International Journal of Mathematical Archive-4(11), 2013, 134-146
Available online through www.ijma.info ISSN 2229 – 5046

# USING TAYLOR EXPANSION TO PARTIAL FRACTIONS DECOMPOSITION

**Mircea Ion Cîrnu***
*Polytechnic University of Bucharest, Romania.*

*(Received on: 26-09-13; Revised & Accepted on: 11-11-13)*

***ABSTRACT***

*The automatic Taylor development, based on discrete convolution and deconvolution, is used to partial fractions decomposition.*

***Keywords and phrases:*** *Differential Taylor transform, discrete linear convolution and deconvolution, partial fractions decomposition.*

***Mathematics Subject Classification:*** *26C15, 41A58, 30B10.*

## 1. INTRODUCTION

Using the automatic Taylor expansion, we give some new numerical methods to partial fractions decomposition of rational functions. Firstly, we consider the Taylor differential transform that associates to a function the sequence of coefficients of its Taylor series or only a finite number of them. The Taylor transform of a product, respective quotient of functions is computed by discrete convolution, respective deconvolution, of the corresponding sequences of coefficients. This eliminates the algebraic calculation and the Taylor development becomes automatic.

As is well known, partial fractions decomposition of rational functions applied to the integration of these functions. Also it is used to determine the inverse Laplace transforms of such functions and thus to physical system theory. So far, there are numerous attempts to find a simple numerical method for this decomposition. In the paper [7] are presented nine such different methods. Using the automatic Taylor expansion, we give a new method for partial fractions decomposition of rational functions. It extend so called "particular values method". These particular values, used to obtain the numerators of the decomposition, are even the poles of the rational function. When the poles are complex numbers, must work in complex, but there are cases in which it is possible to work only with real numbers. See Example 8.3 below. Our method of partial fractions decomposition is simpler than the already known methods, covers all cases and can be applied even to difficult examples. We give two theorems, one for a single pole of arbitrary multiplicity and the second, for a pair of poles of the same multiplicity. The second theorem can be used in the case of two complex or irrational conjugate poles of the same multiplicity. In the latest example 8.6, we do the partial fractions decomposition by both theorems. Based on the convenient decomposition, we compute the indefinite integral and the inverse Laplace transform of the considered function in that example.

Other applications of the discrete convolution and deconvolution were given by the Author in the works [1]-[5].

## 2. TAYLOR DIFFERENTIAL TRANSFORM

We call *Taylor differential transform* of an indefinite differentiable function $f(x)$, the sequence of functions $T(f(x))=\left(\frac{1}{n!}f^{(n)}(x): n=0,1,2\ldots\right)$ and *Taylor transform in a point* $x=x_0$, the sequence of numbers $T_{x_0}(f(x))=(c_n)$, where $c_n=\frac{1}{n!}f^{(n)}(x_0)$, $n=0,1,\ldots$. For a natural number $m$, we also will denote $T^m(f(x))=\left(\frac{1}{n!}f^{(n)}(x): n=0,1,\ldots,m\right)$, $T_{x_0}^m(f(x))=(c_n : n=0,1,\ldots,m)$ and $P_{x_0}^m(f(x))=P_{x_0}^m(c_n)=\sum_{n=0}^{m}c_n(x-x_0)^n$, the Taylor polynomial of degree m.

***Corresponding author: Mircea Ion Cîrnu****

If we denote $ab=(a_n b_n)$ the *usual product* of two sequences $a=(a_n)$ and $b=(b_n)$, the derivatives of the function $f(x)$ are given by the formula $f^{(n)}(x)=(n!)T(f(x))$ and there values in the point $x=x_0$ by the formula $f^{(n)}(x_0)=(n!)T_{x_0}(f(x))$.

**3. EXAMPLES OF TAYLOR TRANSFORMS**

Some examples of Taylor transforms in an arbitrary point and in origin, are: $T(a)=(a,0,0,\ldots)$, where $a$ is a constant function.

$$T(x^m)=\left(x^m, mx^{m-1},\ldots,\underset{n}{\binom{m}{n}x^{m-n}},\ldots,mx,\underset{m}{1},0,0,\ldots\right), \text{ for } m=1,2,\ldots.$$

Particularly, $T(x)=(x,1,0,0,\ldots)$, and $T(x^2)=(x^2,2x,1,0,0,\ldots)$.

$$T\left(\frac{1}{x}\right)=\left(\frac{1}{x},-\frac{1}{x^2},\frac{1}{x^3},-\frac{1}{x^4},\ldots,\frac{(-1)^n}{x^{n+1}},\ldots\right), \text{ for } x\neq 0,$$

$$T(e^x)=e^x\left(1,1,\frac{1}{2},\frac{1}{3!},\ldots,\frac{1}{n!},\ldots\right)$$

$$T(\ln x)=\left(\ln x,\frac{1}{x},-\frac{1}{2x^2},\frac{1}{3x^3},-\frac{1}{4x^4},\ldots,\frac{(-1)^{n-1}}{nx^n},\ldots\right), \text{ for } x>0.$$

$$T(\sin x)=\left(\sin x,\cos x,-\frac{1}{2}\sin x,-\frac{1}{3!}\cos x,\frac{1}{4!}\sin x,\ldots\right),$$

$$T(\cos x)=\left(\cos x,-\sin x,-\frac{1}{2}\cos x,\frac{1}{3!}\sin x,\frac{1}{4!}\cos x,\ldots\right).$$

Particularly, $T_0(a)=(a,0,0,\ldots)$, $T_0(x)=(0,1,0,0,\ldots)$, $T_0(x^2)=(0,0,1,0,0,\ldots)$,

$$T_0(x^m)=(0,\ldots,0,\underset{m}{1},0,0,\ldots),\ T_0(e^x)=\left(1,1,\frac{1}{2},\frac{1}{3!},\ldots,\frac{1}{n!},\ldots\right),$$

$$T_0(\sin x)=\left(0,1,0,-\frac{1}{3!},0,\frac{1}{5!},\ldots\right),\ T_0(\cos x)=\left(1,0,-\frac{1}{2},0,\frac{1}{4!},\ldots\right).$$

**Remark:** These formulas was given in the author book [1], pg. 84. However, unfortunately, in [1] the two formulas for $T\left(\frac{1}{x}\right)$ and $T(\ln x)$ contain mistakes.

**4. OPERATIONS WITH SEQUENCES**

We consider (see [1] and [6]) the *Cauchy product* or (*truncated or short, linear discrete*) *convolution* $c=(c_n)=a*b=\left(\sum_{k=0}^{n}a_k b_{n-k}\right)$ of the sequences $a=(a_n)$ and $b=(b_n)$.

The convolution can be computed by the multiplication algorithm

$$\begin{array}{llll}
a_0 & a_1 & a_2 & \cdots\\
b_0 & b_1 & b_2 & \cdots\\
\hline
a_0b_0 & a_1b_0 & a_2b_0 & \cdots\\
 & a_0b_1 & a_1b_1 & \cdots\\
 & & a_0b_2 & \cdots\\
\hline
c_0=a_0b_0 & c_1=a_1b_0+a_0b_1 & c_2=a_2b_0+a_1b_1+a_0b_2 & \cdots
\end{array}$$

For $m$ natural numbers one denotes $a^{*m} = \underbrace{a * a * \cdots * a}_{m}$.

If the sequences $a = (a_n)$ with $a_0 \neq 0$ and $c = (c_n)$ are given, then the sequence $b = (b_n)$ such that $c = a * b$ can be determined by the *deconvolution* formula (see [1] and [6])

$$b = c / a = \frac{1}{a_0}\left(c_0, c_n - \sum_{k=1}^{n} a_k b_{n-k} : n = 1,2,\ldots\right).$$

The deconvolution can be computed by the division algorithm

$$\begin{array}{ccccc|cccc} c_0 & c_1 & \cdots & c_n & \cdots & a_0 & a_1 & \cdots\ a_n & \cdots \\ \hline \underline{c_0} & \underline{a_1 b_0} & \underline{\cdots} & \underline{a_n b_0} & \underline{\cdots} & b_0 = \dfrac{c_0}{a_0} & b_1 = \dfrac{c_1 - a_1 b_0}{a_0} & \cdots\ b_n & \cdots \\ / & c_1 - a_1 b_0 & \cdots & c_n - a_n b_0 & \cdots & & & & \\ & \underline{c_1 - a_1 b_0} & \underline{\cdots} & \underline{a_n b_1} & \underline{\cdots}\ / & \cdots & & & \end{array}$$

On the base of the usual (long) convolution and deconvolution, which are used to compute the product and the quotient (with rest) of two polynomials, introduced in MATLAB by the instructions $conv$ and $deconv$, it is possible also to consider the truncated convolution and deconvolution by the instructions $tconv$ and $tdeconv$, in the following manner:

```
function  res = tconv(a,b)
l = size(a);
temp = conv(a,b);
res = temp(1,1:l(2));
end
```

```
function  res = tdeconv(a,b)
[l,c] = size(a);
res = zeros(1,c);
for  i = 1:c
   [q r] = deconv(a,b(1,1:c-i+1));
   res(i) = q;
   if  (i < c)
      a = r(1,2:c-i+1);
   end
end
end
```

In the examples of partial fractions decomposition given below, we present several algorithms of convolution and deconvolution, the others being omitted.

## 5. TAYLOR TRANSFORMS OF PRODUCTS AND QUOTIENS OF FUNCTIONS

If a function $f(x)$ contains products and quotients of other functions, then to determine its Taylor transform we can use the discrete convolution and deconvolution, as can see from the following theorem and its consequence.

**Theorem 1:** *If* $f(x)$ *and* $g(x)$ *are indefinite differentiable functions, then*

$$T(f(x)g(x)) = T(f(x)) * T(g(x)). \tag{1}$$

**Proof:** Using Leibniz formula for derivatives of a product of functions,

$$T(f(x)g(x)) = \left(\frac{1}{n!}\left(f(x)g(x)\right)^{(n)}\right) = \left(\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} f^{(k)}(x) g^{(n-k)}(x)\right)$$

$$= \left(\sum_{k=0}^{n}\frac{1}{k!} f^{(k)}(x)\frac{1}{(n-k)!} g^{(n-k)}(x)\right) = \left(\frac{1}{n!} f^{(n)}(x)\right) * \left(\frac{1}{n!} g^{(n)}(x)\right) = T(f(x)) * T(g(x))$$

.

 

**Corollary:** *If* $f(x)$ *and* $g(x) \neq 0$ *are indefinite differentiable functions, then*

$$T(f(x)/g(x)) = T(f(x))/T(g(x)). \tag{2}$$

**Proof:** From theorem, $T(f(x)) = T((f(x)/g(x))g(x)) = T(f(x)/g(x)) * T(g(x))$, and therefore it results the above formula.

**Remarks:** 1) Obviously, the formulas from above Theorem and its Corollary, are also true for Taylor transformation

$T_{x_0}$ Concentrated in a point $x = x_0$.

2) If we want to calculate only derivatives up to order $m$, then are used the Taylor transforms $T^m(f(x))$ and $T^m(g(x))$, sequences of the same finite length $m+1$.

## 6. PARTIAL FRACTIONS DEVELOPMENT

We will use the automatic Taylor development, based on discrete convolution and deconvolution, to get a numerical automatic method for partial fractions development of the rational functions. It is based on the following two theorems.

**Theorem: 2** *If* $f(x)$ *is a function of the form*

$$f(x) = \sum_{k=0}^{m-1} \frac{A_k}{(x-a)^{m-k}} + g(x) = \sum_{k=0}^{m-1} A_k (x-a)^k \frac{1}{(x-a)^m} + g(x), \tag{3}$$

*where* $m \neq 0$ *is a natural number,* $a$ *and* $A_k$ *, for* $k = 0,1,\ldots,m-1$, *are complex numbers and* $g(x)$ *is a function with derivative of* $m-1$ *order in* $a$ *, then the coefficients* $A_k$ *can be computed by the relation*

$$A_k = \frac{1}{k!} \lim_{x \to a} \frac{d^k}{dx^k} \left[ (x-a)^m f(x) \right] = c_k, \; k = 0,1,\ldots,m-1, \tag{4}$$

*hence the numbers* $A_k$ *are the coefficients* $c_k$ *of the Taylor development of the function* $(x-a)^m f(x)$ *in the point* $x = a$ *, namely* $\sum_{k=0}^{m-1} A_k (x-a)^k = P_a^{m-1}((x-a)^m f(x))$.

**Proof:** In conformity with (3), we have

$$c_n = \frac{1}{n!} \lim_{x \to a} \frac{d^n}{dx^n} \left[ (x-a)^m f(x) \right] = \frac{1}{n!} \lim_{x \to a} \frac{d^n}{dx^n} \left[ \sum_{k=0}^{m-1} A_k (x-a)^k + (x-a)^m g(x) \right].$$

For $n = 0$ results $c_0 = A_0$. For $1 \leq n \leq m-1$, it results

$$c_n = \frac{1}{n!} \lim_{x \to a} \sum_{k=0}^{m-1} A_k k(k-1)\cdots(k-n+1)(x-a)^{k-n} + \frac{1}{n!} \lim_{x \to a} \sum_{j=0}^{m-1} \binom{n}{j} m(m-1)\cdots(m-j+1)(x-a)^{m-j} \frac{d^{n-j}}{dx^{n-j}} g(x) = A_n.$$

**Remark:** *As is well known, if* $f(x)$ *is a rational function, its nominator has degree less than the denominator and* $a$ *is a pole of multiplicity* $m$ *of* $f(x)$, *then the function has the form* (3), *therefore the Theorem 2 work.*

**Corollary:** *If* $f(x)$ *is a rational function with degree of the numerator smaller than the denominator, having the distinct poles* $x_1,\ldots,x_p$ *of multiplicities* $m_1,\ldots,m_p$, *then* $f(x)$ *has the partial fractions development*

$$f(x) = \sum_{j=1}^{p} P_{x_j}^{m_j-1} \left[ (x-x_j)^{m_j} f(x) \right] \frac{1}{(x-x_j)^{m_j}}, \tag{5}$$

*where* $P_{x_j}^{m_j-1}\left[(x-x_j) f(x)\right]$ *denotes, for* $j = 1,\ldots,p$, *the Taylor polynomial of degree* $m_j - 1$ *in the point* $x = x_j$

.

**Theorem: 3** *If* $f(x)$ *is a function of the form*

$$f(x)=\sum_{k=0}^{m-1}\frac{A_k x+B_k}{(x-a)^{m-k}(x-b)^{m-k}}+g(x), \tag{6}$$

*where* $m\neq 0$ *is a natural number,* $a$, $b$, $A_k$ *and* $B_k$, *for* $k=0,1,\ldots,m-1$, *are complex numbers and* $g(x)$ *is a function with derivative of* $m-1$ *order in* $a$, *then the coefficients* $A_k$ *and* $B_k$ *can be computed recursively by the relations*

$$c_0=A_0a+B_0, \tag{7}$$

$$c_1=(A_1a+B_1)(a-b)+A_0, \tag{8}$$

$$c_2=(A_2a+B_2)(a-b)^2+A_1a+B_1+A_1(a-b), \tag{9}$$

$$c_3=(A_4a+B_4)(a-b)^3+2(A_2a+B_2)(a-b)+A_2(a-b)^2+A_1, \tag{10}$$

$$c_n=(A_na+B_n)(a-b)^n+(n-1)(A_{n-1}a+B_{n-1})(a-b)^{n-2}+A_{n-1}(a-b)^{n-1}+$$
$$+\sum_{\substack{k=0\\ k\geq\frac{n}{2}}}^{n-2}\frac{k(k-1)\cdots(2k-n+1)}{(n-k)!}(A_ka+B_k)(a-b)^{2k-n}+\sum_{\substack{k=0\\ k\geq\frac{n-1}{2}}}^{n-2}\frac{k(k-1)\cdots(2k-n+2)}{(n-k-1)!}A_k(a-b)^{2k-n+1},$$

$$4\leq n\leq m-1,\ \textit{if}\ m\geq 5 \tag{11}$$

*where* $(c_k: k=0,1,\ldots,m-1)=T_a^{m-1}\left((x-a)^m(x-b)^m f(x)\right)$ *are the coefficients of the Taylor polynomial of degree* $m-1$ *of the function* $(x-a)^m(x-b)^m f(x)$ *in the point* $x=a$.

**Proof:** In conformity with (6), we have

$$c_n=\frac{1}{n!}\lim_{x\to a}\frac{d^n}{dx^n}\left[(x-a)^m(x-b)^m f(x)\right]$$
$$=\frac{1}{n!}\lim_{x\to a}\frac{d^n}{dx^n}\left[\sum_{k=0}^{m-1}A_k(x-a)^k(x-b)^k+(x-a)^m(x-b)^m g(x)\right].$$

For $n=0$, it results (7). For $1\leq n\leq m-1$, it results

$$c_n=\frac{1}{n!}\lim_{x\to a}\sum_{k=0}^{m-1}\sum_{j=k}^{n}\binom{n}{j}k(k-1)\cdots(k-j+1)(x-a)^{k-j}\frac{d^{n-j}}{dx^{n-j}}\left[(A_ka+B_k)(x-b)^k\right]$$
$$+\frac{1}{n!}\lim_{x\to a}\sum_{j=0}^{n}\binom{n}{j}m(m-1)\cdots(m-j+1)(x-a)^{m-j}\frac{d^{n-j}}{dx^{n-j}}\left[(x-b)^m g(x)\right]$$
$$=\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}k!\lim_{x\to a}\frac{d^{n-k}}{dx^{n-k}}\left[(A_kx+B_k)(x-b)^k\right]=(A_na+B_n)(a-b)^n$$
$$+\sum_{k=0}^{n-1}\frac{1}{(n-k)!}\lim_{x\to a}\left[(A_kx+B_k)\frac{d^{n-k}}{dx^{n-k}}(x-b)^k+(n-k)A_k\frac{d^{n-k-1}}{dx^{n-k-1}}(x-b)^k\right],$$

from which results the formulas (8)-(11).

**Particular cases:** From (11), if $m\geq 5$, it results the formulas

$$c_4=(A_4a+B_4)(a-b)^4+3(A_3a+B_3)(a-b)^2+A_3(a-b)^3+A_2a+B_2+2A_2(a-b), \tag{12}$$

$$c_5=(A_5a+B_5)(a-b)^5+4(A_4a+B_4)(a-b)^3+A_4(a-b)^4+3(A_3a+B_3)(a-b)+3A_3(a-b)^2+A_2 \tag{13}$$

**Remark:** *As is well known, if* $f(x)$ *is a rational function, its nominator has degree less than the denominator, the last being a polynomial with real coefficients and* $a$ *is a complex pole of multiplicity* $m$ *of* $f(x)$, *then the function has the form* (6) *with* $b=\overline{a}$ *the complex conjugate of* $a$, *therefore the Theorem 3 work. See Example 8.5. Same situation when the denominator has rational numbers as coefficients and* $a$ *is an irrational pole of the function of multiplicity* $m$. *Then* $b$ *is the irrational conjugate of* $a$ *and the Theorem 3 also work. See Example 8.6.*

## 8. EXAMPLES OF PARTIAL FRACTIONS DEVELOPMENT

**8.1.**
$$f(x)=\frac{x}{(x+1)^2(x-1)^3(x-2)^5}=P_{-1}^1\left(\frac{x}{(x-1)^3(x-2)^5}\right)\frac{1}{(x+1)^2}$$
$$+P_1^2\left(\frac{x}{(x+1)^2(x-2)^5}\right)\frac{1}{(x-1)^3}+P_2^4\left(\frac{x}{(x+1)^2(x-1)^3}\right)\frac{1}{(x-2)^5}$$
$$=P_{-1}^1\left(\frac{(-1,1)}{(-2,1)^{*3}*(-3,1)^{*5}}\right)\frac{1}{(x+1)^2}+P_1^2\left(\frac{(1,1,0)}{(2,1,0)^{*2}*(-1,1,0)^{*5}}\right)\frac{1}{(x-1)^3}$$
$$+P_2^4\left(\frac{(2,1,0,0,0)}{(3,1,0,0,0)^{*2}*(1,1,0,0,0)^{*3}}\right)\frac{1}{(x-2)^5}=P_{-1}^1\left(\frac{(-1,1)}{(1944,-6156)}\right)\frac{1}{(x+1)^2}$$
$$+P_1^2\left(\frac{(1,1,0)}{(-4,16,-21)}\right)\frac{1}{(x-1)^3}+P_2^4\left(\frac{(2,1,0,0,0)}{(9,33,46,30,9)}\right)\frac{1}{(x-2)^5}$$
$$=P_{-1}^1\left(-\frac{1}{1944},-\frac{13}{11664}\right)\frac{1}{(x+1)^2}+P_1^2\left(-\frac{1}{4},-\frac{5}{4},\frac{59}{16}\right)\frac{1}{(x-1)^3}$$
$$+P_2^1\left(\frac{2}{9},-\frac{19}{27},\frac{13}{9},-\frac{593}{243},\frac{2689}{729}\right)\frac{1}{(x-2)^5}=\left[-\frac{1}{1944}-\frac{13}{11664}(x+1)\right]\frac{1}{(x+1)^2}$$
$$+\left[-\frac{1}{4}-\frac{5}{4}(x-1)+\frac{59}{16}(x-1)^2\right]\frac{1}{(x-1)^3}+\left[\frac{2}{9}-\frac{19}{22}(x-2)+\frac{13}{9}(x-2)^2-\frac{593}{243}(x-2)^3\right.$$
$$\left.+\frac{2689}{729}(x-2)^4\right]\frac{1}{(x-2)^5}=-\frac{1}{1944(x+1)^2}-\frac{13}{11664(x+1)}-\frac{1}{4(x-1)^3}-\frac{5}{4(x-1)^2}$$
$$+\frac{59}{16(x-1)}+\frac{2}{9(x-2)^5}-\frac{19}{27(x-2)^4}+\frac{13}{9(x-2)^3}-\frac{593}{243(x-2)^2}+\frac{2689}{729(x-2)}.$$

For the Taylor polynomial of four degree in $x=2$ have been performed the convolutions

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| | 1 | 1 | 0 | 0 |
| 1 | 2 | 1 | 0 | 0 |
| 1 | 1 | 0 | 0 | 0 |
| 1 | 2 | 1 | 0 | 0 |
| | 1 | 2 | 1 | 0 |
| 1 | 3 | 3 | 1 | 0 |

| | | | | |
|---|---|---|---|---|
| 3 | 1 | 0 | 0 | 0 |
| 3 | 1 | 0 | 0 | 0 |
| 9 | 3 | 0 | 0 | 0 |
| | 3 | 1 | 0 | 0 |
| 9 | 6 | 1 | 0 | 0 |

| | | | | |
|---|---|---|---|---|
| 1 | 3 | 3 | 1 | 0 |
| 9 | 6 | 1 | 0 | 0 |
| 9 | 27 | 27 | 9 | 0 |
| | 6 | 18 | 18 | 6 |
| | | 1 | 3 | 3 |
| 9 | 33 | 46 | 30 | 9 |

From which it results

$$(1,1,0,0,0)^{*3} = (1,3,3,1,0),\ (3,1,0,0,0)^{*2} = (9,6,1,0,0)$$

and

$$(3,1,0,0,0)^{*2} * (1,1,0,0,0)^{*3} = (9,33,46,30,9)$$

and the deconvolution

$$\begin{array}{ccccc|ccccc}
2 & 1 & 0 & 0 & 0 & 9 & 33 & 46 & 30 & 9 \\
2 & \frac{22}{3} & \frac{92}{9} & \frac{20}{3} & 2 & \frac{2}{9} & -\frac{19}{27} & \frac{13}{9} & -\frac{593}{243} & \frac{2689}{729} \\
\hline
/ & -\frac{19}{3} & -\frac{92}{9} & -\frac{20}{3} & -2 & & & & & \\
 & -\frac{19}{3} & -\frac{209}{9} & -\frac{874}{27} & -\frac{190}{9} & & & & & \\
\hline
 & / & 13 & \frac{694}{27} & \frac{172}{9} & & & & & \\
 & & 13 & \frac{143}{3} & \frac{598}{9} & & & & & \\
\hline
 & & / & -\frac{593}{27} & -\frac{142}{3} & & & & & \\
 & & & -\frac{593}{27} & -\frac{6523}{81} & & & & & \\
\hline
 & & & / & \frac{2689}{81} & & & & & \\
 & & & & \frac{2689}{81} & & & & & \\
\hline
 & & & & / & & & & &
\end{array}$$

Hence

$$\frac{(2,1,0,0,0)}{(3,1,0,0,0)^{*2} * (1,1,0,0,0)^{*3}} = \left(\frac{2}{9},-\frac{19}{27},\frac{13}{9},-\frac{593}{243},\frac{2689}{729}\right).$$

**8.2.** $f(x) = \dfrac{15625}{(x-2)^3(x^2+1)^6} = \sum_{j=0}^{2} \dfrac{\tilde{c}_j}{(x-2)^{2-j}} + \sum_{k=0}^{5} \dfrac{A_k x + B_k}{(x^2+1)^{6-k}}$. We have

$$(\tilde{c}_0,\tilde{c}_1,\tilde{c}_2) = T_2^2\left(\frac{15625}{(x^2+1)^6}\right) = \frac{15625(1,0,0)}{(5,4,1)^{*6}} = \frac{(15625,0,0)}{(15625,75000,168750)} = (1,-4.8,12.24),$$

$$(c_0,\ldots,c_5) = T_i^5\left(\frac{15625}{(x-2)^3}\right) = \frac{15625(1,0,0,0,0,0)}{(-2+i,1,0,0,0,0)^{*3}} = \frac{(15625,0,0,0,0,0)}{(-2+11i,9-12i,-6+3i,1,0,0)}$$

$$= (-250-1375i,525-1800i,1140-1230i,1170-440i,834+87i,442.68+282.24i).$$

Hence $c_0 = A_0 i + B_0 = -250-1375i$, $A_0 - -1375$, $B_0 = -250$,

 

$c_1 = 2i(A_1 i + B_1) - 1375 = 525 - 1800i$, $A_1 = -950$, $B_1 = -900$,

$c_2 = -4(A_2 i + B_2) - 1900i - 950i - 900 = 1140 - 1230i$, $A_2 = -405$, $B_2 = -510$,

$c_3 = -8i(A_3 i + B_3) + 1620 + 4i(-405i - 510) - 950 = 1170 - 440i$, $A_3 = -140$,

$B_3 = -200$, $c_4 = 16(A_4 i + B_4) + 1132i + 12(140i + 200) - 405i - 510 - 1620i = = 834 + 87i$,

$A_4 = -43$,

$B_4 = -66$, $c_5 = 32i(A_5 i + B_5) - 688 - 32i(-43i - 66) + 2112i + 840 - 1200i + 1680 - 405$

$= 442.68 + 282,24i$,

$A_5 = -12.24$, $B_5 = -19.68$.

Therefore, $$f(x) = \frac{1}{(x-2)^3} - \frac{4.8}{(x-2)^2} + \frac{12.24}{x-2} - \frac{1375x + 250}{(x^2+1)^6} - \frac{950x + 900}{(x^2+1)^5}$$
$$- \frac{405x + 510}{(x^2+1)^4} - \frac{140x + 200}{(x^2+1)^3} - \frac{43x + 66}{(x^2+1)^2} - \frac{12.24x + 19.68}{x^2+1}.$$

**8.3.** $f(x) = \dfrac{x^6 + x^5 + x^3 + 3x^2 + x + 4}{(x^2+1)^5 (x^2+3)^2}$.

**Solution: 1 (complex)** We will obtain $f(x) = \sum_{j=0}^{1} \dfrac{\tilde{A}_j x + \tilde{B}_j}{(x^2+3)^{1-j}} + \sum_{k=0}^{4} \dfrac{A_k x + B_k}{(x^2+1)^{4-k}}$**.** We have

$$(\tilde{c}_0, \tilde{c}_1) = T^1_{i\sqrt{3}} \left( \frac{x^6 + x^5 + x^3 + 3x^2 + x + 4}{(x^2+1)^5} \right) = \frac{(-32 + 7i\sqrt{3}, 37 + 60i\sqrt{3})}{(-2, 2i\sqrt{3})^{*5}}$$

$$= \frac{(-32 + 7i\sqrt{3}, 37 + 60i\sqrt{3})}{-32(1, -5i\sqrt{3})} = -\frac{1}{32}(-32 + 7i\sqrt{3}, -68 - 100i\sqrt{3}),$$

Hence

$\tilde{c}_0 = \tilde{A}_0 i\sqrt{3} + \tilde{B}_0 = 1 - \dfrac{7}{32} i\sqrt{3}$, $\tilde{A}_0 = -\dfrac{7}{32}$, $\tilde{B}_0 = 1$,

$\tilde{c}_1 = 2i\sqrt{3}(\tilde{A}_1 i\sqrt{3} + \tilde{B}_1) - \dfrac{7}{32} = \dfrac{68}{32} + \dfrac{100}{32} i\sqrt{3}$, $A_1 = -\dfrac{25}{64}$, $B_1 = \dfrac{25}{16}$.

$$(c_0, \ldots, c_4) = T_i^4 \left( \frac{x^6 + x^5 + x^3 + 3x^2 + x + 4}{(x^2+3)^2} \right) = \frac{(i, 3 + 12i, 18 - 7i, -9 - 20i, -15 + 5i)}{(2, 2i, 1, 0, 0)^{*2}}$$

$$= \frac{(i, 3 + 12i, 18 - 7i, -9 - 20i, -15 + 5i)}{(4, 8i, 0, 4i, 1)} = \left( \frac{i}{4}, \frac{5}{4} + 3i, \frac{21}{2} - \frac{17}{4} i, -\frac{21}{2} - 26i, -\frac{211}{4} + \frac{335}{16} i \right),$$

$$c_0 = A_0 i + B_0 = \frac{i}{4},\ A_0 = \frac{1}{4},\ B_0 = 0;\ c_1 = 2i(A_1 i + B_1) + \frac{1}{4} = \frac{5}{4} + 3i,\ A_1 = -\frac{1}{2},\ B_1 = \frac{3}{2};$$

$$c_2 = -4(A_2 i + B_2) - i - \frac{i}{2} + \frac{3}{2} = \frac{21}{2} - \frac{17}{4}i,\ A_2 = \frac{11}{16},\ B_2 = -\frac{9}{4};$$

$$c_3 = -8i(A_3 i + B_3) + 4i\left(\frac{11i}{16} - \frac{9}{4}\right) - \frac{11}{4} - \frac{1}{2} = \frac{21}{2} - 26i,\ A_3 = -\frac{9}{16},\ B_3 = \frac{17}{8};$$

$$c_4 = 16(A_4 i + B_4) - 12\left(-\frac{9i}{16} + \frac{17}{8}\right) + \frac{9i}{2} + \frac{11i}{16} - \frac{9}{4} + \frac{11i}{4} = -\frac{211}{4} + \frac{335i}{16},\ A_4 = \frac{25}{64},.\ B_4 = -\frac{25}{16}.$$

Therefore

$$f(x) = \frac{x}{4(x^2+1)^5} + \frac{3-x}{2(x^2+1)^4} + \frac{11x-36}{16(x^2+1)^3} + \frac{34-9x}{16(x^2+1)^2} + \frac{25(x-4)}{64(x^2+1)} + + \frac{32-7x}{32(x^2+3)^2} + \frac{25(x-4)}{64(x^2+3)}.$$

**Solution: 2 (real)** $f(x) = f_1(t) + x f_2(t)$**,** where $t = x^2$ and

$$f_1(t) = \frac{t^3+3t+4}{(t+1)^5(t+3)^2}$$

$$= P_{-1}^4\left(\frac{t^3+3t+4}{(t+3)^2}\right)\frac{1}{(t+1)^5} + P_{-3}^1\left(\frac{t^3+3t+4}{(t+1)^5}\right)\frac{1}{(t+3)^2}$$

$$= P_{-1}^4\left(\frac{(0,6,-3,1,0)}{(2,1,0,0,0)^{*2}}\right)\frac{1}{(t+1)^5} + P_{-3}^1\left(\frac{(-32,30)}{(-2,1)^{*5}}\right)\frac{1}{(t+3)^2}$$

$$= P_{-1}^4\left(\frac{(0,6,-3,1,0)}{(4,4,1,0,0)}\right)\frac{1}{(t+1)^5} + P_{-3}^1\left(\frac{(-32,30)}{(-32,80)}\right)\frac{1}{(t+3)^2}$$

$$= P_{-1}^4\left(0,\frac{3}{2},-\frac{9}{4},\frac{17}{8},-\frac{25}{16}\right)\frac{1}{(t+1)^5} + P_{-3}^1\left(\left(1,-\frac{25}{16}\right)\right)\frac{1}{(t+3)^2}$$

$$= \left[\frac{3}{2}(t+1) - \frac{9}{4}(t+1)^2 + \frac{17}{8}(t+1)^3 - \frac{25}{16}(t+1)^4\right]\frac{1}{(t+1)^5} + \left[1 - \frac{25}{16}(t+3)\right]\frac{1}{(t+3)^2}$$

$$= \frac{3}{2(t+1)^4} - \frac{9}{4(t+1)^3} + \frac{17}{8(t+1)^2} - \frac{25}{16(t+1)} + \frac{1}{(t+3)^2} - \frac{25}{16(t+3)},$$

$$f_2(t) = \frac{t^2+t+1}{(t+1)^5(t+3)^2} = P_{-1}^4\left(\frac{t^2+t+1}{(t+3)^2}\right)\frac{1}{(t+1)} + P_{-3}^1\left(\frac{t^2+t+1}{(t+1)^5}\right)\frac{1}{(t+3)^2}$$

$$= P_{-1}^4\left(\frac{(1,-1,1,0,0)}{(2,1,0,0,0)^{*2}}\right)\frac{1}{(t+1)} + P_{-3}^1\left(\frac{(7,-5)}{(-2,1)^{*5}}\right)\frac{1}{(t+3)^2}$$

$$= P_{-1}^4\left(\frac{(1,-1,1,0,0)}{(4,4,1,0,0)}\right)\frac{1}{(t+1)} + P_{-3}^1\left(\frac{(7,-5)}{(-32,80)}\right)\frac{1}{(t+3)^2}$$

$$= P_{-1}^4\left(\frac{1}{4},-\frac{1}{2},\frac{11}{16},-\frac{9}{16},\frac{25}{64}\right)\frac{1}{(t+1)} - P_{-3}^1\left(\frac{7}{32},\frac{25}{64}\right)\frac{1}{(t+3)^2}$$

$$= \left[\frac{1}{4} - \frac{1}{2}(t+1) + \frac{11}{16}(t+1)^2 - \frac{9}{16}(t+1)^3 + \frac{25}{64}(t+1)^4\right]\frac{1}{(t+1)^5}$$

$$-\left[\frac{7}{32}+\frac{25}{64}(t+3)\right]\frac{1}{(t+3)^2}=\frac{1}{4(t+1)^5}-\frac{1}{2(t+1)^4}+\frac{11}{1\ (t+61)^3}-\frac{9}{1\ (t+61)^2}$$
$$+\frac{25}{64(t+1)}-\frac{7}{32(t+3)^2}-\frac{25}{64(t+3)}.$$

It results that $f(x)$ has the same form as that obtained in the previous solution.

**8.4.** $f(x)=\dfrac{x^6-5x^5+10x^4-9x^3+5x^2-3x+2}{(x^2-2x+2)^5(x^2-2x+4)^2}$.

Performing the change of variables $x=y+1$ we obtain the function $f(y)=\dfrac{y^6+y^5+y^3+3y^2+y+4}{(y^2+1)^5(y^2+3)^2}$, hence the example 8.4 is reduced to example 8.3.

**Remark:** *All the rational functions whose denominator is a product of trinomials whose canonical forms have the same binomial, can be developed in partial fractions by the real method presented in the second solution of the Example 8.3, after a convenient change of variable.*

**8.5** $f(x)=\dfrac{1}{(x^2+1)^2(x^2+2x+2)^3}=\sum_{j=0}^{1}\dfrac{\tilde{A}_j x+\tilde{B}_j}{(x^2+1)^{2-j}}+\sum_{k=0}^{2}\dfrac{A_k x+B_k}{(x^2+2x+2)^{3-k}}$,

$$(\tilde{c}_0,\tilde{c}_1)=T_i^1\left(\frac{1}{(x^2+2x+2)^3}\right)=\frac{(1,0)}{(1+2i,2+2i)^{*3}}=\frac{(1,0)}{(-11-2i,-42+6i)}=$$
$$=(-0.088+0.016i,0.2974-0.1632i), \text{ hence}$$

$\tilde{c}_0=\tilde{A}_0 i+\tilde{B}_0=-0.088+0.016i$, $\tilde{A}_0=0.016$, $\tilde{B}_0=-0.088$,

$\tilde{c}_1=2i(\tilde{A}_1 i+\tilde{B}_1)+0.016=0.2974-0.1632i$, $\tilde{A}_1=-0.1408$, $\tilde{B}_1=-0.0816$;

$$(c_0,c_1,c_2)=T_{-1+i}^2\left(\frac{1}{(x^2+1)^2}\right)=\frac{(1,0,0)}{(1-2i,-2+2i,1)^{*2}}=\frac{(1,0,0)}{(-3-4i,4+12i,2-12i)}$$
$$=(-0.12+0.16i,-0.416+0.288i,-0.7456+0.3008i), \text{ hence}$$

$c_0=A_0(-1+i)+B_0=-0.12+0.16i$, $A_0=0.16$, $B_0=0.04$;

$c_1=2i[A_1(-1+i)+B_1]+0.16=-0.416+0.288i$, $A_1=0.288$, $B_1=0.432$;

$c_2=-4[A_2(-1+i)+B_2]+0.288(-1+i)+0.432+0.576i=-0.7456+0.3005i$,

$A_2=0.1408$, $B_1=0.3632$. Therefore

$$f(x)=\frac{0.016x-0.088}{(x^2+1)^2}-\frac{0.1408x+0.0816}{x^2+1}+\frac{0.16x+0.04}{(x^2+2x+2)^3}+\frac{0.288x+0.432}{(x^2+2x+2)^2}+\frac{0.1408x+0.3632}{x^2+2x+2}.$$

**8.6.** $f(x)=\dfrac{4}{(x+1)^3(x^2-2x-1)^2}$.

**Solution: 1 (Based on Theorem 3)**

$f(x)=\sum_{k=0}^{2}\dfrac{\tilde{c}_k}{(x+1)^{2-k}}+\sum_{n=0}^{1}\dfrac{A_n x+B_n}{(x^2-2x-1)^{1-n}}$. We have

$(\tilde{c}_0,\tilde{c}_1,\tilde{c}_2)=T_{-1}^2\left(\dfrac{4}{(x^2-2x-1)^2}\right)=\dfrac{(4,0,0)}{(2,-4,1)^{*2}}=\dfrac{(1,0,0)}{(1,-4,5)}=(1,4,11)$ and

$(c_0,c_1)=T_{1+\sqrt{2}}^1\left(\dfrac{4}{(x+1)^3}\right)=\dfrac{(4,0)}{(2+\sqrt{2},1)^{*3}}=\dfrac{(4,0)}{(20+14\sqrt{2},18+12\sqrt{2})}= \quad =\left(10-7\sqrt{2},-51+36\sqrt{2}\right),$ in conformity with the algorithms

$$\begin{array}{ll}
2+\sqrt{2} & 1\\
2+\sqrt{2} & 1\\
\hline
6+4\sqrt{2} & 2+\sqrt{2}\\
 & 2+\sqrt{2}\\
\hline
6+4\sqrt{2} & 4+2\sqrt{2}\\
2+\sqrt{2} & 1\\
\hline
20+14\sqrt{2} & 12+8\sqrt{2}\\
 & 6+4\sqrt{2}\\
\hline
20+14\sqrt{2} & 18+12\sqrt{2}
\end{array}
\qquad
\begin{array}{ll|ll}
4 & 0 & 20+14\sqrt{2} & 18+12\sqrt{2}\\
\cline{3-4}
4 & 12-6\sqrt{2} & 10-7\sqrt{2} & -51+36\sqrt{2}\\
\cline{1-2}
/ & -12+6\sqrt{2} & & \\
 & -12+6\sqrt{2} & & \\
\cline{2-2}
 & / & &
\end{array}$$

So, $c_0=A_0\left(1+\sqrt{2}\right)+B_0=10-7\sqrt{2}$, $A_0=-7$, $B_0=17$ and
$c_1=2\sqrt{2}\left[A_1\left(1+\sqrt{2}\right)+B_1\right]-7=-51+36\sqrt{2}$, $A_1=-11$, $B_1=29$.

Therefore the decomposition in partial fractions is

$$f(x)=\frac{1}{(x+1)^3}+\frac{4}{(x+1)^2}+\frac{11}{x+1}+\frac{17-7x}{(x^2-2x-1)^2}+\frac{29-11x}{x^2-2x-1}. \tag{14}$$

**Solution: 2 (Based on Theorem 2)**

$$f(x)=\frac{4}{(x+1)^3\left(x-1-\sqrt{2}\right)^2\left(x-1+\sqrt{2}\right)^2}$$

$$=P_{-1}^2\left(\frac{4}{(x^2-2x-1)^2}\right)\frac{1}{(x+1)^3}+P_{1+\sqrt{2}}^1\left(\frac{4}{(x+1)^3\left(x-1+\sqrt{2}\right)^2}\right)\frac{1}{\left(x-1-\sqrt{2}\right)^2}$$

$$+P^1_{1-\sqrt{2}}\left(\frac{4}{(x+1)^3\left(x-1-\sqrt{2}\right)^2}\right)\frac{1}{\left(x-1+\sqrt{2}\right)^2}$$

$$=P^2_{-1}\left(\frac{(4,0,0)}{(2,-4,1)^{*2}}\right)\frac{1}{(x+1)^3}+P^1_{1+\sqrt{2}}\left(\frac{4(1,0)}{\left(2+\sqrt{2},1\right)^{*3}*\left(2\sqrt{2},1\right)^{*2}}\right)\frac{1}{\left(x-1-\sqrt{2}\right)^2}$$

$$+P^1_{1-\sqrt{2}}\left(\frac{4(1,0,0)}{\left(2-\sqrt{2},1\right)^{*3}*\left(-2\sqrt{2},1\right)^{*2}}\right)\frac{1}{\left(x-1+\sqrt{2}\right)^2}=P^2_{-1}\left(\frac{(1,0,0)}{(1,-4,5)}\right)\frac{1}{(x+1)^3}$$

$$+P^1_{1+\sqrt{2}}\left(\frac{(1,0,0)}{\left(40+28\sqrt{2},64+44\sqrt{2}\right)}\right)\frac{1}{\left(x-1-\sqrt{2}\right)^2}$$

$$+P^1_{1-\sqrt{2}}\left(\frac{(1,0,0)}{\left(40-28\sqrt{2},64-44\sqrt{2}\right)}\right)\frac{1}{\left(x-1+\sqrt{2}\right)^2}=P^2_{-1}(1,4,11)\frac{1}{(x+1)^3}$$

$$+\frac{1}{8}P^1_{1+\sqrt{2}}\left(10-7\sqrt{2},-44+37\sqrt{2}\right)\frac{1}{\left(x-1-\sqrt{2}\right)^2}$$

$$+\frac{1}{8}P^1_{1-\sqrt{2}}\left(10+7\sqrt{2},-44-37\sqrt{2}\right)\frac{1}{\left(x-1+\sqrt{2}\right)^2}=\frac{1}{(x+1)^3}+\frac{4}{(x+1)^2}+\frac{11}{x+1}$$

$$+\frac{10-7\sqrt{2}}{8\left(x-1-\sqrt{2}\right)^2}+\frac{-44+31\sqrt{2}}{8\left(x-1-\sqrt{2}\right)}+\frac{10+7\sqrt{2}}{8\left(x-1+\sqrt{2}\right)^2}+\frac{-44-7\sqrt{2}}{8\left(x-1+\sqrt{2}\right)}. \qquad (15)$$

**Remark:** *A little algebraic calculus shows that the two developments of* $f(x)$ *in partial fractions given by the formulas (14) and (15) are equal. However, for integration and for calculus of the inverse Laplace transform, more convenient is the development (15). Indeed, it gives*

$$\int f(x)\,dx=-\frac{1}{2(x+1)^2}-\frac{4}{x+1}+11\ln|x+1|-\frac{10-7\sqrt{2}}{8\left(x-1-\sqrt{2}\right)}-\frac{44-31\sqrt{2}}{8}\ln\left|x-1-\sqrt{2}\right|$$

$$-\frac{10+7\sqrt{2}}{8\left(x-1+\sqrt{2}\right)}-\frac{44+31\sqrt{2}}{8}\ln\left|x-1+\sqrt{2}\right|+C,$$

*where* $C$ *is an arbitrary constant. If* $x$ *is a complex variable with* $\mathrm{Re}(x)>1+\sqrt{2}$ *, from formula (15) it results the inverse Laplace transform*

$$L^{-1}\left(f(x)\right)=\frac{1}{2}t^2e^{-t}+4te^{-t}+11e^{-t}+\frac{10-7\sqrt{2}}{8}te^{\left(1+\sqrt{2}\right)t}-\frac{44-31\sqrt{2}}{8}e^{\left(1+\sqrt{2}\right)t}$$

$$+\frac{10+7\sqrt{2}}{8}te^{\left(1-\sqrt{2}\right)t}-\frac{44+31\sqrt{2}}{8}e^{\left(1-\sqrt{2}\right)t},\ t\geq 0.$$

## 9. CONCLUSIONS

Besides the well known applications of the Taylor series development – the approximation of the functions by polynomials and power series method for solving different types of equations, the last being now named *Taylor differential transformation method*, we added other two, the automatic high order differentiation in [1] and the partial fractions decomposition in this paper. As can see by the above examples, the new methods given in the present article can be applied to cases that are difficult to solve by the methods known so far.

## REFERENCES

[1] M. I. Cîrnu, Algebraic equations. New methods for old problems, Lambert Academic Publishing, Germany, 2013.

[2] M. I. Cîrnu, First order differential recurrence equations with discrete auto-convolution, International Journal of Mathematics and Computation, 4(2009), 124-128.

[3] M. I. Cîrnu, Initial value problems for first-order differential recurrence equations with auto-convolution, Electronic Journal of Differential Equation, 2011, 2, 1-13.

[4] M. I. Cîrnu, A certain integral-recurrence equation with discrete-continuous auto-convolution, Archivum Mathematicum (Brno), 47(2011), 267-272.

[5] M. I. Cîrnu, Determinantal formulas for sum of generalized arithmetic-geometric series, Boletin de la Asociacion Matematica Venezolana, 43(2011), 1, 25-38.

[6] M. A. Cuenod, Introduction a l'Analyse Impulsionelle. Principe et Application, Dunod, Paris, 1970.

[7] J. F. Mahoney, B. D. Sinclair, Partial fractions expansion: a review of computational methodology and efficiency, Journal of Computational and Applied Mathematics, 9(1983) 3, 247-269.

**Source of support: Nil, Conflict of interest: None Declared**