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RESULTS ON FIXED POINT THEOREM AND MENGER SPACES

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ABSTRACT

In this paper, we prove a common fixed point theorem in Menger spaces by using five compatible mappings.

Keywords: *Menger space, t-norm, Common fixed point, Compatible maps, Weak - compatible maps.*

AMS subject classification (2000): *45H10, 54H25.*

I. INTRODUCTION AND PRELIMINARIES

There have been a number of generalizations of metric space. One such generalization is Menger space in which, used distribution functions instead of nonnegative real numbers as value of metric. A Menger space is a space in which the concept of distance is considered to be a probabilistic, rather than deterministic. For detail discussion of Menger spaces and their applications we refer to Schweizer and Sklar [16]. The theory of Menger space is fundamental importance in probabilistic functional analysis.

The important development of fixed point theory in Menger spaces were due to Sehgal and Bharucha-Reid [13]. A probabilistic metric space shortly PM-Space, is an ordered pair (X, F) consisting of a non empty set X and a mapping F from $X \times X$ to L, where L is the collection of all distribution functions (a distribution function F is non decreasing and left continuous mapping of reals in to $[0,1]$ with properties, $inf\ F(x) = 0$ and $sup\ F(x) = 1$).

The value of F at $(x, y) \in X \times X$ is represented by $F_{x,y}$. The function $F_{x,y}$ are assumed satisfy the following conditions:
(FM-0) $F_{x,y}\ (t)\ = 1$, for all $t > 0$, iff $x = y$;
(FM-1) $F_{x,y}\ (0) = 0$, if $t = 0$;
(FM-2) $F_{x,y}(t) = F_{y,x}(t)$;
(FM-3) $F_{x,y}\ (t)\ =\ 1$ and $F_{y,z}\ (s)\ =\ 1$ then $F_{x,z}\ (t\ +\ s) = 1$.

A mapping $T: [0,1] \times [0,1] \to [0,1]$ is a t-norm, if it satisfies the following conditions:
(FM-4) $T(a, 1) = a$ for every $a \in [0,1]$;
(FM-5) $T(0, 0) = 0$,
(FM-6) $T(a, b) = T(b, a)$ for every $a, b \in [0,1]$;
(FM-7) $T(c, d)\ \geq T(a, b)$for $c \geq a$ and $d \geq b$
(FM-8) $T(T(a, b), c) = T(a, T(b, c))$ where $a, b, c, d \in [0,1]$.

A Menger space is a triplet (X, F, T), where (X, F) is a PM-Space, X is a non-empty set and a $t -$ norm satisfying instead of (FM-8) a stronger requirement.
(FM-9) $F_{x,z}\ (t\ +\ s) \geq T\left(F_{x,y}(t), F_{y,z}(s)\right)$ for all $x \geq 0, y \geq 0$.

For a given metric space (X, d) with usual metric d, one can put $F_{x,y}\ (t)\ =\ H\ (t\ - d(x, y))$ for all $x, y\ \in\ X$ and t > 0. where H is defined as:
$$H(t)\ = \begin{cases} 1\ if\ s > 0, \\ 0\ if\ s \leq 0. \end{cases}$$
and t-norm T is defined as $T(a, b) =\ min\ \{a, b\}$.

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For the proof of our result we required the following definitions.

Definition: 1.1 [11] Let $(X, F,*)$ be a Menger space and $*$ be a continuous t-norm.

a) A sequence $\{x_n\}$ in X is said to be converge to a point x in X (written $x_n \to x$) iff for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n,x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$.

b) (b) A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists an integer $n_0 = n_0(\varepsilon, \lambda)$ such that $F_{x_n,x_{n+p}}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0$ and $p > 0$.

c) A Menger space in which every Cauchy sequence is convergent is said to be complete.

Remark: 1.2 If $*$ is a continuous t-norm, it follows from $(FM-4)$ that the limit of sequence in Menger space is uniquely determined.

Definition: 1.3[15] Self maps A and B of a Menger space $(X, F,*)$ are said to be weakly compatible (or coincidentally commuting) if they commute at their coincidence points, i.e. if $Ax = Bx$ for some $x \in X$ then $ABx = BAx$.

Definition: 1.4[11] Self maps A and B of a Menger space $(X, F,*)$ are said to be compatible if $F_{ABx_m,BAx_n,}(t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Ax_n \to x$, $Bx_n \to x$ for some x in X as $n \to \infty$.

Remark: 1.5 If self maps A and B of a Menger space $(X, F,*)$ are compatible then they are weakly compatible.

The following is an example of pair of self maps in a Menger space which are weakly compatible but not compatible.

Example: 1.6 Let (X, d) be a metric space where $X = [0,2]$ and $(X, F,*)$ be the induced Menger space with $F_{x,y}(t) = H\big(t - d(x,y)\big), \forall\, x, y \in X$ and $\forall\, t > 0$.

Define self maps A and B as follows:

$$Ax = \begin{cases} 2 - x, & if\ 0 \leq x < 1, \\ 2 & if\ 1 \leq x \leq 2, \end{cases} \quad \text{and} \quad Bx = \begin{cases} x, & if\ 0 \leq x < 1, \\ 2 & if\ 1 \leq x \leq 2, \end{cases}$$

Take $x_n = 1 - 1/n$. Then $F_{Ax_{n+1},}(t) = H\,(t - (1/n))$ and $\lim_{n \to \infty} F_{Ax_{n+1},}(t) = H\,(t) = 1$.

Hence $Ax_n \to \infty$ as $n \to \infty$. Similarly, $Bx_n \to \infty$ as $n \to \infty$. Also $F_{ABx_n,BAx_n}(t) = H\,(t - (1 - 1/n))$ and $\lim_{n\to\infty} F_{ABx_n,BAx_n,}(t) \to 1 = H(t - 1) \neq 1, \forall\, t > 0$.

Hence the pair (A, B) is not compatible. Set of coincidence points of A and B is $[1,2]$. Now for any $x \in [1,2]$, $Ax = Bx = 2$, and $AB(x) = A\,(2) = 2 = S(2) = SA(x)$. Thus A and B are weakly compatible but not compatible.

Lemma: 1.7 Let $\{x_n\}$ be a sequence in a Menger space$(X, F,*)$ with continuous t-norm $*$ and $t * t\ \ t$. If there exists a constant $k \in (0,1)$ such that

$$F_{x_n,x_{n+1}}(kt) \geq F_{x_{n-1},x_n}(t)$$

for all $t > 0$ and $n = 1, 2\ldots$, then $\{x_n\}$ is a Cauchy sequence in X.

Lemma: 1.8[15] Let $(X, F,*)$ be a Menger space. If there exists $k \in (0,1)$ such that

$$F_{x,y}(kt) \geq F_{x,y}(t)$$

for all $x, y \in X$ and $t > 0$, then $x = y$.

2. MAIN RESULTS

Theorem: 2.1 Let A, B, S, T and P be self maps on a complete Menger space $(X, F,*)$ with $t * t \geq t$ for all $t \in [0,1]$, satisfying:

(a) $P(X) \subseteq AB(X), P(X) \subseteq ST(X)$;

(b) there exists a constant $k \in (0,1)$ such that

$$M_{Px,Py,}(kt) \geq M_{ABx,Px,}(t) * M_{Px,STy,}(t) * M_{ABx,STy,}(t) * \frac{M_{Px,ABx,}(t) * M_{Px,STy,}(t)}{M_{STy,ABx,}(t)} * M_{ABx,Py,}(3-\alpha)t$$

for all $x, y \in X, \alpha \in (0,3)$ and $t > 0$,

(c) $PB = BP, PT = TP, AB = BA$ and $ST = TS$,

(d) A and B are continuous,

(e) the pair (P, AB) is compatible (if compatible then it is weak compatible)

Then A, B, S, T and P have a common fixed point in X.

Proof: Since $P(X) \subset AB(X)$, for $x_0 \epsilon X$, we can choose a point $x_0 \epsilon X$ such that $Px_0 = ABx_1$. Since $P(X) \subset$ ST(X), for this point x_1, we can choose a point $x_2 \epsilon X$ such that $Px_1 = STx_2$. Thus by induction, we can define a sequence $y_n \epsilon X$ as follows:

$y_{2n} = Px_{2n} = ABx_{2n+1}$ and $y_{2n+1} = Px_{2n+1} = STx_{2n+1}$

for $n = 1,2, \ldots.$ from (b),

For all $t > 0$ and $\alpha = 2 - q$ with $q \in (0, 2)$, we have

$$M_{y_{2n+1},y_{2n+2},}(kt) = M_{Px_{2n+1},Px_{2n+2},}(kt) \geq M_{y_{2n+1},y_{2n+1},}(t) * M_{y_{2n},y_{2n+1},}(t)$$
$$* M_{y_{2n},y_{2n+1},}(t) * \frac{M_{y_{2n+1},y_{2n},}(t) * M_{y_{2n+1},y_{2n+1},}(t)}{M_{y_{2n+1},y_{2n},}(t)}$$
$$* M_{y_{2n},y_{2n+2},}(1+q)t,$$

$$M_{y_{2n+1},y_{2n+2},}(kt) \geq M_{y_{2n},y_{2n+1},}(t) * M_{y_{2n},y_{2n+2},}(1+q)t$$

$$\geq M_{y_{2n},y_{2n+1},}(t) * M_{y_{2n},y_{2n+1},}(t) * M_{y_{2n+1},y_{2n+2},}(qt)$$

$$\geq M_{y_{2n},y_{2n+1},}(t) * M_{y_{2n+1},y_{2n+2},}(t)$$

as $q \to 1$.Since$*$is continuous and $M_{x,y}(*)$ is continuous, letting $q \to 1$ in above eq., we get

$$M_{y_{2n+1},y_{2n+2},}(kt) \geq M_{y_{2n},y_{2n+1},}(t) * M_{y_{2n+1},y_{2n+2},}(t) \ldots\ldots \quad (1)$$

Similarly, we have

$$M_{y_{2n+2},y_{2n+3},}(kt) \geq M_{y_{2n+1},y_{2n+2},}(t) * M_{y_{2n+2},y_{2n+2},}(t) \ldots\ldots \quad (2)$$

Thus from (1) and (2), it follows that

$$M_{y_{n+1},y_{n+2},}(kt) \geq M_{y_n,y_{n+1},}(t) * M_{y_{n+1},y_{n+2},}(t)$$

for $n = 1,2, \ldots$ and then for positive integers n and p,

$$M_{y_{n+1},y_{n+2},}(kt) \geq M_{y_n,y_{n+1},}(t) * M_{y_{n+1},y_{n+2},}\left(\frac{t}{k^p}\right).$$

Thus, since $M_{y_{n+1},y_{n+1},}\left(\frac{t}{k^p}\right) \to 1$ as $p \to \infty$ we have

$$M_{y_{n+1},y_{n+2},}(kt) \geq M_{y_n,y_{n+1},}(t).$$

y_n is Cauchy sequence in X and since x is complete, y_n converges to a point $z \epsilon X$.

Since Px_n, ABx_{2n+1} and STx_{2n+2} are subsequences of y_n, they also converge to the point z, since A, B are continuous and pair $\{P, AB\}$ is compatible and also weak compatible, we have

$lim_{n\to\infty} PABx_{2n+1} = ABz$ and $lim_{n\to\infty}(AB)^2 x_{2n+1} = ABz$, From (b) with $\alpha = 2$, we get

$$M_{PABx_{2n+1},\ Px_{2n+2},}(kt) \geq M_{(AB)^2x_{2n+1},}(t) * M_{PABx_{2n+1},\ STx_{2n+2},}(t) * M_{(AB)^2x_{2n+1},\ STx_{2n+2},}(t) * \frac{M_{PABx_{2n+1},(AB)^2x_{2n+1},}(t) * M_{PABx_{2n+1},\ STx_{2n+2},}(t)}{M_{STx_{2n+2},(AB)^2x_{2n+1},}(t)} * M_{(AB)^2x_{2n+1},\ Px_{2n+2},}(t)$$

which implies that

$$M_{ABz,z}(kt) = lim_{n\to\infty} M_{PABx_{2n+2},}(kt)$$

$$\geq 1 * M_{ABz,z,}(t) * M_{ABz,z,}(t) * \frac{1 * M_{ABz,z}(t)}{M_{z,ABz}(t)} * M_{ABz,z,z}(t)$$

We have

$ABz = z$, since $M_{z,,STz}(t) \geq M_{z,ABz}(t) = 1$ for all $t > 0$,

We get $STz = z$. again by (b) with $\alpha = 2$,

We have

$$M_{PABx_{2n+1},Pz}(kt) \geq M_{(AB)^2x_{2n+1},\ PABx_{2n+1},}(t) * M_{PABx_{2n+1},\ STz,,}(t) * M_{(AB)^2x_{2n+1},\ STz,,}(t) * \frac{M_{PABx_{2n+1},(AB)^2x_{2n+1},}(t) * M_{PABx_{2n+1},\ STz,,}(t)}{M_{STz,(AB)^2x_{2n+1},}(t)} * M_{(AB)^2x_{2n+1},\ Pz,}(t)$$

which implies that

$$M_{ABz,Pz,Pz}(kt) = lim_{n\to\infty} M_{PABx_{2n+1},Pz,}(kt)$$

$$\geq 1 * 1 * 1 * 1 * M_{ABz,Pz,}(t)$$

$$\geq M_{ABz,Pz,}(t).$$

We have $ABz = Pz$.Now, we show that $Bz = z$.Infact, from (b) with $\alpha = 2$, and (c) we get,

$$M_{Bz,z,}(kt) = M_{BPz,Pz,}(kt)$$

$$= M_{PBz,Pz,}(kt)$$

$$M_{PBz,Pz,}(kt) \geq M_{PBz,STz,}(t) * M_{ABBz,STz,}(t) * \frac{M_{PBz,ABBz,}(t) * M_{PBz,z,z}(t)}{M_{z,PBz,}(t)} * M_{PBz,z}(t)$$

$$= 1 * M_{Bz,z,}(t) * M_{Bz,z,}(t) * 1 * M_{Bz,z,}(t)$$

$$= M_{Bz,z,}(t).$$

which implies that $Bz = z$.Since $ABz = z$,

we have $Az = z$.Next, we show that $Tz = z$. Indeed from (b) with $\alpha = 2$, and (c) we get

$$M_{Tz,z,}(kt) = M_{TPz,Pz,}(kt) = M_{Pz,Pz,}(kt)$$

$$\geq 1 * M_{z,Tz,}(t) * M_{z,Tz,}(t) * 1 * M_{z,Tz,}(t)$$

$$\geq M_{Tz,z,}(t),$$

which implies that $Tz = z$.Since $STz = z$, we have $Sz = STz = z$. Therefore, by combining the above results we obtain,

$$Az = Bz = Sz = Tz = Pz.$$

Therefore z is the common fixed point of A, B, S, T and P.

Finally, the uniqueness of the fixed point of A, B, S, T and P.

Corollary: 2.2 Let A, B, S, T and P be self maps on a complete Menger space $(X, F, *)$ with $t * t \geq t$ for all $t \in [0, 1]$, satisfying;

(a) $P(X) \subseteq A(X)$ and $P(X) \subseteq S(X)$

(b) there exists a constant $k \in (0, 1)$ such that

$$M_{Px,Py,}(kt) \geq M_{Ax,Px,}(t) * M_{Px,Sy,}(t) * M_{Ax,Sy,}(t) * \frac{M_{Px,Ax,}(t) * M_{Px,Sy,}(t)}{M_{Sy,Ax,}(t)} * M_{Ax,Py,}(3-\alpha)t$$

for all $x, y \in X, \alpha \in (0,3)$ and $t > 0$,

(c) A or P are continuous,

(d) the pair $\{P, A\}$ is compatible,

Then A, S and P have a common fixed point in X.

CONCLUSION

In this present article we prove a common fixed point theorem in Menger spaces by using five compatible mappings. In fact our main result is more general then other previous known results.

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