\alpha_0\xi k_3(\sigma)$$
$$+ic_2\left[\frac{9\pi^2}{4}\frac{1}{\sigma}-\frac{2}{3\pi^2}\left(R_{T0}-\frac{1}{\gamma^2}R_{S0}\right)\right]-i\alpha_0\frac{3\pi^2}{2}\left[R_{T1}\int_0^1V^*\bar{\theta}_1\,dy-R_{S1}\int_0^1V^*\bar{\phi}_1\,dy\right] \tag{66}$$

where

$$k_1(\sigma)=\frac{3}{\sigma}\int_0^1\left[\left\{\bar{V}_1D^2h-h\left(D^2-\frac{\pi^2}{2}\right)\bar{V}_1-\sigma D\bar{\theta}_1\right\}V^*-h\bar{\theta}_1\theta^*-h\bar{\phi}_1\phi^*\right]dy \tag{67}$$

$$k_2(\sigma)=\frac{3}{\sigma}\int_0^1\left[\left\{\bar{V}_1D^2h-h\left(D^2-\frac{\pi^2}{2}\right)\bar{V}_1-\sigma D\bar{\phi}_1\right\}V^*-h\bar{\theta}_1\theta^*-h\bar{\phi}_1\phi^*\right]dy \tag{68}$$

$$k_3(\sigma)=\frac{1}{\sigma}\int_0^1\left[\left\{2\sigma\left(D^2-\frac{\pi^2}{2}\right)\bar{V}_1+\sigma R_{T0}\,\bar{\theta}_1-\sigma R_{S0}\bar{\phi}_1\right\}V^*+\bar{\theta}_1\theta^*+\gamma\bar{\phi}_1\phi^*\right]dy \tag{69}$$

At the neutral state, c_2 must be real, since $R_{T2}-\frac{1}{\gamma}R_{S2}$ is real, imaginary and real parts of equations becomes

$$R_{T2}-\frac{1}{\gamma}R_{S2}=\zeta\left[\frac{9\pi^2}{2}+\frac{4}{3\pi^2}\left(R_{T0}-\frac{1}{\gamma}R_{S0}\right)\right]+\frac{9}{2}\xi^2-\alpha_0^2R_{T0}k_1(\sigma)+\alpha_0^2R_{S0}k_2(\sigma) \tag{70}$$

And

$$c_2\left[\frac{9\pi^2}{4}\frac{1}{\sigma}-\frac{2}{3\pi^2}\left(R_{T0}-\frac{1}{\gamma^2}R_{S0}\right)\right]=3\alpha_0\xi k_3(\sigma)+\alpha_0\frac{3\pi^2}{2}\left[R_{T1}\int_0^1V^*\bar{\theta}_1\,dy-R_{S1}\int_0^1V^*\bar{\phi}_1\,dy\right] \tag{71}$$

Since $R_{T0}-\frac{1}{\gamma}R_{S0}=\frac{27\pi^4}{8}$

$$R_{T2}-\frac{1}{\gamma}R_{S2}=\frac{9}{2}\xi^2+9\pi^2\zeta-\alpha_0^2R_{T0}k_1(\sigma)+\alpha_0^2R_{S0}k_2(\sigma) \tag{72}$$

5. RESULTS AND DISCUSSION

This is the problem of buoyancy driven convection when both the temperature and solutal concentration suffer from the differences in the lower and upper surfaces and also convection caused by the effect of inclination of the tilted surfaces from the horizontal direction. The critical Rayleigh number at neutral state depends on various parameters on the wave number, solutal Rayleigh number and the angle of inclination etc. The minimum critical Rayleigh number value is $R_{T0}=\frac{1}{\gamma}R_{S0}+\frac{27\pi^4}{8}$ for a steady longitudinal roll disturbance. For other disturbances, when $\alpha_0\neq 0$, the critical Rayleigh number is given by equation (72) shows that oscillatory motions can be excluded. The results obtained shows interesting prediction that if the Rayleigh number increased past the critical value, the convective instability for small values of $\sigma\ and\ \gamma$ will lead to steady parallel rolls having definite wave length and with their axis in the x-direction rather than being oscillatory. The critical Rayleigh number sharply increases with solutal Rayleigh number and decreases due to the presence of Lewes number.

The critical Rayleigh number is the same as that for double diffusive convection for longitudinal disturbances having their axes aligned in the direction of mean flow. i.e. $\alpha_0 = 0$. For other disturbances ($\alpha_0 \neq 0$) which generally lead to oscillatory instability in single as well as double diffusive components for horizontal, inclined fluid layers and inclined slots as well. The critical Rayleigh number expression found to depend on various parameters and remains stationary for all other disturbance wave numbers. For the disturbance with $\alpha_0 \neq 0$ shows sharp dependence on the various parameters such as Prandtl numbers, R_{S0}, R_{T0} and $K1 = K1(\sigma, R_{S0}, \gamma)$ and $K2 = K2(\sigma, R_{S0}, \gamma)$. It can be readily seen from the graphs $K1$ *and* $K2$ they reach the asymptotic values and thus the critical Rayleigh number also reaches asymptotic values and are negative and positive. The numerical solution obtained for $\bar{V}_1(y), \bar{\theta}_1(y)$ and $\bar{\phi}_1(y)$ are depicted graphically in figures 2 to 27. As requires by their governing equations and boundary conditions show that $\bar{V}_1(y)$and $\bar{\theta}_1(y)$ are antisymmetric and $\bar{\theta}_1(y)$ and $\bar{\phi}_1(y)$ are symmetric but not identically similar. It is apparent that for Prandtl numbers higher than 1.0 the magnitude of $2V_1$ is much less than one would expect $\bar{\theta}_1(y)$ and $\bar{\phi}_1(y)$ are intensive to the Prandtl number.

The predicted flow pattern at the neutral state seems to depend primarily on both the hydrodynamic as well as thermal convection. Thermal instability occurs when the layer is nearly horizontal and is heated from below. In contrast when the mechanism of instability is hydrodynamic when it is vertical or when it is positioned at an angle tshat it corresponds to two opposing convective streams. Within the transition range of the angle of inclination both mechanisms are active and lead to rather complicated dependence of the critical Rayleigh number ϕ and σ and solutal Rayleigh number which leads to the special case of transverse rolls.

The present analysis and the principle conclusions from this study would not have been affected by the use of more realistic boundary conditions.

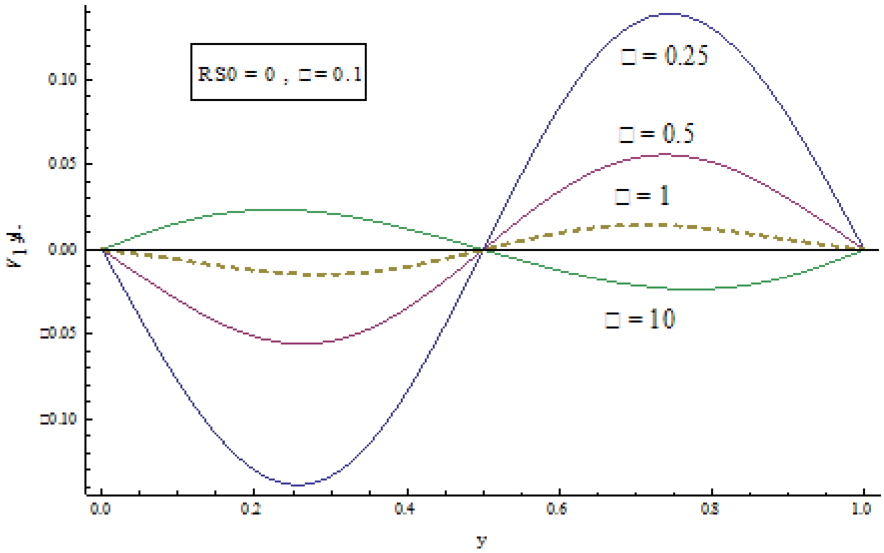

Figure: 2 Variation of $\bar{V}_1(y)$ with y for different values of σ

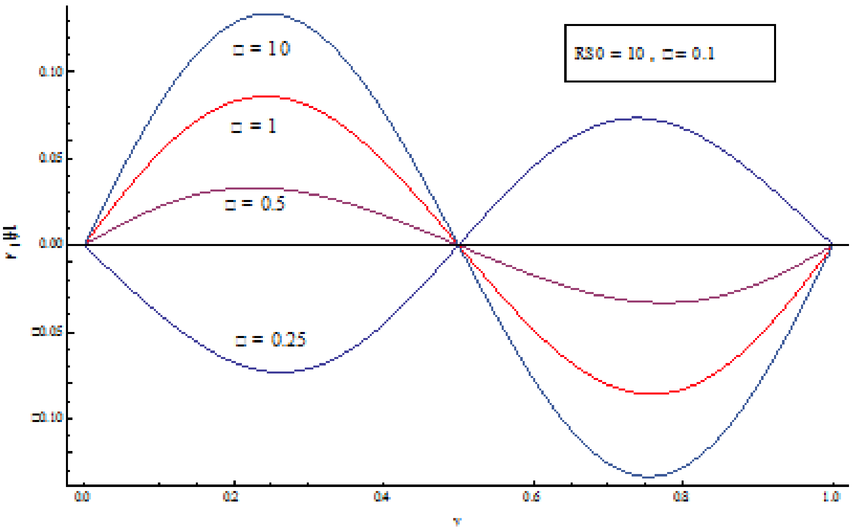

Figure: 3 Variation of $\bar{V}_1(y)$ with y for different values of σ

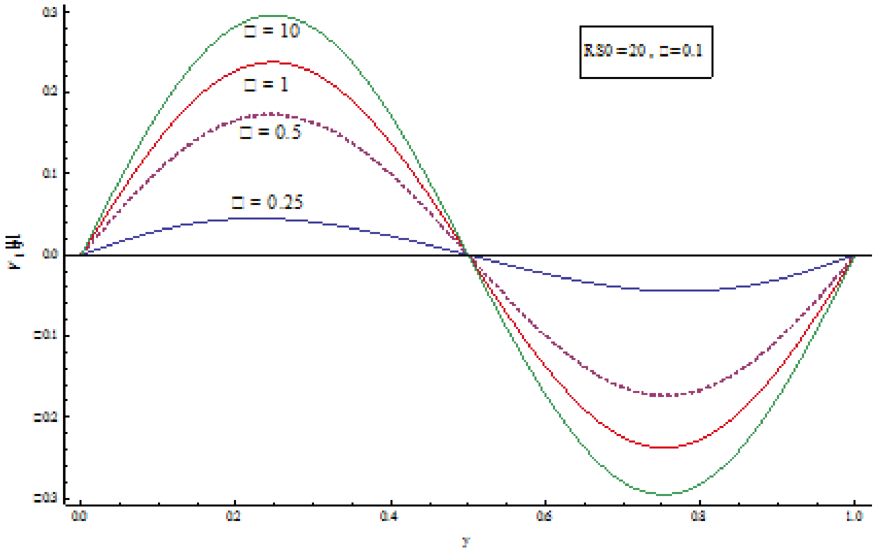

Figure: 4 Variation of $\bar{V}_1(y)$ with y for different values of σ

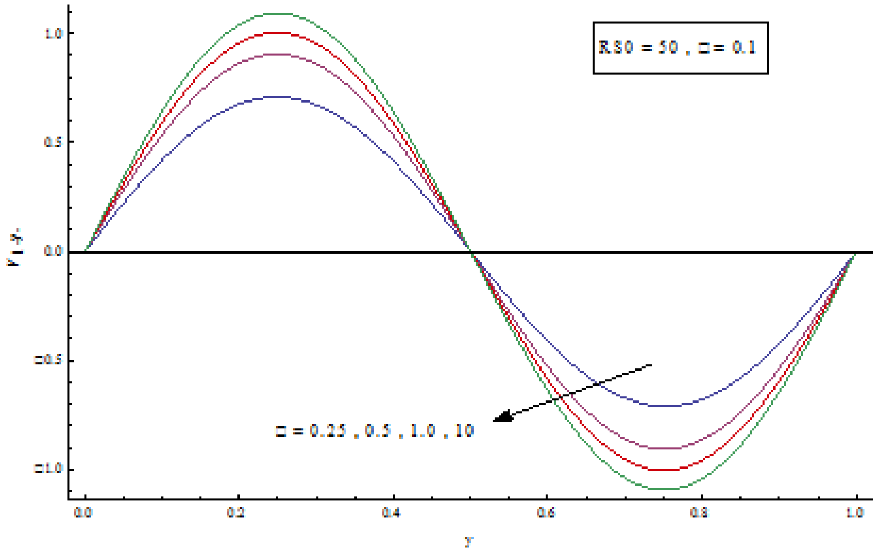

Figure: 5 Variation of $\bar{V}_1(y)$ with y for different values of σ

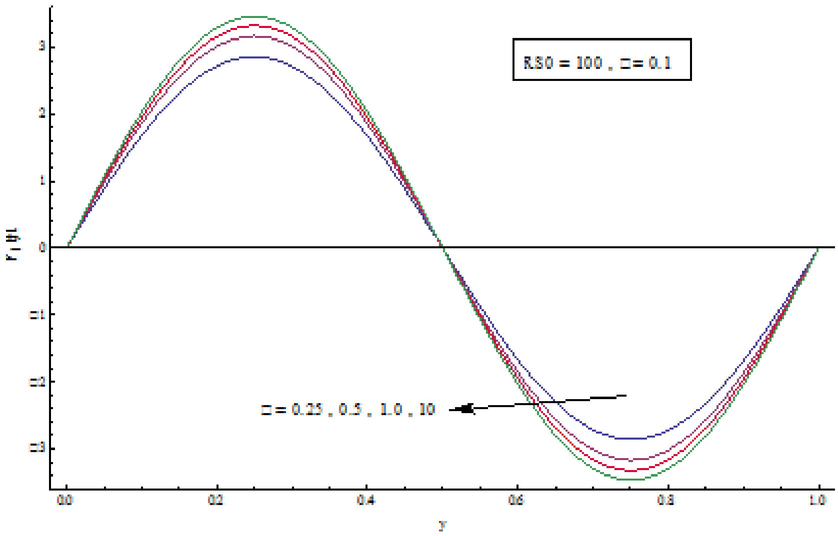

Figure: 6 Variation of $\bar{V}_1(y)$ with y for different values of σ

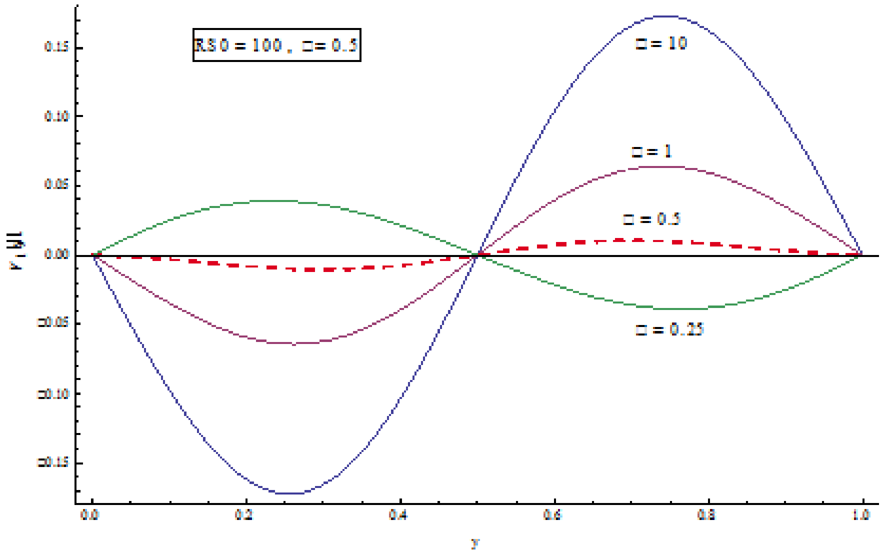

Figure: 7 Variation of $\bar{V}_1(y)$ with y for different values of σ

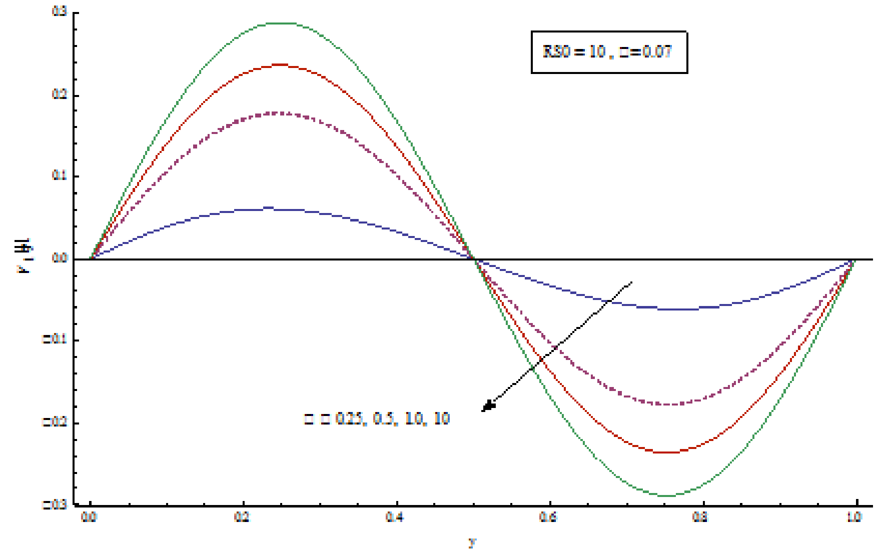

Figure: 8 Variation of $\bar{V}_1(y)$ with y for different values of σ

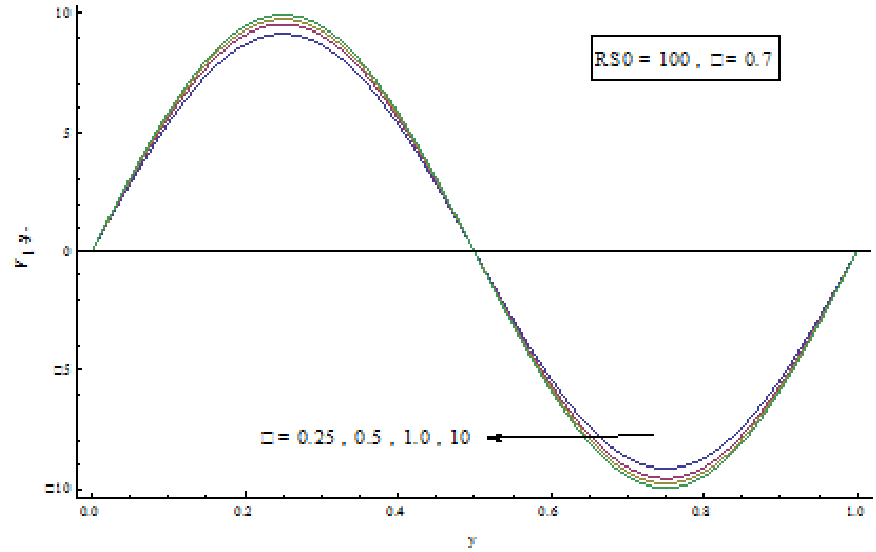

Figure: 9 Variation of $\bar{V}_1(y)$ with y for different values of σ

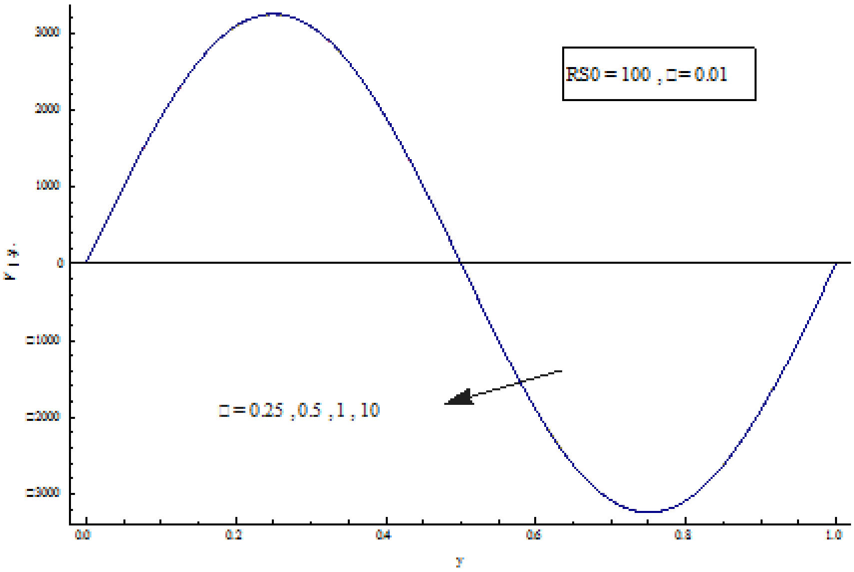

Figure: 10 Variation of $\bar{V}_1(y)$ with y for different values of σ

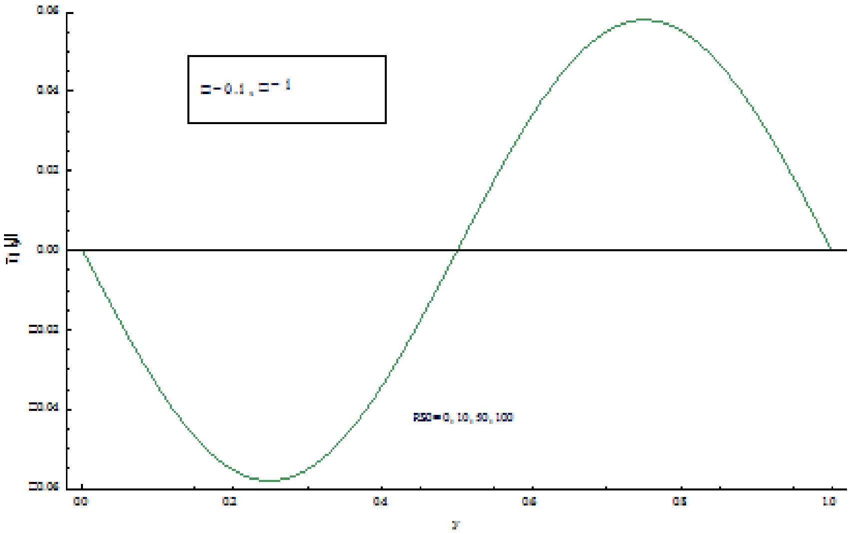

Figure: 11 Variation of $\bar{\theta}_1(y)$ with y for different values of R_{S0}

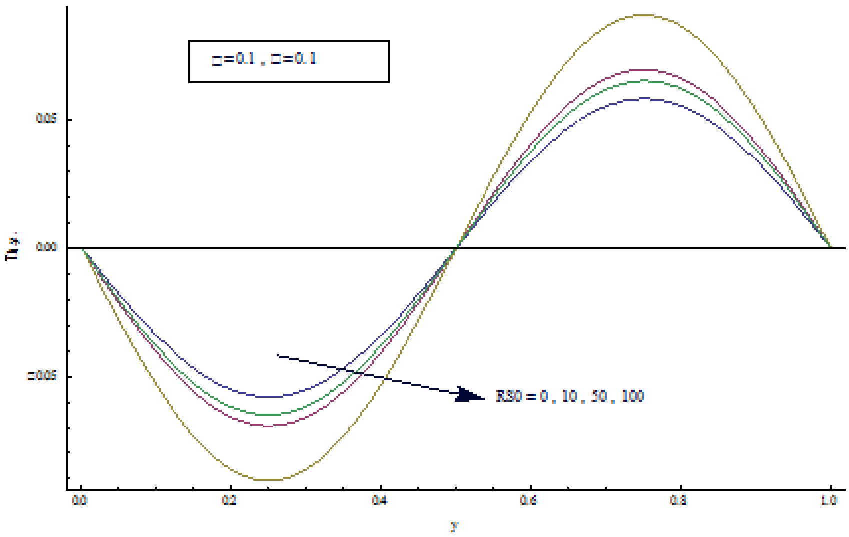

Figure: 12 Variation of $\bar{\theta}_1(y)$ with y for different values of R_{S0}

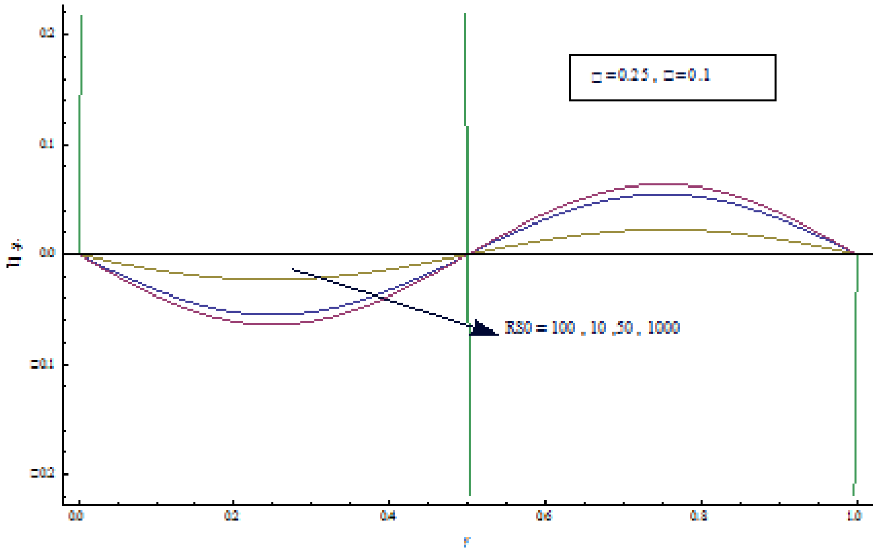

Figure: 13 Variation of $\bar{\theta}_1(y)$ with y for different values of R_{S0}

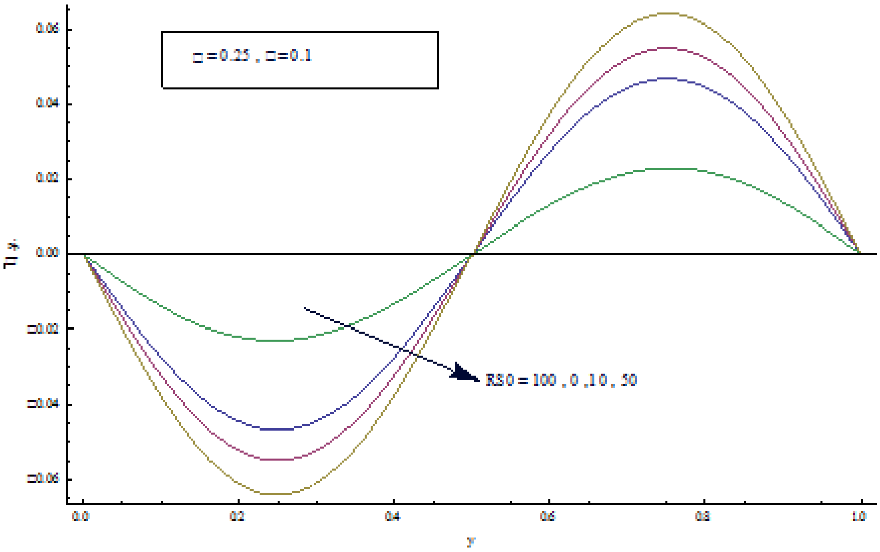

Figure: 14 Variation of $\bar{\theta}_1(y)$ with y for different values of R_{S0}

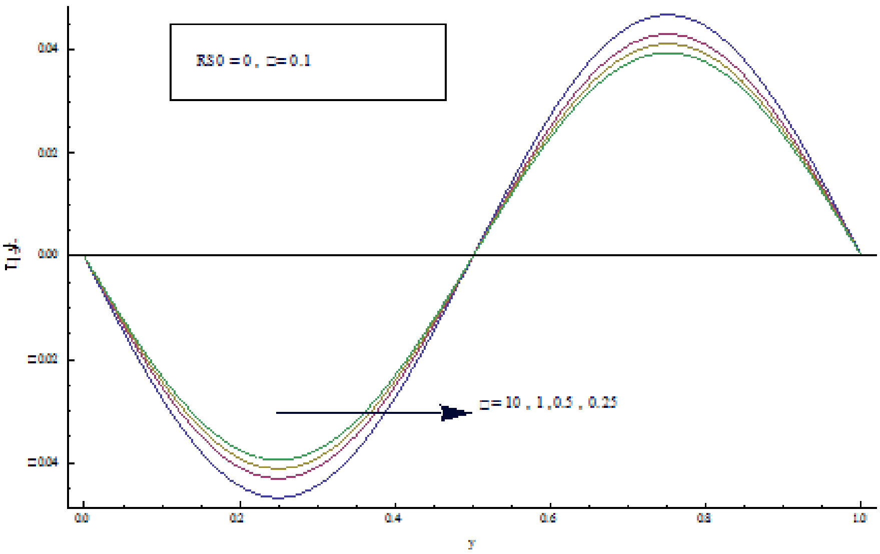

Figure: 15 Variation of $\bar{\theta}_1(y)$ with y for different values of σ

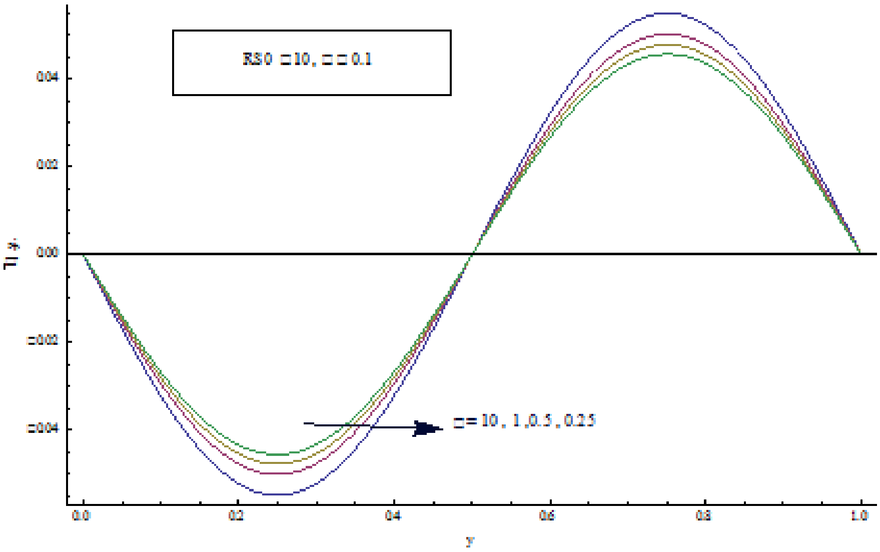

Figure: 16 Variation of $\bar{\theta}_1(y)$ with y for different values of σ

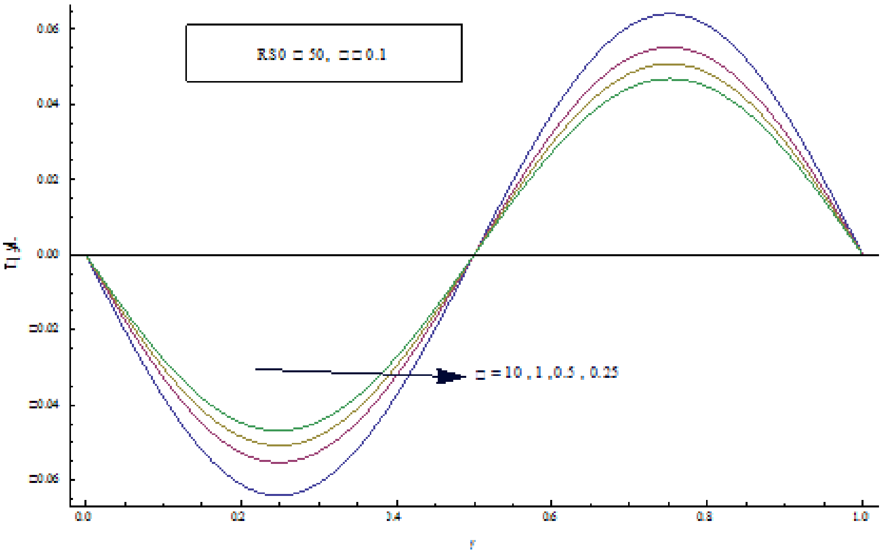

Figure: 17 Variation of $\bar{\theta}_1(y)$ with y for different values of σ

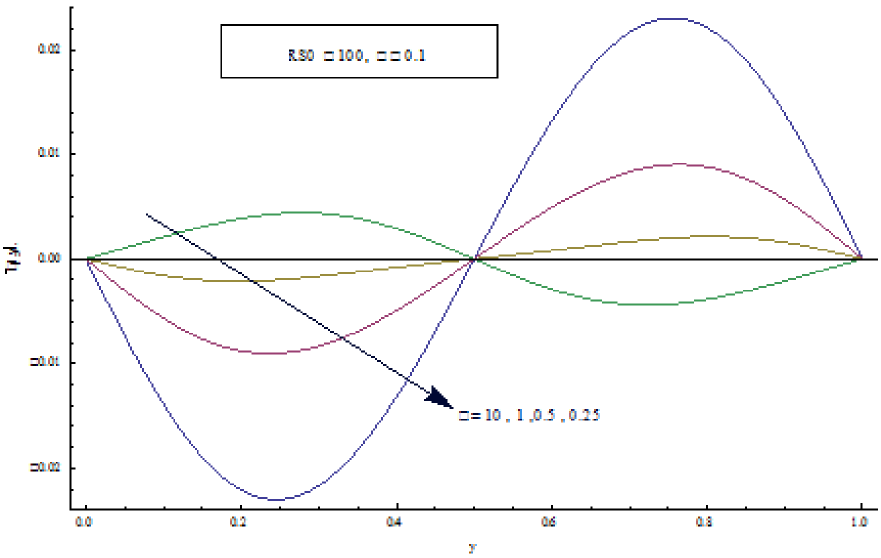

Figure: 18 Variation of $\bar{\theta}_1(y)$ with y for different values of σ

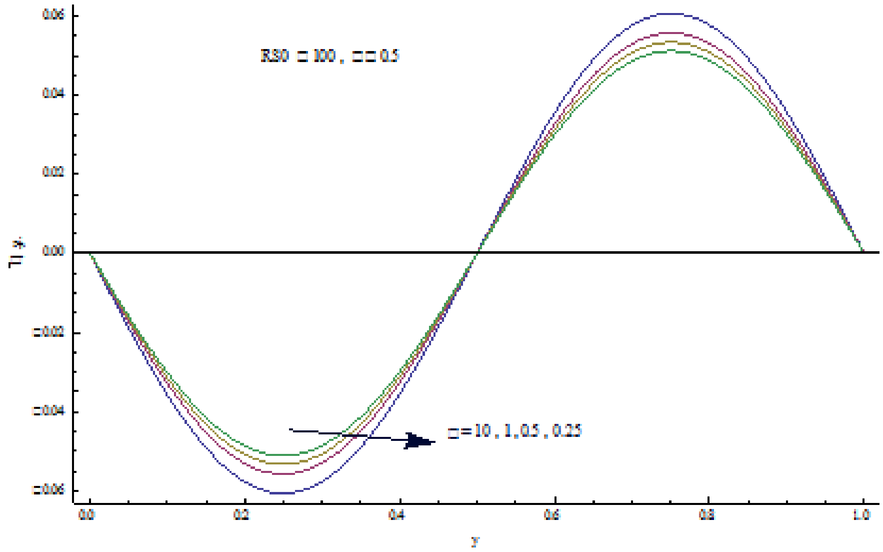

Figure: 19 Variation of $\bar{\theta}_1(y)$ with y for different values of σ

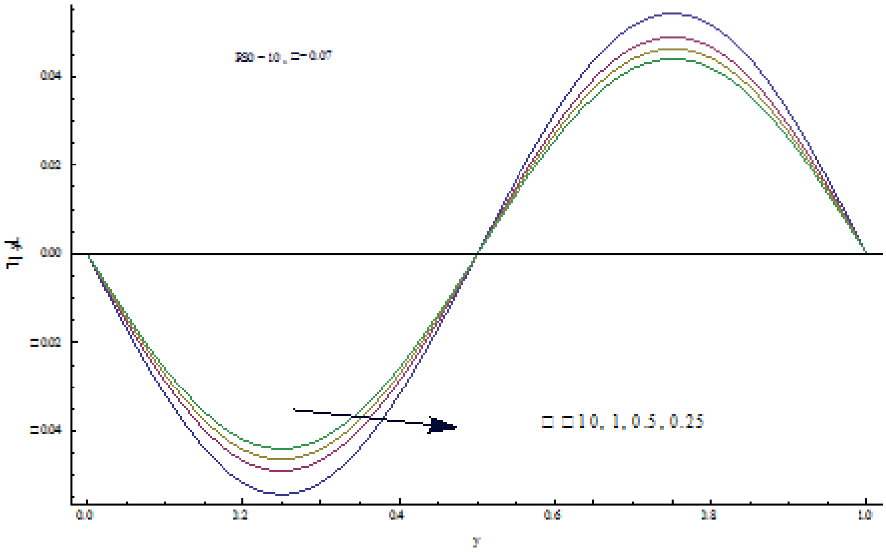

Figure: 20 Variation of $\bar{\theta}_1(y)$ with y for different values of σ

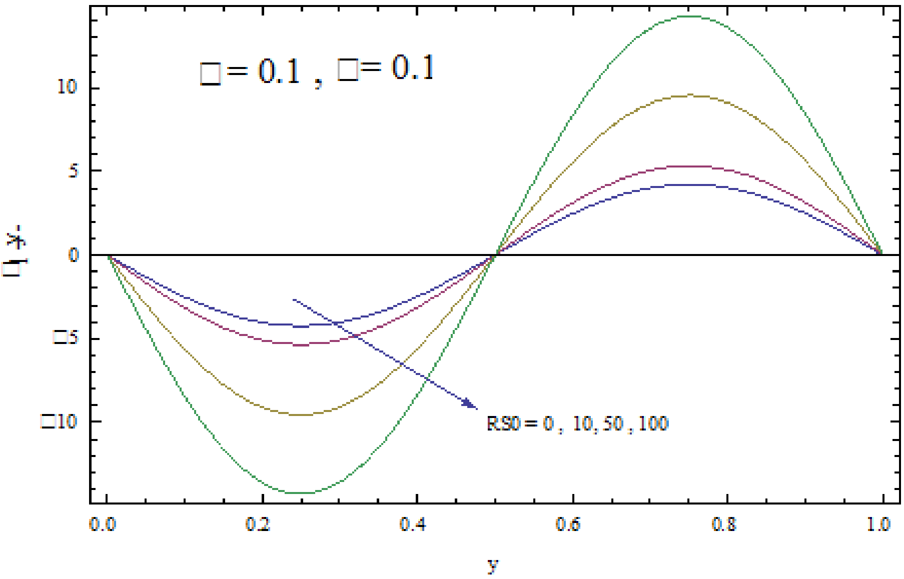

Figure: 21 Variation of $\bar{\phi}_1(y)$ with y for different values of RS0

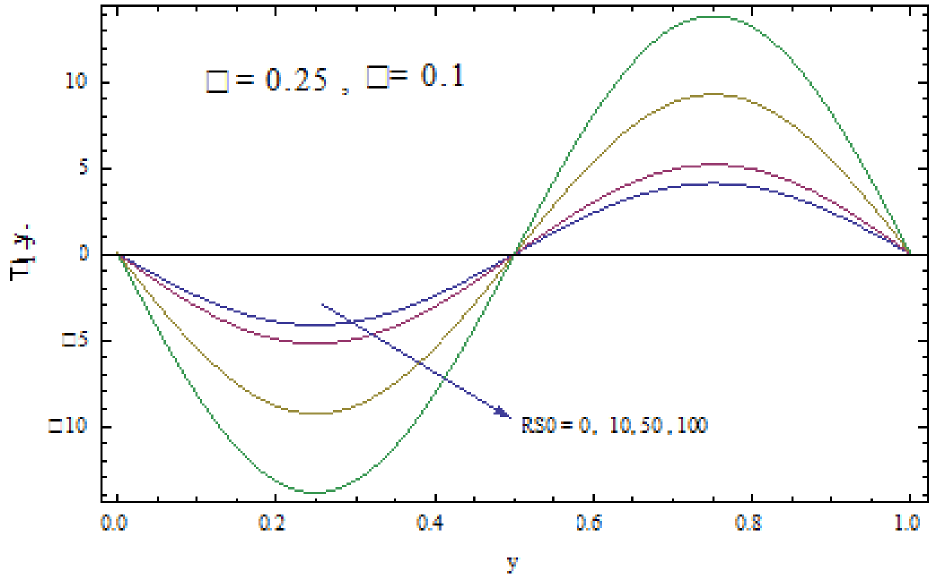

Figure: 22 Variation of $\bar{\phi}_1(y)$ with y for different values of RS0

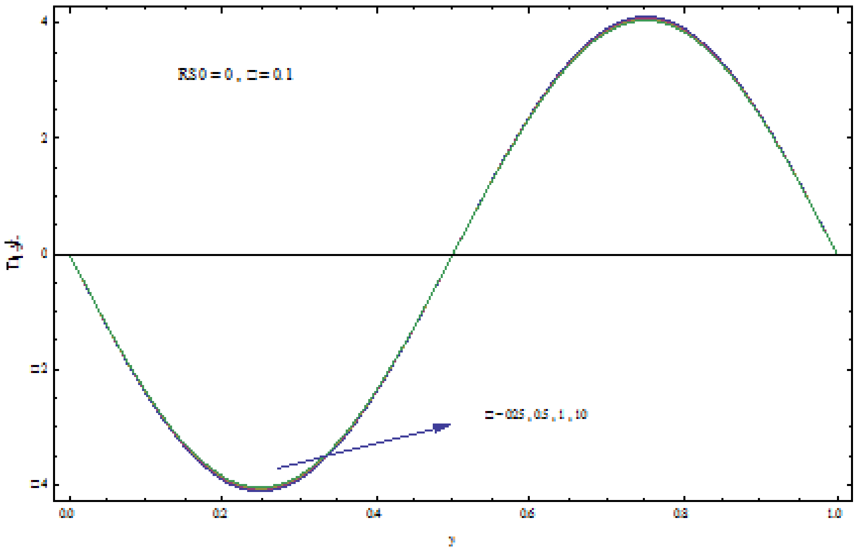

Figure: 23 Variation of $\bar{\phi}_1(y)$ with y for different values of σ

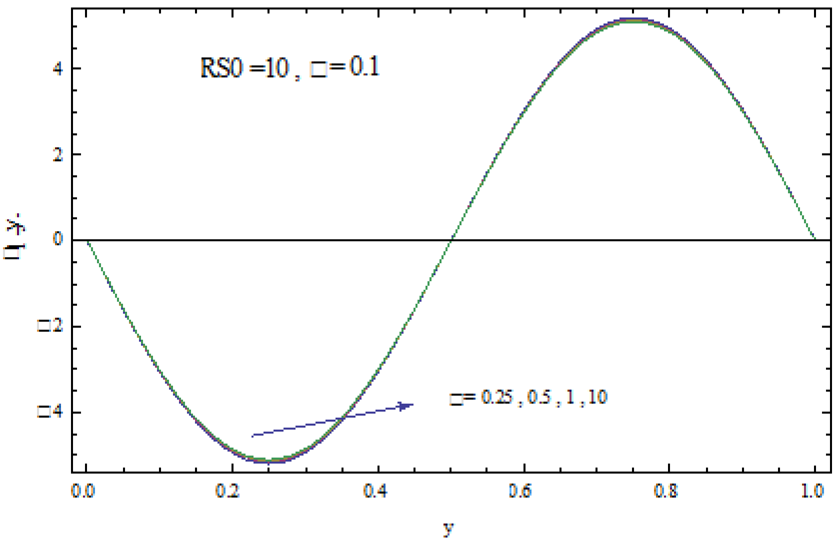

Figure: 24 Variation of $\bar{\phi}_1(y)$ with y for different values of σ

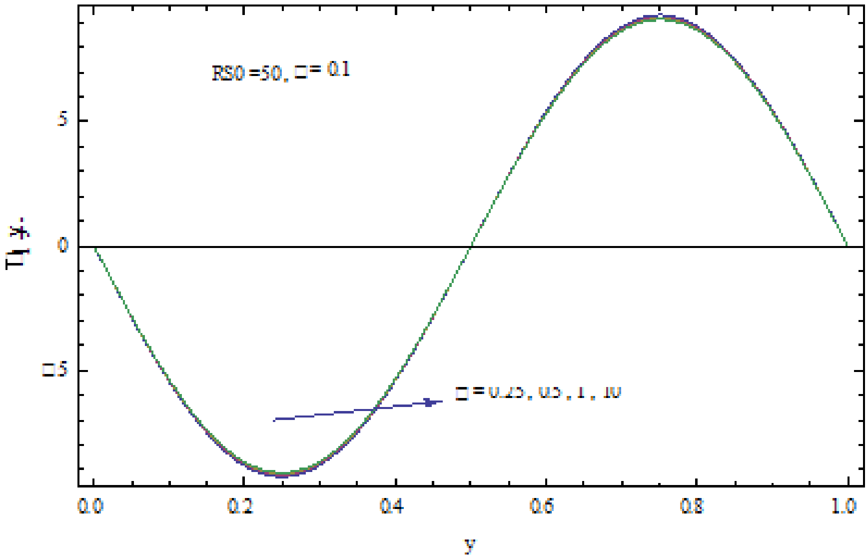

Figure: 25 Variation of $\bar{\phi}_1(y)$ with y for different values of σ

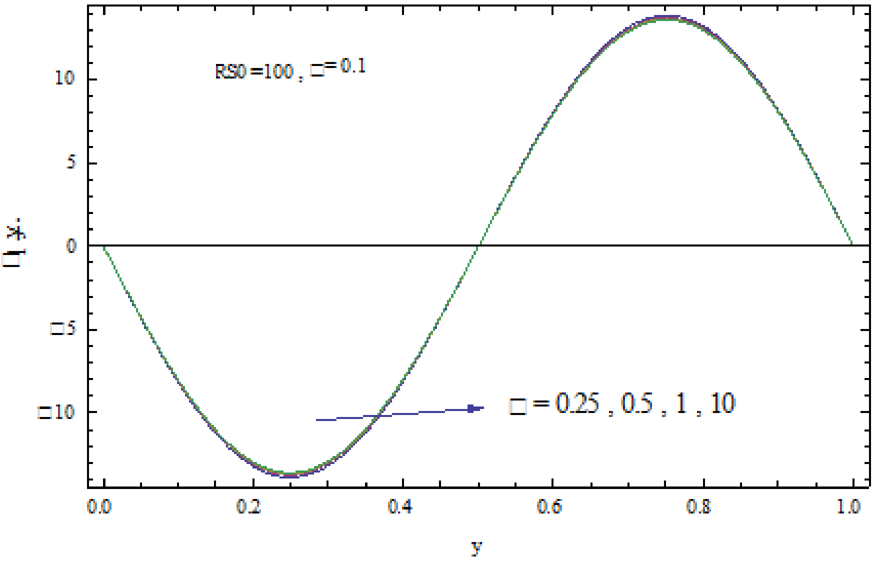

Figure: 26 Variation of $\bar{\phi}_1(y)$ with y for different values of σ

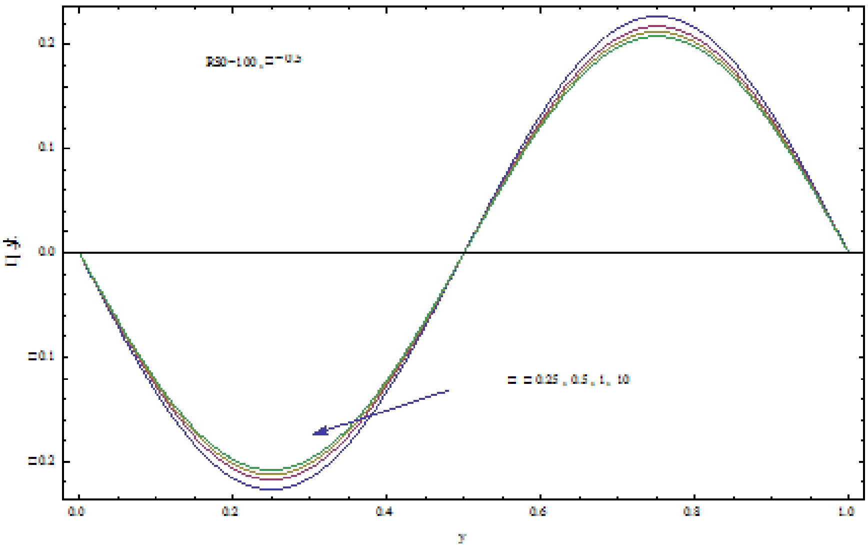

Figure: 27 Variation of $\bar{\phi}_1(y)$ with y for different values of σ

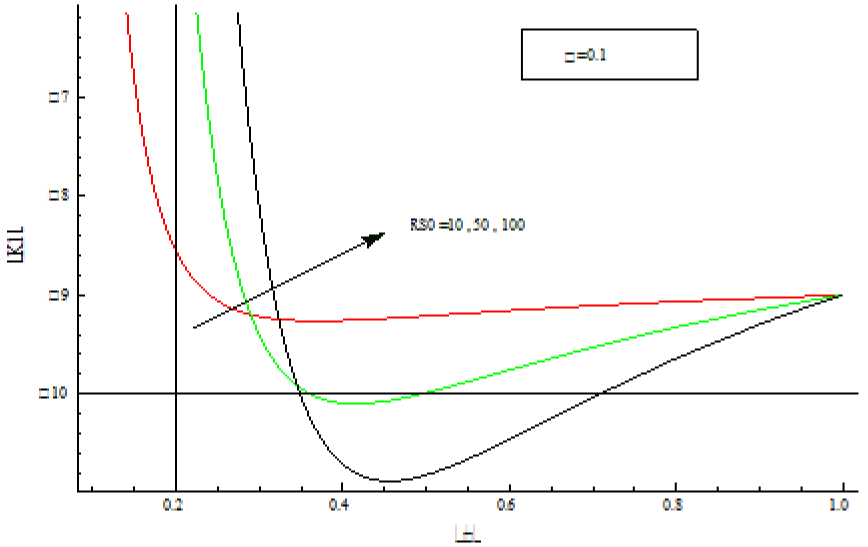

Figure: 28 Variation of K1 verses ratio of diffusivities

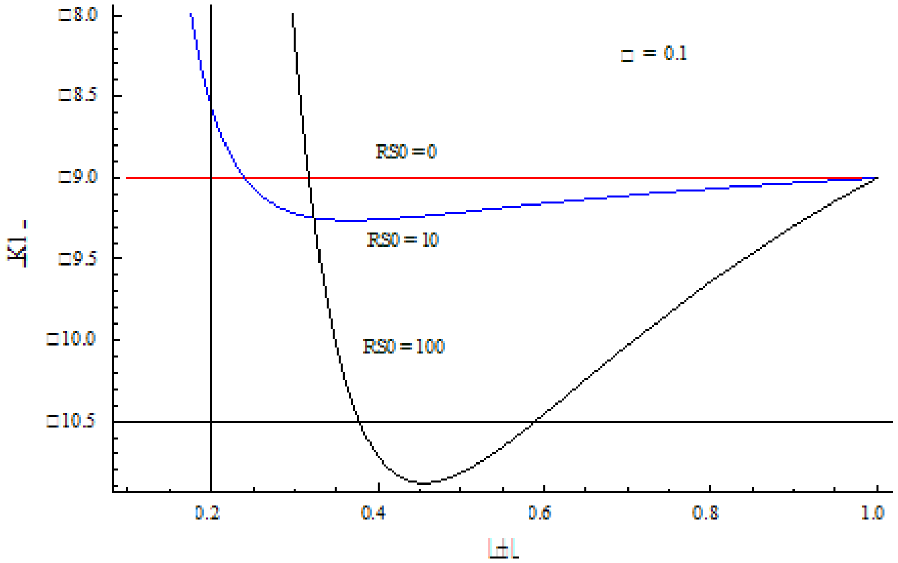

Figure: 29 Variation of K1 verses ratio of diffusivity

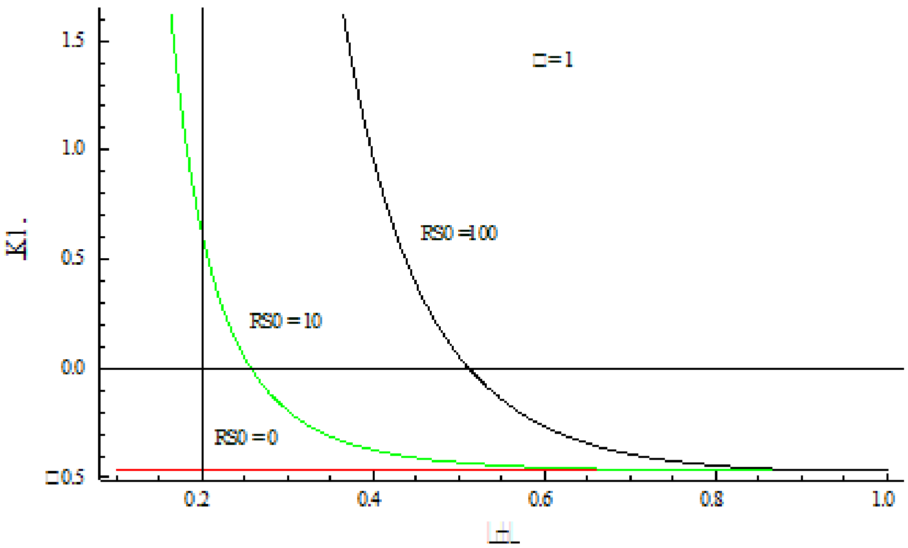

Figure: 30 Variation of K1 verses ratio of diffusivity

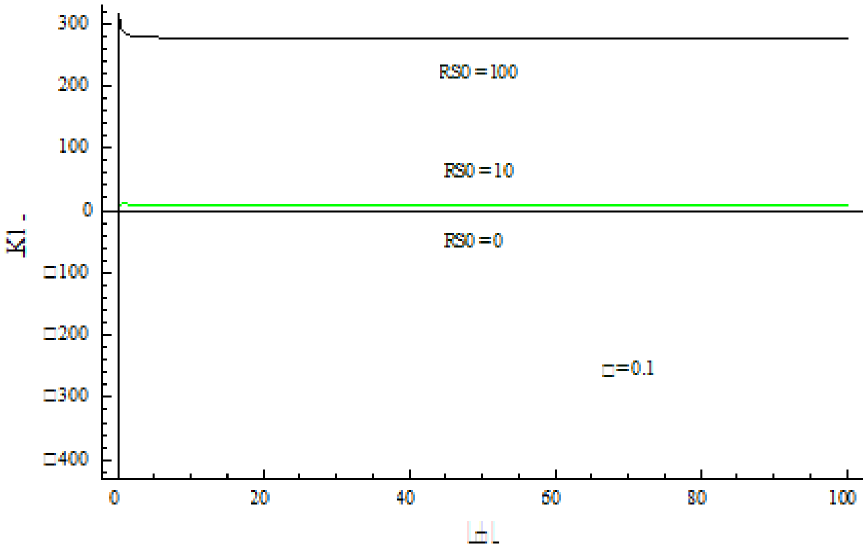

Figure: 31 Variation of K1 verses Prandtl number

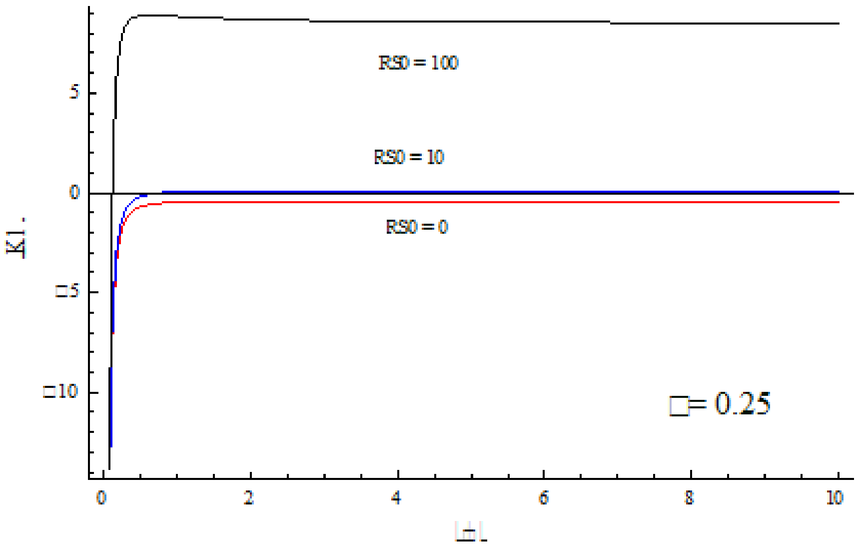

Figure: 32 Variation of K1 verses Prandtl number

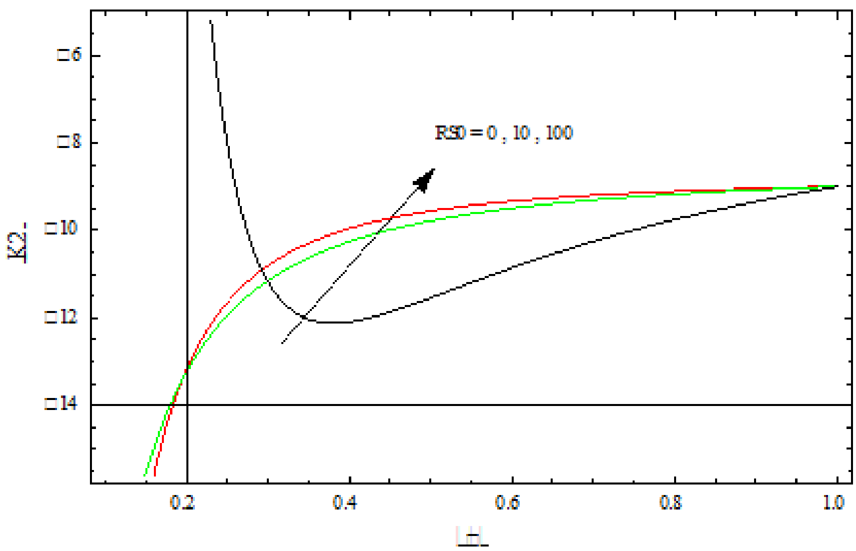

Figure: 33 Variation of K2 verses ratio of diffusivity

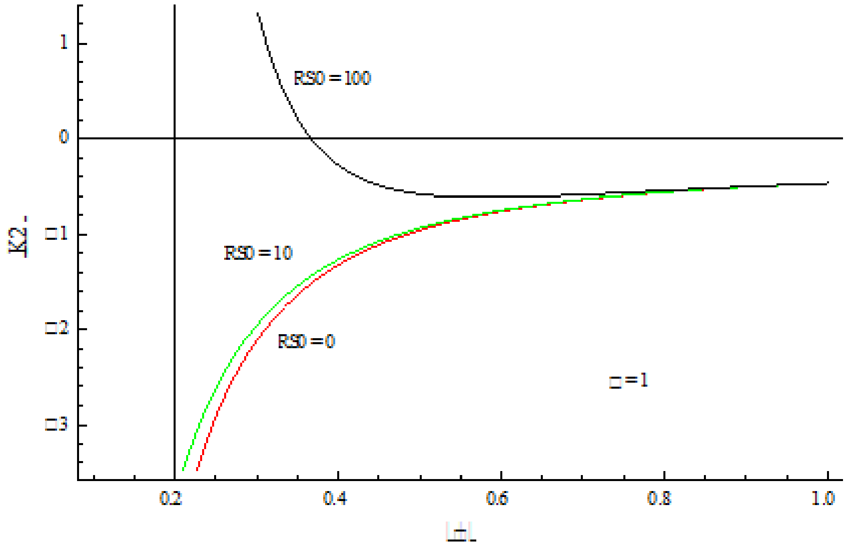

Figure: 34 Variation of K2 verses ratio of diffusivity

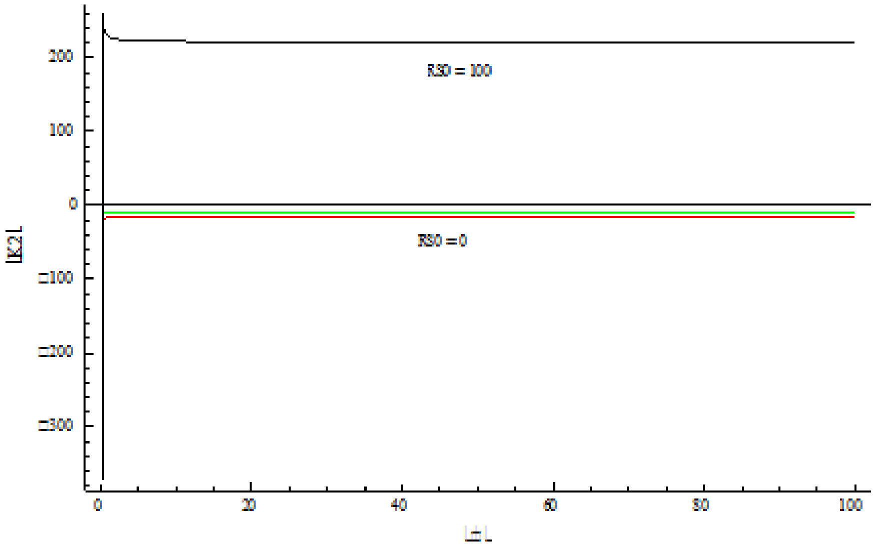

Figure: 35 Variation of K2 verses Prandtl number

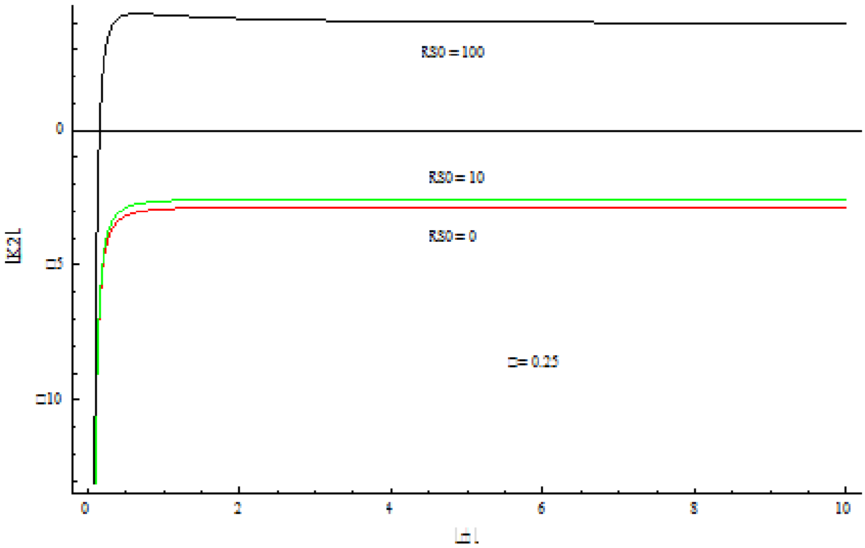

Figure: 36 Variation of K2 verses Prandtl number

REFERENCES

1. S. CHANDRASEKHAR, Hydrodynamic and Hydromagnetic Stability. Clarendon Press, Oxford (1961).
2. A. SCHLUTER, D. LORTZ and F. H. BUSSE, on the stability of steady finite amplitude convection, J. Fluid Mech. **23**, 129 (1965).
3. Baines, P.G. & Gill, A. E. On thermohaline convection with linear gradients. J. Fluid Mech. 37, 289-306.(1969)
4. Schechter, R.S., Velarde, M.G. & Platten, J. K. The two-component Benard problem. Adv. Chem. Phys. 26, 265-301.(1974).
5. J. S. Turner Double diffusive phenomena. Annual Review of Fluid Mechanics. 6, 37-54. (1974).
6. Stommel H. &Federov, K. N. Small-Scale structure in temperature and salinity near Timor and Mindanao. Tellus.19, 2, 306-325(1967).
7. E. PALM, On the tendency towards hexagonal cells in steady convection, J. Fluid Mech. **8**, 183 (1960).
8. ERNST SCHMIDT and P. L. SILVERSTONE, Natural convection in horizontal liquid layers, Chem. Engng Prog. Symp. Ser. **55**, No. 29, 163 (1959).
9. E. L. KOSCHMIEDER, On convection on a uniformly heated plane, Beitr. Phys. Atmos. **39**, 1 (1969).
10. C. Q. HOARD, C. R. ROBERTSON and A. ASTIVOS, Experiments on the cellular structure in Benard convection, Int. J. Heat and Mass Transfer. **13**, 849-856, 1970.
11. S. H. DAVIS, Convection in a box: Linear theory, J. Fluid Mech. **30**, 465 (1967).
12. L. A. SEGEL, Distant side-walls cause slow amplitude modulation of cellular convection, J. Fluid Mech. 38, 203 (1969).
13. C. M. VEST and V. S. ARPACI, Stability of natural convection in a vertical slot, J. Fluid Mech. **36**, 1 (1969).
14. A. P. GALLAGHER and A. McD. MERCER, On the behaviour of small disturbances in plane Coutte flow with a temperature gradient, Proc. R. Soc. Lond.**A286**, 117 (1965).
15. R. V. BIRIKH, G.Z. GERSHUNI, E. M. ZHUKHOVITSKII and R. N. RUDAKOV, Hydrodynamic and thermal instability of a steady convective flow, Prikl. Mat. Mekh.**32**, 256 (1968).
16. SCHMITT, R.W. The growth rate of super-critical salt fingers. Deep-Sea Res. 264, 23-40. (1979).
17. GREGG, M. (University of Washington, Seattle, W A 98105, U. S. A.)Measurements of oceanic double-diffusive events from a dropped platform.
18. GARGETT, A. E. (Institute of Ocean Sciences, Sidney, B. C. V8L4B2, Canada) Direct observation of an oceanic salt fingering interface.
19. STERN M. E. Inequalities and variational principles in double-diffusive turbulence. J. Fluid Mech. **114**, 105-121.
20. RUDDICK, B.R. & TURNER, J.S. The vertical length scale of double-diffusive intrusions. Deep-Sea Res. **26**, 903-913. (1979).
21. HOYLER, J. Y. On the collective stability of salt fingers. J. Fluid Mech. **110**, 195-207.
22. THORPE, S. A., HUTT, P. K. & SOULSBY, R.The effect of horizontal gradients on thermohaline convection. J. Fluid Mech., **38**, 375-400. (1969).
23. HART, J. E. On sideways diffusive instability. J. Fluid Mech. **49**, 279-288. (1971).
24. PALIWAL, R. C. & CHEN, C. F. Double-diffusive instability in an inclined fluid layer. Part 2.Theoritical investigation. J. Fluid Mech. **98**, 769-785. (1980).
25. THANGAM, S., ZEBIB, A. & CHEN, C. F. Double-diffusive convection in an inclined fluid layer. J. Fluid Mech. **116**, 363-378. (1982).
26. S. F. LIANG & A. ACRIVOS Stability of Buoyancy-driven convection in a tilted slot. Int. J. Heat Mass Transfer. **13**, 449-458. (1969).

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