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CONNECTED DOMINATION POLYNOMIAL OF A GRAPH

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ABSTRACT

Let $G = (V, E)$ be a simple connected graph of order n. A connected dominating set of G is a set S of vertices of G such that every vertex in $V - S$ is adjacent to some vertex in S and the induced subgraph $\langle S \rangle$ is connected. The connected domination number $\Upsilon_c(G)$ is the minimum cardinality of a connected dominating set of G. In this paper we introduce the connected domination polynomial of G. The connected domination polynomial of a connected graph G of order n is the polynomial $D_c(G, x) = \sum_{i=\Upsilon_c(G)}^{n} d_c(G,i)x^i$, where $d_c(G, i)$ is the number of connected dominating set of G of size i and $\Upsilon_c(G)$ is the connected domination number of G. We obtain some basic properties of the connected domination polynomial and compute this polynomial and its roots for some standard graphs.

Keywords: *Connected dominating set, connected domination polynomial, connected domination roots.*

Mathematics Subject Classification: *05C69.*

1. INTRODUCTION

Throughout this paper we will consider only a simple connected graphs finite and undirected, without loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G, respectively. In general, we use $\langle X \rangle$ to denote the subgraph induced by the set of vertices X. N(v) and N[v] denote the open and closed neighbourhood of a vertex v, respectively. A set D of vertices in a graph G is a dominating set if every vertex in $V-D$ is adjacent to some vertex in D. The domination number $\Upsilon(G)$ is the minimum cardinality of a dominating set of G. For terminology and notations not specifically defined here we refer reader to [4]. For more details about domination number and its related parameters, we refer to [5], [7], and [9].

A dominating set S of G is called a connected dominating set if the induced subgraph $\langle S \rangle$ is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\Upsilon_c(G)$ [8].

A dominating set with cardinality $\Upsilon_c(G)$ is called Υ_c-set. We denote the family of dominating sets of a graph G with cardinality i by $D_c(G, i)$.

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Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then their **union** $G = G_1 \cup G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The **join** $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets X_1 and X_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. A **spider** is a tree T with the property that the removal of all end paths of length two of T results in an isolated vertex, called the head of spider. For any real number x, $\lceil x \rceil$ denotes the smallest integer greater than or equal to x, and $\lfloor x \rfloor$ denotes the largest integer less than or equal to x.

The domination polynomial of a graph is introduced by Saeid Alikhani and Yee- hock Peng [1].

In this paper motivated by domination polynomial of a graph [1], we introduce the connected domination polynomial of G, we obtain some properties of the connected domination polynomial and compute this polynomial and its roots for some standard graphs.

2. CONNECTED DOMINATION POLYNOMIAL OF A GRAPH

Definition: 2.1 Let G be a connected graph of order n and let $d_c(G, i)$ denoted the number of connected dominated sets with cardinality i. Then the connected domination polynomial $D_c(G, x)$ of G is defined as:

$$D_c(G,x) = \sum_{i=\gamma_c(G)}^{n} d_c(G,i)\, x^i,$$

where $\Upsilon_c(G)$ is the connected domination number of G. The roots of the connected domination polynomial are called the connected domination roots of G and denoted by $Z(D_c(G, X))$.

Example: 2.2 Let G be the graph in the Figure 1, V(G) = {1, 2, 3, 4}. Then the connected domination number is one and the connected dominating set of size two are {1, 2}, {2, 3}, {2, 4}, the connected dominating sets of size three are {1, 2, 3}, {1, 2, 4}, {2, 3, 4} and one connected dominating set of size four. Hence

$$D_c(G, x) = x(x^3 + 3x^2 + 3x + 1),$$

and the connected dominating roots of G are 0 and −1 with three multiplicities.

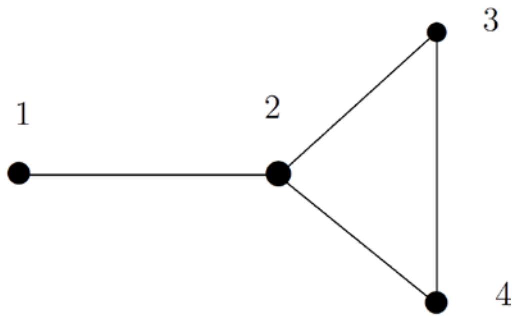

Figure 1: G

Theorem: 2.3 For any path P_n on $n \geq 3$ vertices,

$$D_c(P_n, x) = x^n + 2x^{n-1} + x^{n-2},$$

and the connected dominating roots are 0 with multiplicity n – 2 and –1 with multiplicity 2.

Proof: Let G be path P_n with $n \geq 3$ and let $P_n = \{v_1, v_2, ..., v_n\}$. The connected domination number of P_n is n – 2 and there is only one connected domination set of order n − 2. That means $d_c(P_n, n-2) = 1$.

Also there are only two connected dominating sets of order n–1 namely $\{v_2, ..., v_n\}$ and $\{v_1, v_2, ..., v_{n-1}\}$. Therefore $d_c(Pn, n-1) = 2$ and clearly there is only one connected dominating set of order n. Hence $D_c(P_n, x) = x^n + 2x^{n-1} + x^{n-2}$ and it is clear that the roots of the polynomial $x^n + 2x^{n-1} + x^{n-2}$ are 0 with multiplicity n – 2 and –1 with multiplicity 2.

Theorem: 2.4 For any cycle C_n with n vertices,

$$D_c(C_n, x) = x^{n-2} (x^2 + nx + n),$$

and the connected dominating roots are 0 with multiplicity (n – 2) and $\frac{-n+\sqrt{n^2-4n}}{2}$, $\frac{-n-\sqrt{n^2-4n}}{2}$

Proof: Let G be a cycle C_n with n and let $C_n = \{v_1, v_2, ..., v_n, v_1\}$. The connected domination number of C_n is n–2 and there are n possibilities for the connected dominating set of size (n – 2). That means $d_c(C_n, n-2) = n$.

Also there are only n connected dominating sets of order n–1 namely $\{v_2, v_3, ..., v_n\}$, $\{v_2, v_3, v_4, ..., v_n\}$,..., $\{v_1, v_2, v_3, ..., v_{n-1}\}$. Therefore $d_c(C_n, n-1) = n$ and clearly there is only one connected dominating set of order n. Hence $D_c(C_n, x) = x^n + nx^{n-1} + nx^{n-2} = x^{n-2} (x^2 + nx + n)$ and the roots of this polynomial are 0 with multiplicity n − 2 and

$$\frac{-n+\sqrt{n^2-4n}}{2}, \frac{-n-\sqrt{n^2-4n}}{2}$$

Theorem: 2.5 For any star graph $K_{1,n}$ with n + 1 vertices, where $n \geq 2$, $D_c(K_{1,n}, x) = x(1 + x)^n$ and the connected dominating roots are 0 and –1 with multiplicity n.

Proof: Let G be star graph of size n + 1 and $n \geq 2$. By labeling the vertices of G as $v_0, v_1, v_2, ...v_n$, where v_0 is the vertex of degree n, then clearly there is only one connected dominating set of size one and there are n connected dominating set of size two namely $\{v_0, v_1\}, \{v_0, v_2\}, \{v_0, v_3\}, ..., \{v_0, v_n\}$. Similarly for the connected dominating set of size three we need to select the vertex v_0 and two vertices from the set of vertices $\{v_1, v_2, ...v_n\}$.

That means there are $\binom{n}{2}$ connected dominating sets. In general

$d_c(G, i) = \binom{n}{i-1}$

Hence

$$\begin{aligned} D_c(K_{1,n,}x) &= x + nx^2 + \binom{n}{2} x^3 + \binom{n}{3} x^4 + ... + \binom{n}{n} x^{n+1} \\ &= x\left[1 + nx + \binom{n}{2} x^2 + \binom{n}{3} x^3 + ... + \binom{n}{n} x^2\right] \\ &= x \sum_{k=0}^{n} \binom{n}{k} x^k \\ &= x\ (1+x)^n \end{aligned}$$

Thus, the connected dominating roots are 0 and -1 with multiplicity n.

Theorem: 2.6 For any complete graph K_n of n vertices,

$$D_c(K_n, x) = (1 + x)^n - 1.$$

Proof: Let G be a complete graph K_n. Then for any $1 \leq i \leq n$, it is easy to see that $d_c(K_n, i) = \binom{n}{i}$.

Therefore

$$D_c\left(K_{n,}x\right)=\sum_{i=1}^{n}\binom{n}{i}x^i$$
$$= nx+\binom{n}{2}x^2+\binom{n}{3}x^3+\binom{n}{4}x^4+\ldots+\binom{n}{n}x^n$$
$$=\left[\sum_{i=0}^{n}\binom{n}{i}x^i\right]-1$$
$$=\left(x+1\right)^n-1$$

Theorem: 2.7 Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs of orders n_1 and n_2 respectively. Then
$$D_c((G_1 + G_2), x) = D_c(G_1, x) + D_c(G_2, x) + [(1+x)^{n_1} - 1]\,[(1+x)^{n_2} - 1].$$

Proof: From the definition of $(G_1 + G_2)$ if D1 be any connected dominating set of G_1, then D_1 is connected dominating set of $(G_1 + G_2)$, similarly if D2 be any connected dominating set of G_2, then D_2 is connected dominating set of $(G_1 + G_2)$ and also $D_1 \cup D_2$ is connected dominating set of $(G_1 + G_2)$,

then

$$D_c\left(\left(G_1+G_2\right),x\right)=D_c\left(G_1,x\right)+D_c\left(G_2,x\right)$$
$$+\binom{n_1}{1}\binom{n_2}{1}x^2+\binom{n_1}{1}\binom{n_2}{1}x^3$$
$$+\binom{n_1}{2}\binom{n_2}{1}x^3+\binom{n_1}{1}\binom{n_2}{3}x^4+\binom{n_2}{2}\binom{n_2}{2}x^4$$
$$+\binom{n_1}{3}\binom{n_2}{1}x^4+\binom{n_1}{1}\binom{n_2}{4}x^5+\binom{n_1}{2}\binom{n_2}{3}x^5$$
$$+\binom{n_1}{3}\binom{n_2}{2}x^5+\binom{n_1}{4}\binom{n_2}{1}x^5$$
$$+\ldots+\left[\binom{n_1}{1}\binom{n_2}{n_1+n_2-1}+\ldots+\binom{n_1}{n_1+n_2-1}\binom{n_2}{1}\right]x^{n_1+n_2}$$
$$=\left[\binom{n_1}{1}x+\binom{n_1}{2}x^2+\ldots+\binom{n_1}{n_1}x^{n_1}\binom{n_2}{1}x+\binom{n_2}{2}x^2+\ldots+\binom{n_2}{n_2}x^{n_2}\right]$$
$$+D_c\left(G_1,\ x\right)+D_c\left(G_2,x\right)$$
$$=D_c\left(G_1,x\right)+D_c\left(G_2,x\right)+\sum_{i=1}^{n_1}\binom{n_1}{i}x^i\sum_{i=1}^{n_2}\binom{n_2}{i}x^i$$
$$=D_c\left(G_1,x\right)+D_c\left(G_2,x\right)+\left[\left(1+x\right)^{n_1}-1\right]\left[\left(1+x\right)^{n_2}-1\right]$$

Corollary: 2.8 Let G be any wheel graph W_n with n vertices. Then

$$D_c(G, x) = x(1 + x)^{n-1} + x^{n-1} + (n - 1)x^{n-2} + (n - 1)x^{n-3}.$$

Proof: It is known that $W_n \cong C_n - 1 + K_1$ and by using Theorem 2.7, we have

$$D_c((C_{n-1} + K_1), x) = D_c(C_{n-1}, x) + D_c(K_1, x) + [(1+x)^{n-1} - 1]\,[(1+x)^1 - 1]$$

Also by using Theorems 2.4 and 2.6, we get

$$\begin{aligned}D_c((C_{n-1} + K_1), x) &= D_c(C_{n-1}, x) + D_c(K_1, x) + [(1+x)^{n-1} - 1]\,[(1+x)^1 - 1]\\ &= x^{n-1} + (n - 1)x^{n-2} + (n - 1)x^{n-3} + x + x\,[(1 + x)^{n-1} - 1]\\ &= x(1 + x)^{n-1} + x^{n-1} + (n - 1)x^{n-2} + (n - 1)x^{n-3}.\end{aligned}$$

Theorem: 2.9 Let G be any complete bipartite graph $K_{m,n}$, where $1 < m \leq n$. Then

$$D_c(G, x) = [(1 + x)^m - 1]\ [(1 + x)^n - 1]$$

Proof: If G is complete bipartite graph with partite sets V_1 and V_2, then any connected dominating set of G contains atleast one vertex from V_1 and at least one vertex from V_2 . So as in Theorem 2.7, we have

$$\begin{aligned} D_c(G,x) &= \binom{m}{1}\binom{n}{1}x^2 + \binom{m}{1}\binom{n}{2}x^3 + \binom{m}{2}\binom{n}{1}x^3 \\ &\quad + \ldots + \left[\binom{m}{1}\binom{n}{m+n-1} + \ldots + \binom{m}{n+m-1}\binom{n}{1}\right]x^{m+n} \\ &= \left[\binom{m}{1}x + \binom{m}{2}x^2 + \ldots + \binom{m}{m}x^m\right]\left[\binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n\right] \\ &= \sum_{i=1}^{m}\binom{m}{i}x^i \sum_{i=1}^{n}\binom{n}{i}x^i \\ &= \left[(1+x)^m - 1\right]\left[(1+x)^n - 1\right]. \end{aligned}$$

Theorem: 2.10 For any connected graph G with n vertices,

$$D_c(G \circ K_1, x) = (x^2 + x)^n$$

Proof: Let $H = G \circ K_1$. Then clearly $\Upsilon_c(H) = n$ and it is easy to see that there are $\binom{n}{i}$ possibilities to extend the connected dominating into connected dominating set of size n + i. Hence

$$\begin{aligned} D_c(H,x) &= x^n + \binom{n}{1}x^{n+1} + \binom{n}{2}x^{n+2} + \ldots + x^{2n} \\ &= x^n\left(1 + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + x^n\right) \\ &= x^n\left(\sum_{k=0}^{n}\binom{n}{k}x^k\right) \\ &= x^n\sum_{k=0}^{n}\binom{n}{k}x^k \\ &= (x^2 + x)^n. \end{aligned}$$

Theorem: 2.11 For any connected graph G with n vertices,

$$Dc(G \circ \overline{K_m},\ x) = x^n(1 + x)^{mn}.$$

Proof: Let $H = G \circ \overline{K_m},$. Then clearly $\Upsilon_c(H) = n$ and it is obvious that $\binom{mn}{i}$ possibilities to extend the connected dominating into connected dominating set of size n + i, where $1 \leq i \leq mn$. Hence

$$D_c(H,x) = x^n + \binom{mn}{1}x^{n+1} + \binom{mn}{2}x^{n+2} + \ldots + x^{n(m+1)}$$
$$= x^n\left(1 + \binom{mn}{1}x + \binom{mn}{2}x^2 + \ldots + x^{mn}\right)$$
$$= x^n\left(\sum_{i=1}^{mn}\binom{mn}{i}x^i\right)$$
$$= x^n(1+x)^{mn}.$$

A bi-star is a tree obtained from the graph K_2 with two vertices u and v by attaching m pendant edges in u and n pendant edges in v and denoted by B(m, n).

Theorem: 2.12 Let G be a bi-star graph B(m, n) as in Figure 2. Then

$$D_c(B(m, n), x) = x^2(1 + x)^{m+n}.$$

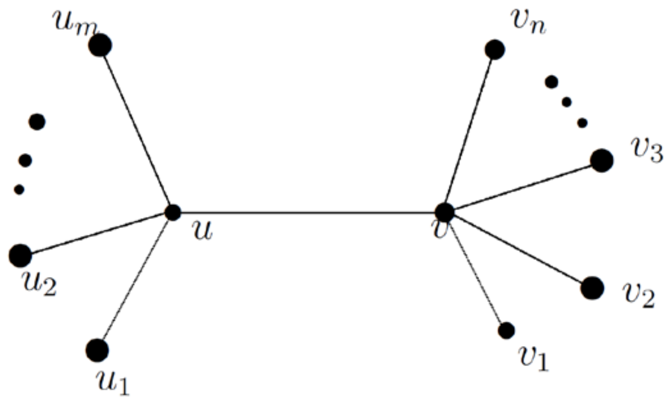

Figure 2:

Proof: Let G be a bi-star and with labeling as in Figure 2. Then $\gamma_c(B(m, n)) = 2$ namely the set {u, v} is the only unique minimum connected dominating set. Hence $d_c(B(m, n), \gamma_c) = 1$, and it is obvious that any other connected dominating set must be contain the two vertices u and v. Hence there are $\binom{m+n}{1}$ possibilities to extend the connected dominating into connected dominating set of size 3, and there are $\binom{m+n}{2}$ possibilities to extend the connected dominating into connected dominating set of size 4. In general it is easy to see that there are $\binom{m+n}{i-2}$ possibilities to extend the connected dominating into connected dominating set of size i. Therefore

$$D_c(B(m,n),x) = x^2 + \binom{m+n}{1}x^3 + \binom{m+n}{2}x^4 + \ldots + x^{n+m+2}$$
$$= x^2\left(1 + \binom{m+n}{1}x + \binom{m+n}{2}x^2 + \binom{m+n}{3}x^3 \ldots + x^{m+n}\right)$$
$$= x^2(1+x)^{m+n}.$$

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