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A RELATED FIXED POINT THEOREM OF INTEGRAL TYPE ON TWO SEMI - METRIC SPACES

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ABSTRACT

A related fixed point theorem for two pairs of mappings on two semi-metric spaces satisfying integral type inequality is obtained. The result extends a result of R.K. Namdeo, N.K. Tiwari, B. Fisher and K. Tas [6].

Keywords: *Symmetric space, semi-metric space, fixed point, related fixed point, integral type inequality.*

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1. INTRODUCTION

Many mathematicians have studied and developed a number of generalizations of a metric space which play a variety of roles in the study of fixed point theories. One of the generalizations is semi-metric space. Reference and background are given in [8]. Related fixed point theories are studied in [1- 6] and many others.

2. PRELIMINARIES

We recall some basic concepts.

Definition: 2.1 A symmetric function on a set X is a non-negative real valued function d on $X \times X$ such that for $x, y \in X$,
(i) $d(x, y) = 0$ if and only if $x = y$.
(ii) $d(x, y) = d(y, x)$.

Let d be a symmetric on set X. For $r > 0$ and $x \in X$, let $B(x, r) = \{ y \in X : d(x, y) < r\}$. A topology $\tau(d)$ on X is defined as $U \in \tau(d)$ if and only if for each $x \in U$, $B(x, r) \subseteq U$. A subset S of X is a neighbourhood of $x \in X$ if there exists $U \in \tau(d)$ such that $x \in U \subset S$.

Definition: 2.2 A symmetric d is semi-metric if for each $x \in X$ and for each $r > 0$, $B(x, r)$ is a neighbourhood of x in the topology $\tau(d)$.

We note that for every $\{x_n\} \subseteq X$ and $x \in X$, $\lim_{n\to\infty} d(x_n, x) = 0$ if and only if $x_n \to x$ in the topology $\tau(d)$.

Now, some axioms are stated as follows:

Let (X, d) be a semi-metric space. Then,

(W_3) Given $\{x_n\}$, x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y) = 0$ imply $x = y$.

(W_4) Given $\{x_n\}$, $\{y_n\}$ and x in X, $\lim_{n\to\infty} d(x_n, x) = 0$ and $\lim_{n\to\infty} d(x_n, y_n) = 0$ imply $\lim_{n\to\infty} d(y_n, x) = 0$

$(1C)$ A symmetric d on a set X is said to be 1–continuous if $\lim_{n\to\infty} d(x_n, x) = 0$ implies $\lim_{n\to\infty} d(x_n, y) = d(x, y) = 0$ for all $y \in X$.

(W_3), (W_4) and $(1C)$ are respectively found in [8], [8] and [7].

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Definition: 2.3 A sequence in X is *d*-Cauchy if it satisfies the usual metric condition with respect to d.

* (X, d) is (Σ) *d*-complete if for every sequence $\{x_n\}$, $\sum_{n=1}^{+\infty} d(x_n, x_{n+1}) < \infty$ implies that $\{x_n\}$ is convergent in the topology $\tau(d)$.

∗ (X, d) is *d*-Cauchy complete if for every *d*-Cauchy sequence $\{x_n\}$, there exists $x \in X$ with $x_n \to x$ in the topology $\tau(d)$.

∗ (X, d) is *S*-complete if for every *d*-Cauchy sequence $\{x_n\}$, there exists $x \in X$ with $\lim_{n\to\infty} d(x_n, x) = 0$.

The following was proved by R.K. Namdeo, N.K. Tiwari, B. Fisher and K. Tas in [6].

Theorem: 2.4 Let (X, d) and (Y, ρ) be complete metric spaces. Let T be a mapping of X into Y and S be a mapping of Y into X satisfying the inequalities

$$d(Sy, Sy')\, d(STx, STx') \le c \max\{ d(Sy, Sy')\, \rho(Tx, Tx'),\ d(x', Sy)\, \rho(y', Tx)\ d(x, x')\, d(Sy, Sy'),\ d(Sy, STx)\, d(Sy', STx')\}$$

$$\rho(Tx, Tx')\, \rho(TSy, TSy') \le c \max\{ d(Sy, Sy')\, \rho(Tx, Tx'),\ d(x', Sy)\, \rho(y', Tx),\ \rho(y, y')\, \rho(Tx, Tx'),\ \rho(Tx, TSy)\, \rho(Tx', TSy')\}$$

for all x, x' in X and y, y' in Y, where $0 \le c < 1$. If either S or T is continuous, then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Now, theorem 2.4 is extended to two pairs of mappings on semi-metric spaces in integral setting as follows.

3. MAIN RESULT

Theorem: 3.1 Let (X, d) and (Y, υ) be two 1- continuous semi-metric spaces. Let A, B be mappings of X into Y and S, T be mappings of Y into X satisfying the inequalities

$$\int_0^{d(Sy,\,Ty')\,d(SAx,\,TBx')} \varphi(t)\,dt \le \int_0^{c\max\{d(Sy,\,Ty')\,\upsilon(Ax,\,Bx'),\,d(x',\,Sy)\,\upsilon(y',\,Ax),\ d(x,\,x')\,d(Sy,\,Ty'),\,d(Sy,\,SAx)\,d(Ty',\,TBx')\}} \varphi(t)\,dt \qquad (1)$$

$$\int_0^{\upsilon(Ax,\,Bx')\,\upsilon(BSy,\,ATy')} \varphi(t)\,dt \le \int_0^{c\max\{d(Sy,\,Ty')\,\upsilon(Ax,\,Bx'),\,d(x',\,Sy)\,\upsilon(y',\,Ax),\ \upsilon(y,\,y')\,\upsilon(Ax,\,Bx'),\,\upsilon(Ax,\,BSy)\,\upsilon(Bx',\,ATy')\}} \varphi(t)\,dt \qquad (2)$$

for all x, x' in X and y, y' in Y, where $0 \le c < 1$. If either X is (Σ) *d*- complete and Y satisfies (W_4) or Y is $(\Sigma)\,\upsilon$- complete and X satisfies (W_4) and one of the mappings A, B, S and T is continuous, then SA and TB have a unique common fixed point z in X and BS and AT have a unique common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof: Let y_1 be any arbitrary point in X. We define sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively as follows:

$$Sy_{2n-1} = x_{2n-1},\ Bx_{2n-1} = y_{2n},\ Ty_{2n} = x_{2n},\ Ax_{2n} = y_{2n-1}, \text{ for } n = 1, 2, 3, \ldots$$

Applying inequality (1), we get

$$\int_0^{d(Sy_{2n-1},\,Ty_{2n})\,d(SAx_{2n},\,TBx_{2n-1})} \phi(t)\,dt = \int_0^{d^2(x_{2n-1},\,x_{2n})} \phi(t)dt$$

$$\le \int_0^{c\max\{d(Sy_{2n-1},\,Ty_{2n})\,\upsilon(Ax_{2n},\,Bx_{2n-1}),\ d(x_{2n-1},\,Sy_{2n-1})\,\upsilon(y_{2n},\,Ax_{2n}),\ d(x_{2n},\,x_{2n-1})\,d(Sy_{2n-1},\,Ty_{2n}),\ d(Sy_{2n-1},\,SAx_{2n})\,d(Ty_{2n},\,TBx_{2n-1})\}} \varphi(t)\,dt$$

$$= \int_0^{c\max\{d(x_{2n-1},x_{2n})\,\upsilon(y_{2n-1},y_{2n}),\, d(x_{2n-1},x_{2n-1})\upsilon(y_{2n},y_{2n-1}),\, d(x_{2n},x_{2n-1})d(x_{2n-1},x_{2n}),\, d(x_{2n-1},x_{2n-1})d(x_{2n},x_{2n})\}} \phi(t)\,dt$$

From which it follows that

$$\int_0^{d(x_{2n-1},\,x_{2n})} \varphi(t)\,dt \le \int_0^{c\max\{\upsilon(y_{2n-1},y_{2n}),d(x_{2n-1},x_{2n})\}} \varphi(t)\,dt \tag{3}$$

Applying inequality (2), we get

$$\int_0^{\upsilon(Ax_{2n},Bx_{2n-1})\,\upsilon(BSy_{2n-1},ATy_{2n})} \varphi(t)\,dt = \int_0^{\upsilon^2(y_{2n-1},\,y_{2n})} \varphi(t)\,dt$$

$$\le \int_0^{c\max\{d(Sy_{2n-1},Ty_{2n})\,\upsilon(Ax_{2n},Bx_{2n-1}),\,d(x_{2n-1},Sy_{2n-1})\,\upsilon(y_{2n},Ax_{2n}),\,\upsilon(y_{2n-1},y_{2n})\,\upsilon(Ax_{2n},Bx_{2n-1}),\,\upsilon(Ax_{2n},BSy_{2n-1})\,\upsilon(Bx_{2n-1},ATy_{2n})\}} \varphi(t)\,dt$$

$$= \int_0^{c\max\{d(x_{2n-1},x_{2n})\,\upsilon(y_{2n-1},y_{2n}),\,d(x_{2n-1},x_{2n-1})\,\upsilon(y_{2n},y_{2n-1}),\,\upsilon(y_{2n-1},y_{2n})\,\upsilon(y_{2n-1},y_{2n}),\,\upsilon(y_{2n-1},y_{2n})\,\upsilon(y_{2n},y_{2n-1})\}} \varphi(t)\,dt$$

From which it follows that

$$\int_0^{\upsilon(y_{2n-1},\,y_{2n})} \varphi(t)\,dt \le \int_0^{c\max\{\upsilon(y_{2n-1},y_{2n}),d(x_{2n-1},x_{2n})\}} \varphi(t)\,dt \tag{4}$$

(3) and (4) can be written as

$$\int_0^{d(x_{n-1},\,x_n)} \varphi(t)\,dt \le \int_0^{c\max\{\upsilon(y_{n-1},y_n),d(x_{n-1},x_n)\}} \varphi(t)\,dt$$

$$\int_0^{\upsilon(y_{n-1},\,y_n)} \varphi(t)\,dt \le \int_0^{c\max\{\upsilon(y_{n-1},y_n),d(x_{n-1},x_n)\}} \varphi(t)\,dt$$

which can be again written as

$$\int_0^{d(x_{n+1},\,x_n)} \varphi(t)\,dt \le \int_0^{c\max\{\upsilon(y_{n+1},y_n),d(x_{n+1},x_n)\}} \varphi(t)\,dt \tag{5}$$

$$\int_0^{\upsilon(y_{n+1},\,y_n)} \varphi(t)\,dt \le \int_0^{c\max\{\upsilon(y_{n+1},y_n),d(x_{n+1},x_n)\}} \varphi(t)\,dt \tag{6}$$

From (5) and (6), by induction, we get

$$\max\{\int_0^{d(x_{n+1},\,x_n)} \varphi(t)\,dt\,,\ \int_0^{\upsilon(y_{n+1},\,y_n)} \varphi(t)\,dt\ \} \le \int_0^{c^n\max\{\upsilon(y_1,\,y_2),d(x_1,\,x_2)\}} \varphi(t)\,dt$$

$$= \int_0^{c^n M_{d,\upsilon}} \varphi(t)\,dt,$$

where $M_{d,\upsilon} = \max\{\upsilon(y_1,\,y_2),d(x_1,\,x_2)\}$

Therefore, $\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} \upsilon(y_{n+1}, y_n) = 0$.

Suppose that X is (Σ) d- complete. We have,

$$\sum_{k=1}^{n} d(x_k, x_{k+1}) \le M_{d,\upsilon} \sum_{k=1}^{n} c^k, \quad n \ge 1.$$

which implies that $\sum_{k=1}^{+\infty} d(x_k, x_{k+1}) < \infty$. Therefore, $x_n \to z$ for some $z \in X$. Let A be continuous and $w = Az$. Then, $\lim_{n\to\infty} \upsilon(y_{2n-1}, w) = \lim_{n\to\infty} \upsilon(Ax_{2n}, Az) = 0$ and therefore, $\lim_{n\to\infty} \upsilon(y_{2n}, w) = 0$ since

$\lim_{n\to\infty} \upsilon(y_{2n-1}, y_{2n}) = 0$ and Y satisfies (W_4). Hence, $\lim_{n\to\infty} \upsilon(y_n, w) = 0$.

Using (3), we have

$$\int_0^{d(Sw, x_{2n})} \varphi(t)dt \le \int_0^{c\max\{\upsilon(y_{2n-1}, y_{2n}), d(Sw, z)\}} \varphi(t)dt$$

On letting $n \to \infty$ and using 1 – continuity of d, we have

$$\int_0^{d(Sw, z)} \varphi(t)dt \le \int_0^{c\, d(Sw, z)} \varphi(t)dt$$

$$\Rightarrow d(Sw, z) \le c\, d(Sw, z)$$

which implies that

$Sw = z = SAz$

Using (4), we have

$$\int_0^{\upsilon(y_{2n-1}, Bz)} \varphi(t)dt \le \int_0^{c\max\{d(x_{2n-1}, x_{2n}), \upsilon(w, Bz)\}} \varphi(t)dt$$

On letting $n \to \infty$ and using 1 – continuity of υ, we have

$$\int_0^{\upsilon(w, Bz)} \varphi(t)dt \le \int_0^{c\, \upsilon(w, Bz)} \varphi(t)dt$$

$$\Rightarrow \upsilon(w, Bz) \le c\, \upsilon(w, Bz)$$

which implies that

$Bz = w = BSw$

Again, using (3), we have

$$\int_0^{d(z, Tw)} \varphi(t)dt \le \int_0^{c\max\{\upsilon(w, w), d(Tw, z)\}} \varphi(t)dt$$

$$\Rightarrow d(z, Tw) \le c\, d(Tw, z)$$

which implies that

$Tw = z = TBz$

As, Az = w, we have

$ATw = w$.

The same results hold if one of the mappings B, S and T is continuous instead of A.

To prove uniqueness, let SA and TB have a second fixed point z' in X.

On using (3), we have

$$\int_0^{d(z,z')} \varphi(t)dt \le \int_0^{c\max\{\upsilon(w,w'),\, d(z,z')\}} \varphi(t)dt$$

which implies that

$$\int_0^{d(z,z')} \varphi(t)dt \le \int_0^{c\,\upsilon(w,w')} \varphi(t)dt \tag{7}$$

On using (4), we have

$$\int_0^{\upsilon(w,w')} \varphi(t)dt \le \int_0^{c\max\{d(z,z'),\, \upsilon(w,w')} \varphi(t)dt$$

which implies that

$$\int_0^{\upsilon(w,w')} \varphi(t)dt \le \int_0^{c\,d(z,z')} \varphi(t)dt \tag{8}$$

From (7) and (8), we have

$$\int_0^{d(z,z')} \varphi(t)dt \le \int_0^{c\,\upsilon(w,w')} \varphi(t)dt \le \int_0^{c^2 d(z,z')} \varphi(t)dt$$

$$\Rightarrow \int_0^{d(z,z')} \varphi(t)dt \le \int_0^{c^2 d(z,z')} \varphi(t)dt$$

$$\Rightarrow \int_0^{d(z,z')} \varphi(t)dt \le \int_0^{c^2 d(z,z')} \varphi(t)dt \le \dots \le \int_0^{c^n d(z,z')} \varphi(t)dt$$

$$\Rightarrow 0 \le \int_0^{d(z,z')} \varphi(t)dt \le \lim_{n\to\infty} \int_0^{c^n d(z,z')} \varphi(t)dt$$

$$\Rightarrow d(z, z') = 0.$$

which implies that

$z = z'$

This proves the uniqueness of z. Similarly, the uniqueness of w can be proved. This completes the proof.

Corollary: 3.2 Let (X, d) and (Y, υ) be two 1-complete semi-metric spaces. Let S be a mapping of X into Y and T be a mapping of Y into X satisfying the inequalities

$$\int_0^{d(Ty,\, Ty')\, d(TSx,\, TSx')} \varphi(t)\,dt \le \int_0^{\substack{c\max\{d(Ty,\, Ty')\,\upsilon(Sx,\, Sx'),\, d(x',Ty)\,\upsilon(y',Sx) \\ d(x,x')\,d(Ty,\, Ty'),\, d(Ty,TSx)\,d(Ty',TSx')\}}} \varphi(t)\,dt$$

$$\int_0^{\upsilon(Sx,Sx')\,\upsilon(STy,\, STy')} \varphi(t)\,dt \le \int_0^{\substack{c\max\{d(Ty,\, Ty')\,\upsilon(Sx,\, Sx'),\, d(x',\, Ty)\,\upsilon(y',\, Sx) \\ \upsilon(y,y')\,\upsilon(Sx,\, Sx'),\, \upsilon(Sx,\, STy)\,\upsilon(Sx',\, STy')\}}} \varphi(t)\,dt$$

for all x, x' in X and y, y' in Y, where $0 \le c < 1$. If either X is (Σ) d- complete and Y satisfies (W_4) or Y is (Σ) υ- complete and X satisfies (W_4) and one of the mappings S and T is continuous, then TS has a unique fixed point z in X and ST has a unique fixed point w in Y. Further, $Sz = w$ and $Tw = z$.

Proof: Putting A = B = S and S = T = T in theorem 3.1, we can obtain the result.

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