International Journal of Mathematical Archive-2(4), Apr. - 2011, Page: 543-548
Available online through www.ijma.info ISSN 2229 – 5046

X-DOMATIC PARTITION OF GRAPH $VV^{+}(G)$ AND CHROMATIC NUMBER OF G

***Y. B. Venkatakrishnan[1] and [2] V. Swaminathan**

[1] *Department of Mathematics, Sastra University, Tanjore 613 402. India*

[2] *Coordinator, Ramanujan Research Centre, S. N. College, Madurai 625 002. India*

E-mail: [1] sulanesri@yahoo.com, [2] Venkatakrish2@maths.sastra.edu

(Received on: 10-04-11; Accepted on: 17-04-11)

ABSTRACT

Let $G^1 = (X, Y, E)$ be a bipartite graph. A X- domatic partition of G^1 is a partition of X, all of whose classes are X-dominating set in G^1. The X-domatic number of G^1 is the maximum number of classes of X-domatic partition of G^1. The X-domatic number of G^1 is denoted by $d_X(G^1)$.

Let G be a simple undirected graph. We obtain a sharp upper bound for the sum of chromatic number of G and X-domatic number of the bipartite graph VV^{+} obtained from G and characterize the corresponding extremal graphs.

***Keywords:** Bipartite graphs, Chromatic number, X-domatic number.*

***MSC 2000:** 05C69.*

---

1. INTRODUCTION:

Let $G = (V, E)$ be a simple undirected graph. The chromatic number $\chi(G)$ is defined to be the minimum number of colors required to color all the vertices such that adjacent vertices do not receive the same color. A domatic partition of G is a partition of $V(G)$, all of whose classes are dominating set in G. The domatic number of G is the maximum number of classes of a domatic partition of G. The domatic number is denoted by $d(G)$.

Given a graph $G = (V, E)$, the bipartite graph $VV^{+}(G) = (V, V^1, E^1)$ is defined by the edges $E^1 = \{(u, v^1)/(u, v) \in E\}$ together with $\{(u, u^1)/u \in V\}$. Let $G^1 = (X, Y, E)$ be a bipartite graph. A subset $D \subseteq X$ is an X-dominating set if for every $x \in X - D$, there exists atleast one vertex $u \in D$ such that x and u are adjacent to a common vertex $y \in Y$. The minimum cardinality taken over all the minimal X-dominating set is called X-domination number and is denoted by $\gamma_X(G^1)$. A X-domatic partition of G^1 is a partition of X, all of whose classes are X-dominating set in G^1. The X-domatic number of G^1 is the maximum number of classes of a X-domatic partition of G^1. The X-domatic number of G^1 is denoted by $d_X(G^1)$.

Bipartite theory of graphs was proposed in [2] and [3]. Given any problem say A, on an arbitrary graph G, there is a corresponding problem B on a bipartite graph G^1, such that solution for B provides a solution for A.

Several authors have studied the problem of obtaining an upper bound for the sum of a domination parameter and a graph

---

****Corresponding author: *Y.B. Venkatakrishnan[1], E-mail: Venkatakrish2@maths.sastra.edu***

theoretic parameter and characterized the corresponding extremal graphs. In [5] Paulraj Joseph J and Arumugam proved $\gamma_c + \chi \leq p+1$. They also characterized the class of graphs for which the upper bound is attained. In [4] Mahadevan et all proved $\gamma_{dd}(G) + \chi(G) \leq 2n$, and characterized the corresponding extremal graphs. In this paper, we obtain sharp upper bound for the sum of the chromatic number of a graph G and the X-domatic number of bipartite graph $VV^+(G)$ constructed from G and characterize the corresponding extremal graphs. By the theory of bipartite graph, the above gives an upper bound for the sum of the chromatic number of a graph G and the domatic number of G^2. [2] defines square of a graph $G^2 = (V, E^2)$ as $(u,v) \in E^2$ if and only if distance $d(u,v) \leq 2$ in G.

Proposition 1.1: $d_X(VV^+(K_n)) = n$.

Proof: K_n is a complete graph on n vertices. Every vertex is adjacent to other vertices in G. In VV^+, every vertex is X-adjacent to other vertices. Every vertex is a X-dominating set. Therefore, $d_X(VV^+(K_n)) = n$.

Theorem: 1.2 In a bipartite graph $G = (X, Y, E)$,$|X| = n$ then $d_X(G) \leq n$.

Theorem: 1.3 [1] For any connected graph $G,\ \chi(G) \leq \Delta + 1$.

2. MAIN RESULT:

Theorem: 2.1 For any connected graph G, $d_X(VV^+(G)) + \chi(G) \leq 2n$ and equality holds if and only if $G \cong K_n$.

Proof: $d_X(VV^+(G)) + \chi(G) \leq n + \Delta + 1 \leq n + n - 1 + 1 = 2n$. If $d_X(VE(G)) + \chi(G) = 2n$, then the possible case is $d_X(VV^+) = n$ and $\chi(G) = n$. Since, $\chi(G) = n$, $G \cong K_n$ and $d_X(VV^+) = n$. Hence, $G \cong K_n$. The converse of the above is obvious.

G_1 is the family of graphs on n vertices which contains a clique K=K_{n-1} and the other vertex is adjacent to α vertices of K, where $1 \leq \alpha \leq n-2$.

Theorem: 2.2 For any connected graph G, $d_X(VV^+(G)) + \chi(G) = 2n-1$ if and only if $G \in G_1$.

Proof: Assume $d_X(VV^+(G)) + \chi(G) = 2n-1$. This is possible only if $d_X(VV^+(G)) = n$ and $\chi(G) = n-1$ or $d_X(VV^+(G)) = n-1$ and $\chi(G) = n.$

Case: (a) $d_X(VV^+(G)) = n$ and $\chi(G) = n-1$.

Since, $\chi(G) = n-1$, G contains a clique $K = K_{n-1}$. Let x be any vertex of G other than the vertices of K. Since G is connected x is to a vertex of K. Let us assume x is adjacent to α vertices of $K, 1 \leq \alpha \leq \text{n - 2}$. The vertex in $N(x)$ of the graph G is of degree n in $VV^+(G)$. Hence, $G \in G_1$

Case :(b) $d_X(VV^+(G)) = n-1$ and $\chi(G) = n$.

Since, $\chi(G) = n; G \cong K_n$ and $d_X(VV^+(G)) = n \neq n-1$, a contradiction. So no graph exists.

If $G \in G_1$, then $d_X(VV^+(G)) + \chi(G) = 2n-1$.

G_2 is a family of graphs on n vertices which contains a clique $K = K_{n-2}$, and the other vertices are adjacent to same vertex of K and degree of two vertices are α, β where $1 \le \alpha, \beta \le n-3$.

G_3 is a family of graph on n vertices which contains a clique $K = K_{n-2}$ and the other vertices of G is K_2 and the ends of K_2 are adjacent to same vertex of K, the degree of two vertices are α, β where $1 \le \alpha, \beta \le n-2$.

Theorem: 2.3 For a connected graph G, $d_X(VV^+) + \chi(G) = 2n-2$ if and only if G belongs to G_2 or G_3.

Proof: If G belongs to G_2 or G_3, then $d_X(VV^+) + \chi(G) = 2n-2$ is obvious. Conversely, assume that $d_X(VV^+) + \chi(G) = 2n-2$. This is possible only if

(a) $d_X(VV^+) = n; \chi(G) = n-2$ or

(b) $d_X(VV^+) = n-1; \chi(G) = n-1$ or

(c) $d_X(VV^+) = n-2; \chi(G) = n.$

Case: (a) $d_X(VV^+) = n; \chi(G) = n-2$

Since $\chi(G) = n-2,$ G contains a clique $K = K_{n-2}$. Let $S = \{x, y\}$ be other vertices of $G.$ $\langle S \rangle = K_2$ or $\overline{K_2}$.

Sub case: (i) $\langle S \rangle = \overline{K_2}$.

Since G is connected. Vertices of $\langle S \rangle$ are adjacent to vertices of K. If x and y are adjacent to the same vertex of K, we get $d_X(VV^+) = n.$ We also get $d_X(VV^+) = n,$ if x and y are adjacent to α and β vertices of K where $1 \le \alpha, \beta \le n-3$. $G \in G_2$.

Sub case: (ii) $\langle S \rangle = K_2$

G is connected, one vertex of K_2 is adjacent to a vertex of K, $d_X(VV^+) < n$ a contradiction.

If both ends of K_2 are adjacent to same vertex of K, we get $d_X(VV^+) = n$. The vertices x and y can be of adjacent to α and β $1 \le \alpha, \beta \le n-2$. Which is graph G_3.

Case: (b) $d_X(VV^+) = n-1; \chi(G) = n-1$

Since $\chi(G) = n-1,$ G contains a clique $K = K_{n-1}$. Let x be a vertex of G other than the vertices of K. Since G is connected, x should be adjacent to atleast one vertex of K. Hence, $d_X(VV^+(G)) = n \ne n-1$. A contradiction. Therefore, no such graph exists.

Case: (c) $d_X(VV^+) = n-2; \chi(G) = n.$

Since $\chi(G) = n, G \cong K_n$. But $d_X(VV^+(K_n)) = n \ne n-2.$ A contradiction.

We define certain family of graphs

G_4 is the family of graphs on n vertices, which contain a clique $K = K_{n-3}$ and other vertices of G are adjacent to a vertex of K and the degree of the vertices are α, β, γ where $1 \le \alpha, \beta, \gamma \le n-4$.

G_5 is the family of graphs on n vertices, which contain a clique $K = K_{n-3}$ and other vertices of G forms a P_3 and all the vertices of P_3 are adjacent to same vertex of K. The degree of the vertices of P_3 are α, β, γ where $1 \le \alpha, \gamma \le n-3; 1 \le \beta \le n-2$.

G_6 is the family of graphs on n vertices, which contain a clique $K = K_{n-3}$ and other vertices of G form $K_1 \cup K_2$. All the vertices in $K_1 \cup K_2$ are adjacent to a vertex of K. The degree of these vertices are α, β, γ where $1 \le \alpha \le n-4; 1 \le \beta, \gamma \le n-3$.

G_7 is the family of graphs on n vertices, which contain a clique $K = K_{n-3}$ and other vertices of G form a K_3 and the vertices of K_3 are adjacent to different vertices of K.

G_8 is the family of graphs on n vertices, which contain a clique $K = K_{n-2}$ and the other two vertices are adjacent to different vertices of K.

G_9 is the family of graphs on n vertices, which contain a clique $K = K_{n-2}$ and the other vertices forms a K_2 in which the ends are adjacent to different vertices of K.

Theorem: 2.4 For a connected graph G, $d_X(VV^+) + \chi(G) = 2n-3$ if and only if $G \in G_i$ $4 \le i \le 9$.

Proof: If $G \in G_i$ $4 \le i \le 9$, then clearly $d_X(VV^+) + \chi(G) = 2n-3$. Conversely, assume that $d_X(VV^+) + \chi(G) = 2n-3$. This is possible only if

(a) $d_X(VV^+) = n-3; \chi(G) = n$ or

(b) $d_X(VV^+) = n-2; \chi(G) = n-1$ or

(c) $d_X(VV^+) = n-1; \chi(G) = n-2$ or

(d) $d_X(VV^+) = n; \chi(G) = n-3$.

Case (a): $d_X(VV^+) = n-3; \chi(G) = n$

Since, $\chi = n$ $G \cong K_n$. But $d_X(VV^+(K_n)) = n \ne n-3$ a contradiction. Hence, no such graph exists.

Case: (b) $d_X(VV^+) = n-2; \chi(G) = n-1$

Since $\chi(G) = n-1$, G contains a clique $K = K_{n-1}$. Let x be a vertex of G other than the vertices of K. Since G is connected, x is adjacent to atleast one vertex of K. $d_X(VV^+) = n \ne n-2$, a contradiction. Hence, no such graphs exist.

Case: (c) $d_X(VV^+) = n-1; \chi(G) = n-2$

Since $\chi(G) = n-2$, G contains a clique $K = K_{n-2}$. Let $S = \{x, y\}$ be vertices of G other than the vertices of K. Then $\langle S \rangle = K_2 or \overline{K_2}$.

Sub case: (i) If $\langle S \rangle = \overline{K_2}$.

If vertices in $\langle S \rangle$ are adjacent to same vertex of K, we get a contradiction to $d_X(VV^+) = n-1$. Hence no such graphs exists. If the vertices of $\langle S \rangle$ are adjacent to different vertices of K, we get $G \in G_8$.

Sub case: (ii) If $\langle S \rangle = K_2$.

If the end of K_2 are adjacent to a vertex of K, we get a contradiction to $d_X(VV^+) = n-1$. Hence no such graph exists. If the vertices of $\langle S \rangle$ are adjacent to different vertices of K, we get $G \in G_9$.

Case: (d) $d_X(VV^+) = n; \chi(G) = n-3$.

Since $\chi(G) = n-3$, G contains a clique $K = K_{n-3}$. Let $S = \{x, y, z\}$ be vertices of G other than the vertices of K. Then $\langle S \rangle = K_3; \overline{K_3}; P_3; K_2 \cup K_1$.

Sub case: (i) $\langle S \rangle = K_3$

Since G is connected, a vertex of K_3 is adjacent to a vertex of K. If the vertices of $\langle S \rangle$ are adjacent to different vertices of K, we get $d_X(VV^+) = n$ and therefore, $G \in G_7$.

Sub case: (ii) $\langle S \rangle = \overline{K_3}$.

Since G is connected, all the vertices of $\overline{K_3}$ are adjacent to a vertex of K. If the vertices x, y, z are of degree α, β, γ where $1 \le \alpha, \beta, \gamma \le n-4$, then $d_X(VV^+) = n.$ Therefore, $G \in G_4$.

Sub case: (iii) $\langle S \rangle = P_3$

Since G is connected, a vertex of P_3 is adjacent to a vertex of K, a contradiction to $d_X(VV^+) = n.$ If all the vertices of P_3 are adjacent to a vertex of K. If the degree of the vertices are α, β, γ where $1 \le \alpha, \gamma \le n-3; 1 \le \beta \le n-2.$ Then, $d_X(VV^+) = n.$ Hence, $G \in G_5$.

Sub case :(iv) $\langle S \rangle = K_2 \cup K_1$

If a vertex of K_2 and K_1 are adjacent to same vertex of K. If the degree of these vertices are α, β, γ where $1 \le \alpha \le n-4; 1 \le \beta, \gamma \le n-3.$ Then, $d_X(VV^+) = n.$ Hence $G \in G_6$.

Theorem: 2.5 For a connected graph G, $d(G^2) = d_X(VV^+(G))$.

Proof: Let $V_1, V_2, ..., V_n$ be domatic partition of $G.^2$ Each V_i is a dominating set in G^2. Therefore, V_i is a X-dominating set in the bipartite graph $VV^+(G)$. Hence, $V_1, V_2, ..., V_n$ is a X-domatic partition of $VV^+(G)$. Hence, $d_X(VV^+(G)) \ge d(G^2)$.

Conversely, $V_1, V_2, ..., V_n$ be X-domatic partition of $VV^+(G)$. Then, $V_1, V_2, ..., V_n$ is a domatic partition in G^2. Therefore, $d(G^2) \geq d_X(VV^+)$. Hence, $d(G^2) = d_X(VV^+(G))$.

REFERENCES:

[1] F. Harary, Graph Theory, Addison -Wesley, Reading, MA 1972.

[2] Stephen Hedetniemi, Renu Laskar, A Bipartite Theory of Graphs I, Congressus Numerantium Volume 55 (1986), 5-14.

[3] Stephen Hedetniemi, Renu Laskar, A Bipartite Theory of Graphs II, Congressus Numernatium Volume 64 (1988), 137-146.

[4] Mahadevan G , Selvam Avadayappan A, Mydeen Azmal Hussain, "Double Domination number and Chromatic number of a Graph", Mathematical and computational models, edited by R.Nadarajan et all, published by Narosa, India. Pages 182-190.

[5] Paulraj Joseph J and Arumugam S, Domination and Colouring in Graphs, International Journal of management and systems, Vol 15, n0.1, 37-44, 1999.

[6] Teresa W.Haynes, Stephen T. Hedetniemi and Peter J. Slater, Fundamentals of Domination in graphs, Marcel Dekker, Newyork, 1998.

****

