International Journal of Mathematical Archive-4(11), 2013, 35-40
IJMA Available online through www.ijma.info ISSN 2229 – 5046

SOME TYPES OF PAIRWISE NORMAL SPACES

P. Padma[1]*, K. Chandra Sekhara Rao[2] and S. Udayakumar[3]

[1]Department of Mathematics, PRIST University, Kumbakonam, India.

[2]Department of Mathematics, SASTRA University, Kumbakonam, India.

[3]Department of Mathematics, A.V.V.M. Sri Puspam College, Poondi, Tanjore, India.

(Received on: 03-10-13; Revised & Accepted on: 13-11-13)

ABSTRACT

The purpose of this paper is to formulate and establish some results on pairwise normal spaces.

Keywords: *Pairwise lightly normal spaces, pairwise mildly normal spaces, pairwise mildly completely normal, pairwise lightly completely normal.*

Subject classification: *54D15.*

1. INTRODUCTION

We introduce the concept of pairwise lightly normal spaces. Implications between the class of pairwise normal spaces and the class of pairwise almost normal spaces between the class of pairwise almost normal spaces and the class of pairwise mildly normal spaces between the class of pairwise lightly normal spaces and the class of pairwise mildly normal spaces are investigated among other results.

2. PRELIMINARIES

If A is a subset of X with a topology τ, then the closure of A is denoted by τ - cl (A) or cl (A), the interior of A is denoted by τ - int (A) or int (A) and the complement of A in X is denoted by A^c. Now we shall require the following known definitions and prerequisites.

Definition: 2.1A subset S of (X,τ) is said to be an **α-set** if $S \subset \alpha S$ where $\alpha S = int(cl(int(S)))$.

Definition: 2.2 Let f: $(X,\tau) \rightarrow (Y,\mu)$. Then f is said to be **α - closed** if each closed set is mapped into an α - closed set.

Definition: 2.3 (X, τ) is **normal** if given any two disjoint closed sets A and B there exist disjoint open neighborhoods U and V of A and B respectively.

Definition: 2.4 A topological space (X, τ) is **almost normal** if for each pair of disjoint sets A & B one of which is closed and other is regularly closed, there exists open sets U & V such that $A \subset U$, $B \subset V$ & $U \cap V = \phi$.

3. PAIRWISE MILDLY NORMAL

Definition: 3.1 A bitopological space (X, τ_1, τ_2) is said to be **pairwisemildly normal** if given any two disjoint τ_1 - regularly closed set A and τ_2 - regularly closed set B in X, there exists an τ_2 - open neighborhood U of A and there exists an τ_1 - open neighborhood V of B such that $U \cap V = \phi$.

Example: 3.1 Let X = {a, b, c} and let $\tau_1 = \{\phi, X, \{a\}, \{a, c\}\}$ &$\tau_2 = \{\phi, X, \{a\}, \{b, c\}\}$. Then τ_1 - regularly closed sets are ϕ, X, {b, c} and τ_2 - regularly closed sets are ϕ, X, {a}, {b, c}. Take A = {b, c} and B = {a}. Then A is τ_1 - regularly closed set and B is τ_2 - regularly closed set. Also $A \cap B = \phi$. Then there exists an τ_2 - open U = {b, c} and τ_1 - open V = {a} such that $U \cap V = \phi$. Hence (X, τ_1, τ_2) is said to be pairwise mildly normal.

Corresponding author: P. Padma*
[1]Department of Mathematics, PRIST University, Kumbakonam, India.

We write τ_S = {all regularly open sets in X}.

Definition: 3.2 Let τ_{1S} be the semi regularization of $\tau_1$and τ_{2S} be the semi regularization of τ_2. If f: $(X, \tau_{1S}, \tau_{2S}) \to (Y, \mu_{1S}, \mu_{2S})$ is pairwise continuous then f : $(X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ is said to be **pairwise δ - continuous**.

Theorem: 3.1: Let f: $(X, \tau_1, \tau_2) \to (Y, \mu_1, \mu_2)$ be a pairwise δ - continuous, pairwise α - closed, surjection. If X is pairwise mildly normal then Y is pairwise mildly normal.

Proof: Suppose that A and B are τ_1 - regularly closed set, τ_2 - regularly closed set in X such that $A \cap B = \phi$.

But f is δ - continuous.

Hence $f^{-1}(A)$ and $f^{-1}(B)$ are τ_1 - regularly closed and τ_2 - regularly closed in X such that $f^{-1}(A) \cap f^{-1}(B) = \phi$

But X is pairwise mildly normal.

Consequently, there exists an τ_1-open neighborhood U of $f^{-1}(A)$ and there exists an τ_2 - open neighborhood V of $f^{-1}(B)$ such that $U \cap V = \phi$.

By [1], there exist disjoint α - sets C and D such that $A \subset C$ and $B \subset D$ with $\alpha(C) \cap \alpha(D) = \phi$.

Also α (C) and α (D) are τ_1 - regularly open set and τ_2 - regularly open set, and $A \subset \alpha(C)$ and $B \subset \alpha(D)$.

But α (C) and α (D) are τ_1 - open set, τ_2 - open set.

Accordingly, Y is pairwise mildly normal.

Definition: 3.3 A topological space (X, τ_1, τ_2) is said to be **pairwise lightly normal** if given any two disjoint τ_1 - regularly closed set A and τ_2 - regularly closed set B of X, there exists a τ_1 - regularly open neighborhood U of A and there exists a τ_2 - regularly open neighborhood V of B such that $U \cap V = \phi$.

Example: 3.2 In example 2.1, τ_1 - regularly closed sets are ϕ, X, {b, c} and τ_2 - regularly closed sets are ϕ, X, {a}, {b, c}. Take A = {b, c} and B = {a}. Then A is τ_1 - regularly closed set and B is τ_2 - regularly closed set. Also $A \cap B = \phi$. Then there exists an τ_2 - regularly open U = {b, c} and τ_1- regularly open V = {a} such that $U \cap V = \phi$. Hence (X, τ_1, τ_2) is said to be pairwise lightly normal.

Theorem: 3.2 A bitopological space X is pairwise lightly normal if and only if for each τ_1 - regularly closed set F of X and each τ_2 - regularly open neighborhood U of V there exists a τ_1 - regularly open neighborhood V of F such that $F \subset V \subset \tau_2\text{ - cl}(V) \subset U$.

Proof:
Step 1: Suppose that X is pairwise almost normal.

Let F be a τ_1 - regularly closed subset of X.

Let U be a τ_2 - regularly open neighborhood of F.

Then we have $F \cap (X - U) = \phi$ and F, (X – U) are τ_1 - regularly closed and τ_2 – regularly closed subsets of X.

But X is pairwise almost normal.

Accordingly, there exist τ_1 - regularly open set V and τ_2 - regularly open set W such that

$F \subset V$ and $X - U \subset W$.

$\Rightarrow V \subset X - V$ and $(X - W) \subset U$.

$\Rightarrow F \subset V \subset cl(V) \subset cl(X - W) \subset X - W \subset U$.

Step 2: Suppose that for each τ_1 - regularly closed set F and each τ_2 - regularly open neighborhood of F, there exists a τ_1 - regularly open neighborhood V of F such that $F \subset V \subset \tau_2\text{ - cl}(V) \subset U$.

Let F_1 and F_2 be two disjoint τ_1 - regularly closed and τ_2 - regularly closed sets.

But then $X - F_2$ is τ_2 - regularly open and $F_1 \subset X - F_2$.

Thus, there exists a τ_1 - regularly open neighborhood V of F_1 such that

$F_1 \subset V \subset \tau_2$ - cl (V) $\subset X - F_2$.

$\Rightarrow F_1 \subset V$ and $F_2 \subset X - [\tau_2$ - cl (V)].

But V $\subset \tau_2$ - cl (V).

$\Rightarrow$ V and (X– V) are disjoint.

$\Rightarrow$ X is pairwise almost normal.

Theorem: 3.3 If (X, τ_1, τ_2) is pairwise normal then (X, τ_1, τ_2) is pairwise almost normal.

Proof: Let Abe a τ_1 - closed set in X.

Let B be a τ_2 - regularly closed set in X with $A \cap B = \phi$.

But then B is τ_2- closed set.

Since (X, τ_1, τ_2) is pairwise normal, there exist τ_2 - open set U and τ_1 - open set V such that $A \subset U$ and $B \subset V$ with $U \cap V = \phi$.

Thus, (X, τ_1, τ_2) is almost normal.

Theorem: 3.4 If (X, τ_1, τ_2) is pairwise almost normal then (X, τ_1, τ_2) is pairwise mildly normal.

Proof: Suppose that (X, τ_1, τ_2) is pairwise almost normal.

Let A and B be two disjoint τ_1 - regularly closed set and τ_2 - regularly closed in X.

Since A is τ_1 - regularly closed, A is τ_1 - closed.

Also B is τ_2 - regularly closed with $A \cap B = \phi$.

But (X, τ_1, τ_2) is pairwise almost normal.

Hence there exists τ_1 - open set U and τ_2 - open set V such that $A \subset U$ and $B \subset V$ with $U \cap V = \phi$. .

Therefore, (X, τ_1, τ_2) is pairwise mildly normal.

Theorem: 3.5 If (X, τ_1, τ_2) is pairwise lightly normal then (X, τ_1, τ_2) is pairwise mildly normal.

Proof: Suppose that X is pairwise lightly normal.

Let A and B be τ_1 - regularly closed set and τ_1 - regularly closed set with $A \cap B = \phi$.

But (X, τ_1, τ_2) is pairwise lightly normal.

Hence there exists τ_2 - regularly open set U and τ_1 - regularly open set V such that $A \subset U$ and $B \subset V$ with$U \cap V = \phi$.

Similarly, V is τ_1 - open and U is τ_2 - open.

Hence (X, τ_1, τ_2) is pairwise mildly normal.

Theorem: 3.6 Every pairwise regularly closed, pairwise continuous image of a pairwise mildly normal space is pairwise mildly normal space.

Proof: Let X be a pairwise mildly normal space

Let f: $(X, \tau_1, \tau_2) \xrightarrow{onto} (Y, \mu_1, \mu_2)$ be a pairwise regularly closed and pairwise continuous mapping.

Let A and B be disjoint subsets of Y, where A is μ_1 - regularly closed and B is μ_2 - regularly closed.

Then $f^{-1}(A)$ is τ_1 - regularly closed and $f^{-1}(B)$ is τ_2 - regularly closed.

Also $A \cap B = \phi \Rightarrow f^{-1}(A \cap B) = f^{-1}(\phi) = \phi$.

$\Rightarrow f^{-1}(A) \cap f^{-1}(B) = \phi$.

Since X is pairwise mildly normal, there exist disjoint sets G_A and G_B such that $f^{-1}(A) \subset G_A$ and $f^{-1}(B) \subset G_B$, where G_A is τ_2 - open and G_B is τ_1 - open.

Let $G_A^* = \{y: f^{-1}(A) \subset G_A\}$ and $G_B^* = \{y: f^{-1}(B) \subset G_B\}$.

Then $G_A^* \cap G_B^* = \phi$, $A \subset G_A^*$, $B \subset G_B^*$ and since $G_A^* = y - f(X - G_A)$ and $G_B^* = y - f(X - G_B)$.

G_A^* is μ_2 - open and G_B^* is μ_1 – open.

Hence (Y, μ_1, μ_2) is pairwise mildly normal space.

Theorem: 3.7 Every bi - regular closed subspace of a pairwise mildly normal space is pairwise mildly normal.

Proof: Let $(Y, \tau_{1y}, \tau_{2y})$ be a bi - regular closed subspace of a pairwise mildly normal space (X, τ_1, τ_2).

Let A be a τ_{1y} - regular closed set and B be a τ_{2y} - regular closed set disjoint from A.

Since the space Y is bi - regular closed.

A is τ_1- regular closed and B is τ_2 - regular closed.

By pairwise mildly normality of (X, τ_1, τ_2) there exists a τ_2 - open set U and τ_1 - open set V such that $A \subset U$ and $B \subset V$,

$U \cap V = \phi$.

Thus, $A = A \cap Y \subset Y \cap U$.

$B = B \cap Y \subset V \cap Y$

$\Rightarrow (U \cap Y) \cap (V \cap Y) = \phi$.

Also $U \cap Y$ is τ_{2y} - open and $V \cap Y$ is τ_{1y} - open.

Thus, there exists a τ_{1y} - open set $V \cap Y$ and τ_{2y} - open set $U \cap Y$ such that

$A \subset U \cap Y$, $B \subset V \cap Y$, $(U \cap Y) \cap (V \cap Y) = \phi$.

Hence $(Y, \tau_{1y}, \tau_{2y})$ is pairwise mildly normal.

Definition: 3.4 A pairwise mildly normal, pairwise - T_1 space is called a pairwise - T_{MN} space.

Definition: 3.5 A pairwise lightly normal, pairwise - T_1 space is called a pairwise – T_{LN} space.

Theorem: 3.8 Let (X, τ_1, τ_2) be a pairwise mildly normal bitopological space. Let A be a τ_1 - regular closed set as well as τ_2 - regular closed set. Then $(X, \tau_1(A), \tau_2(A))$ is pairwise mildly normal $\Leftrightarrow$ $(X–A, \tau_1 \cap (X – A), \tau_2 \cap (X– A))$ is pairwise mildly normal.

Proof:

Step 1: If (X, τ_1 (A), τ_2 (A)) is pairwise mildly normal then (X–A, $\tau_1 \cap$(X–A), $\tau_2 \cap$ (X – A)) is pairwise mildly normal.

Since A be a τ_1 - regular closed and τ_2 - regular closed and hence also τ_1 (A) - regular closed and τ_2 (A) - regular closed.

But (X – A, $\tau_1 \cap$ (X – A), $\tau_2 \cap$ (X – A)) = (X – A, τ_1 (A) $\cap$ (X – A), τ_2 (A) $\cap$ (X – A)).

Hence the result.

Step 2: Let (X – A , $\tau_1 \cap$ (X – A)) be pairwise mildly normal.

Let F $\cap$ G = ϕ, where F is τ_1 (A) - regular closed, G is τ_2 (A) - regular closed.

Then F $\cap$ A is (τ_1 (A) $\cap$ A) - regular closed and hence ($\tau_1 \cap$ A) - regular closed and G $\cap$ A is (τ_2 (A) $\cap$ A) - regular closed and hence ($\tau_2 \cap$ A) - regular closed in (A, $\tau_1 \cap$ A, $\tau_2 \cap$ A).

Also since A is τ_1 - regular closed and τ_2 - regular closed.

Therefore, F $\cap$ A is τ_1 - regular closed and G $\cap$ A is τ_2 - regular closed.

But (X, τ_1, τ_2) is pairwise normal.

Therefore, there exists disjoint sets U and V such that F $\cap$ A $\subset$ U, G $\cap$ A $\subset$ V, U is τ_2 - open and V is τ_1- open.

Also, F$\cap$(X – A) and G $\cap$ X_A are disjoint, F $\cap$ (X – A) is τ_1 (A) $\cap$ (X – A) - regular closed and also ($\tau_1 \cap$ (X– A))- regular closed and G $\cap$ (X – A) is τ_2 (A) $\cap$(X – A) - regular closed and also ($\tau_2 \cap$ (X – A)) - regular closed subsets of (X – A, τ_1(A) $\cap$ (X – A), τ_2 (A) $\cap$ (X – A)) which is pairwise mildly normal.

Therefore, there exists disjoint sets U^* and V^* such that F $\cap$ (X – A) $\subset$ U^*, G $\cap$ (X – A) $\subset$ V^*, where U^* is $\tau_2 \cap$ (X – A) - open, V^* is $\tau_1 \cap$ (X – A) - open.

But since X – A belongs to τ_1 (A) as well as τ_2(A), U^* is τ_2 - open, V^* is τ_1- open.

Then F = (F $\cap$ A) $\cup$ F $\cap$ (X_A) $\subset$ (U $\cap$ A) $\cup$ U^*.

Take U^{**} = (U$\cap$ A) $\cup$ U^*
V^{**} = (V$\cap$ A) $\cup$ V^*.

Then $U^{**} \cap V^{**} = \phi$.

Hence F $\cap$(X – A) $\subset$ U^{**}.
$\Rightarrow$ F $\subset$ U^{**}.

Similarly, G $\subset$ V^{**}, U^{**} is τ_2 (A) - open and V^{**} is τ_1 (A) - open.

Thus, (X, τ_1 (A), τ_2 (A)) is pairwise mildly normal.

4. PAIRWISE MILDLY COMPLETELY NORMAL

Definition: 4.1 A bitopological space (X, τ_1, τ_2) is said to be pairwise mildly completely normal provided that whenever A and B are subsets of X such that τ_{1S}- cl (A)$\cap$ B = ϕ and A $\cap \tau_{2S}$- cl (B) = ϕ, there exists a τ_2- open set U and a τ_1- open set V such that A $\subset$ U and B $\subset$ V, U $\cap$ V = ϕ.

Definition: 4.2 A pairwise T_1, pairwise mildly completely normal bitopological space is called pairwise T_{MC} space.

Theorem: 4.1 Every pairwise mildly completely normal space is pairwise mildly normal.

Proof: Let (X, τ_1, τ_2) be a pairwise mildly completely normal space.

Let A be a τ_1 - regular closed set, B be a τ_2 - regular closed set such that A $\cap$ B = ϕ.

Then τ_{1S}- cl (A)$\cap$ B = A $\cap$ B = ϕ and A $\cap\tau_{2S}$- cl (B) = A $\cap$ B = ϕ.

By pairwise mildly completely normal, there exists a τ_2- open set U and a τ_1- open set V such that A $\subset$ U and B $\subset$ V, U $\cap$ V = ϕ.

Hence X is pairwise mildly normal.

5. PAIRWISE LIGHTLY COMPLETELY NORMAL

Definition: 5.1A bitopological space (X, τ_1, τ_2) is said to be pairwise lightly completely normal provided that whenever A and B are subsets of X such that τ_{1S}- cl (A)$\cap$B=ϕ and A $\cap\tau_{2S}$- cl (B) = ϕ, there exists a τ_2- regular open set U and a τ_1- regular open set V such that A $\subset$ U and B $\subset$ V, U $\cap$ V = ϕ.

Definition: 5.2 A pairwise T_1 , pairwise lightly completely normal bitopological space is called pairwise T_{LC} space .

Theorem: 5.1 Every pairwise lightly completely normal space is pairwise lighly normal.

Proof: Let (X, τ_1, τ_2) be a pairwise lightly completely normal space.

Let A be a τ_1 - regular closed set, B be a τ_2 - regular closed set such that A $\cap$ B =ϕ.

Then τ_{1S}- cl (A)$\cap$ B = A $\cap$ B = ϕ and A $\cap\tau_{2S}$- cl (B) = A $\cap$ B = ϕ.

By pairwise lightly completely normal, there exists a τ_2- regular open set U and a τ_1- regular open set V such that A $\subset$ U and B $\subset$ V, U $\cap$ V = ϕ.

Hence X is pairwise lightly normal.

REFERENCES

[1] N. Bourbaki, Elements of Mathematics. General topology, Part I, Addison Wesley, Reading, Mass., 1966.
[2] K.Chandrasekhararao, Topology, Alpha Science, Oxford 2009.
[3] K.Chandrasekhararao, ON almost regular spaces, Int. Journal of Math. Analysis, Vol. 7, 2013, no 38, 1857-1862.
[4] K.Chandrasekhararao, Some types of normal spaces, International journal of mathematical Archive, (Accepted).
[5] M.K.Singal and S.P.Arya, On Almost Regular spaces, Glasnik. Matematicki, Series III, 4 (1969), 89 - 99.
[6] A.S.Mashhoan, I.A.Hasanein and S.N.El - Deeb, α - continuous and α - open mappings, Acta Math. Hung., 41 (1983), 213 - 218.
[7] T.Noiri, A note on Mildly normal spaces, Kyungpook Math J. 13 (1973), 225 - 228.
[8] T.Noiri, On δ - continuous functions, J. Korean Math. Soc., 16 (1980), 161 - 166.
[9] M.K.Singal and S.P.Arya, Almost normal and almost completely regular spaces, Glasnik Mat., 5 (25) (1970), 141 – 152.
[10] M.K.Singal and S.P.Arya , On Almost Regular spaces, Glasnik. Matematicki, Series III, 4 (1969), 89 - 99.
[11] D.Sivaraj, Semi open set Characterizations of almost regular spaces, Glasnik. Matematicki, 21 (1986), 437 - 440.
[12] G.B.Navalagi, "Definition Bank in General Topology", Department of math., G. H. college, Haveri-581110, Karanataka, India.

Source of support: Nil, Conflict of interest: None Declared